PARAMETRIC MODULATION OF INSTABILITIES OF A NONLINEAR DISCRETE SYSTEM

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The effect of modulation on the first instability of the logistic map is determined. Similarities with the parametrically modulated anharmonic oscillator are discussed. We also discuss small-amplitude modulation of the period-doubling bifurcations and the structural similarity with bifurcations of Taylor vortex flow in finite length annuli.

The response of a physical system, e.g. a fluid [1], towards a periodically modulated driving and in particular parametric modulation of an instability has received increasing attention recently. A representative example for this situation and a model for many systems [2] [‡1] with stability exchange is a classical particle moving with friction in the potential $\frac{1}{2}[1-r(t)] \times x^2 + \frac{1}{4}x^4$ varying, e.g., like $r(t) = r(1 + \Delta \cos \Omega t)$. Its reaction is typical for parametric modulation: Dynamical stabilization of a statically unstable state [4] and vice versa, generation of harmonic and subharmonic motion [5] and more complicated behavior [6]. That contrasts the simple bifurcation in the unmodulated system, $\Delta = 0$, where the particle starting at not too large $x > 0$ is attracted for $r > 1$ to the minimum at $\sqrt{r-1}$ and to the origin if $r < 1$.

Also the difference equation

$$x_{n+1} = r_n x_n (1 - x_n), \quad r_n = r(1 + \Delta \cos \Omega n), \qquad (1)$$

shows for $\Delta = 0$ such an exchange of stability at $r = 1$ between the origin and another fixpoint at $(r - 1)/r$. Here we determine the effect of modulation on this instability and indicate similarities with the response of the damped, anharmonic oscillator. We also discuss small-amplitude modulation of the higher bifurcations [7,8] of the unmodulated map. We consider mostly rational (or integer) modulation "times" $2\pi/\Omega = T = N/N'$ and mention briefly results for irrational T.

The trivial fixed point $x^* = 0$ of (1) is linearly stable if the geometric mean $|r_1 \ldots r_N|^{1/N}$ of the control parameters is less than one. At the threshold $r_c^\pm(\Delta; N)$ determined by

$$\pm 1 = \partial x_N(x_0)/\partial x_0|_{x_0=0} = (r_c^\pm)^N \prod_{j=0}^{N-1} (1 + \Delta \cos \Omega j). \qquad (2)$$

$x^* = 0$ exchanges stability with a limit cycle: a slope of +1 (−1) of $x_N(x)$ at the origin causes generation of a period N ($2N$) cycle. Furthermore $x_{j+N} = -x_j$ in the latter case. Floquet theory [9] yields the same for the oscillator: the bifurcation of $x = 0$ is either harmonic, $x(t + T) = x(t)$ or subharmonic with $x(t + T) = -x(t)$ [5]. Fig. 1 shows the stability boundaries for $N = 10, 11$ in the first quadrant of the r–Δ plane. Elsewhere they obtain similarly. We restrict ourselves henceforth to positive r. For even N the product entering (2) is positive (negative) if $\Delta^2 < 1$ ($\Delta^2 > 1$), hence the trivial fixed point bifurcates into an N-cycle ($2N$-cycle). Similarly, for odd N the bifurcation is harmonic if $\Delta > -1$ and subharmonic otherwise. While the periodic orbit does depend in general on $T = N/N'$ the threshold $r_c(\Delta; N)$ for its appearence depends only on the modulation period but not on N'.

Note the tendency to stabilize $x^* = 0$. The enhancement of the stability domain by small-amplitude modulation, $r_c^+(\Delta; N) - r_c(\Delta = 0) = \frac{1}{4}\Delta^2$, is proportional

[‡1] It also describes the first instability in the Rayleigh–Bénard problem [3] where x denotes a Fourier amplitude of the velocity field and $r(t)$ is related to the Rayleigh number.

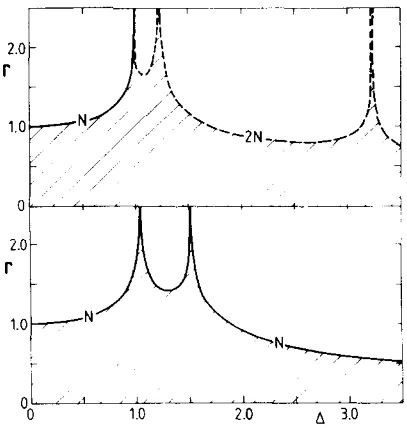

Fig. 1. Linear stability domain (hatched area) of $x^* = 0$ for modulation period $N = 10$ (top) and $N = 11$ (bottom). Full (dashed) curve denotes threshold r_c^+ (r_c^-) for harmonic (subharmonic) bifurcation.

to Δ^2 as for the mechanical model system. Whereas there the threshold shift increases with increasing modulation period [3] here the prefactor of Δ^2 is independent of N (for $N = 2$, however, it is $\frac{1}{2}$). Indeed r_c^+ depends only weakly on N for all $|\Delta| < 1$. For $|\Delta| > 1$, however, $r_c^\pm$ depends more sensitively on the period: There are singularities at modulation amplitudes $\Delta_j = -1/\cos \Omega j$ ($j = 0, 1, \ldots, |\frac{1}{2}N|$) for which *any* finite fluctuation x_0 decyas to zero for any finite r since $r_j = 0$. Also for other Δ there are parameters $r > 1$ for which $x^* = 0$ attracts a finite set $|x_N(x_0)/x_0| < 1$ of initial values around the origin. This basin of attraction shrinking to zero at r_c can be larger than that of the unmodulated map.

With increasing N the number of singularities of r_c at Δ_j (around which stabilization occurs) increases but their widths decrease rapidly, so that $r_c < 1$ for many modulation amplitudes. Also the oscillator experiences similar dynamical destabilization for some $\Delta > 1$. In both cases the stability boundary r_c does neither cross nor touch the Δ axis for finite Δ.

To evaluate r_c for large periods $N \to \infty$ and for irrational "times" T we use the equality

$$\prod_{j=1}^{N} (a^2 - 2ab \cos \Omega j + b^2) = (a^N - b^N)^2 .$$

For $|\Delta| < 1$ we set $-2ab = \Delta = -\sin 2\alpha$ and $a/b = \tan \alpha$ with $|\alpha| < \pi/4$. Then

$$r_c(|\Delta| < 1; N) = (1/\cos^2\alpha)(1 - \tan^N\alpha)^{-2/N} \xrightarrow[N\to\infty]{} 2/(1 + \sqrt{1 - \Delta^2}) . \tag{3}$$

The case $|\Delta| > 1$ is more subtle since r_c diverges at $\Delta_j = -1/\cos(2\pi N'j/N)$. With $-2ab = \Delta = -1/\cos(2\pi\varphi)$ and $a/b = e^{i2\pi\varphi}$ we obtain for irrational φ

$$r_c(|\Delta| > 1; N) = (2/|\Delta|)|4 \sin^2(\pi N\varphi)|^{-1/N} \xrightarrow[N\to\infty]{} 2/|\Delta| \tag{4}$$

The stability boundary for irrational $T = N/N'$ with $N, N' \to \infty$ is given by the above $N \to \infty$ limits.

To construct the bifurcating periodic orbits close to r_c and to determine their stability we generalize the Poincaré–Hopf technique for differential equations. Looking for solutions $\boldsymbol{x} = \{x_0, x_1, \ldots, x_{N-1}\}$ of (1) with period N – the $2N$-periodic case can be treated similarly – we expand

$$\boldsymbol{x} = \lambda \boldsymbol{x}^{(1)} + \lambda^2 \boldsymbol{x}^{(2)} + \ldots \tag{5a}$$

for small distances

$$r - r^{(0)} = \lambda r^{(1)} + \lambda^2 r^{(2)} + \ldots \tag{5b}$$

from the stability threshold in terms of the small parameter λ. The critical value $r^{(0)} = r_c^+$ is defined by the condition (2) that the solution $x_n^{(1)}$ of (1) to linear order in λ

$$\mathcal{L}_0 x_n^{(1)} := x_{n+1}^{(1)} - r^{(0)}(1 + \Delta \cos \Omega n) x_n^{(1)} = 0$$

is periodic $x_{n+N}^{(1)} = x_n^{(1)}$. Then also the solution $y_n^{(1)}$ of the adjoint problem

$$\mathcal{L}_0^+ y_n^{(1)} = y_{n-1}^{(1)} - r^{(0)}(1 + \Delta \cos \Omega n) y_n^{(1)} = 0$$

is periodic and $\mathcal{L}_0^+$ is defined with respect to the scalar product $(\boldsymbol{x}|\boldsymbol{y}) = N^{-1} \sum x_n y_n$ between sequences of period N. For the subharmonic bifurcation one finds $x_{n+N}^{(1)} = -x_n^{(1)}$, $y_{n+N}^{(1)} = -y_n^{(1)}$, and $r^{(0)} = r_c^-$. To order λ^2 one has

$$\mathcal{L}_0 x_n^{(2)} = (1 + \Delta \cos \Omega n) x_n^{(1)} (r^{(1)} - r^{(0)} x_n^{(1)}) .$$

After scalar multiplication with $\boldsymbol{y}^{(1)}$ one finds that $r^{(1)} = r^{(0)} N^{-1} \sum_{n=1}^{N} x_n^{(1)}$ is finite. Thus the N-cycle bifurcates from $x^* = 0$ with a norm that increases linearly with the distance from the stability threshold: $(\boldsymbol{x}|\boldsymbol{x}) = (\boldsymbol{x}^{(1)}|\boldsymbol{x}^{(1)})(r - r_c)^2/r^{(1)2} + \ldots$. For the subharmonic bifurcation $r^{(1)}$ vanishes. There the order parameter increases according to (5) with the square root of the distance. Also with expansion (5) one obtains

$$\partial x_N(x_0)/\partial x_0 = 1 \pm N(r - r_c)/r_c + O(\lambda^2)\,,$$

where the upper sign holds for the trivial fixed point and the lower for the periodic orbit. Hence they exchange stability at r_c.

While the above analysis applies for arbitrary Δ, Ω in the neighborhood of r_c it is difficult to determine the evolution of the bifurcated periodic orbit further away. We now discuss only small-amplitude modulation. In that case the N-cycle which has bifurcated from the origin stays close to the fixed point $1 - 1/r$ of the unmodulated system. Then, if N is odd, the basic cycle transfers its stability, in a pitchfork bifurcation at

$$r_{\mathrm{pf}} = 3[1 + \tfrac{1}{4}\Delta^2(5 + \cos\Omega)/(1 + \cos\Omega)]\,,$$

to a $2N$-cycle which undergoes with increasing r a Feigenbaum cascade [8] of period doublings. If, however, N is even a new stable N-cycle (II) is born (in a tangent bifurcation together with an unstable one $\mathrm{II_u}$ near the threshold $r = 3$ of the unmodulated map) while the original one (I) continues to live. Coexisting in some r interval they die, in general at different values of r each generating a Feigenbaum sequence of subharmonic bifurcations.

To understand the situation one has to investigate (only) one of the N *different* maps $f_n^{(N)}(x)$ identified by the iteration sequence $n, n+1, \ldots, N-1, 0, \ldots, n-1$ (or equivalently by a phase of the modulation) of the maps $f_n(x) = r(1 + \Delta\cos\Omega n)x(1-x)$. Having found its fixed point(s) $x_n = f_n^{(N)}(x_n)$ the other points of the corresponding periodic orbit(s) obtain via (1). Furthermore the N fixed points defining the N-cycle change their stability simultaneously since $\partial f_n^{(N)}(x_n)/\partial x_n$ is independent of n by virtue of the chain rule. We consider henceforth the iteration sequence 0, 1, ... and leave the subscript away.

Since $f^{(N)}(x;\Delta)$ and its iterates are smooth (polynomials) for small Δ one can infer the bifurcation scheme of (1) for sufficiently small Δ from the unmodulated map. One merely has to determine (i) the number of fixed points of $f^{(N)}(x;\Delta = 0)$ and of its iterates and (ii) the slopes at which they change their stability to deduce the above mentioned response of the small-amplitude modulated system: If N is odd $f^{(N)}$ has no stable fixed point slightly above $r = 3$ whereas $f^{(N)})$ has two, say $x_\pm$. Being connected by $f^{(N)}(x_\pm) = x_\mp$ (N is the periodicity length of the modulation!) they are members of a $2N$-cycle. Furthermore, the unstable N-cycle is born in a pitchfork (tangent) bifuraction if N is odd (even) since the derivative at the corresponding fixpoint $f^{(N)\prime}(x;\Delta = 0) = f^{\prime N}(x;\Delta = 0)$ goes through -1 $(+1)$. In the latter case there is no period doubling since $f^{(N)}$ itself has two stable fixed points generating under modulation two *different* coexisting stable N-cycles. Period doubling occurs only if $f^{(N)\prime} = -1$ at a fixed point. For even N and infinitesimal Δ that happens near the kth bifurcation of the static map for which $2N$ is an odd multiple of 2^k. Until then new additional N-cyles are born as shown schematically for $N = 4$ in fig. 2.

With increasing Δ different fixed points of $f^{(4)}(x;\Delta)$ change their positions in the x, r plane. For example bII (fixed point IV) moves together with the adhering period-doubling cascade quickly towards smaller (larger) r. Also bI (fixed point III) moves inwards (outwards), however, slowlier. That leads already for $\Delta = 0.05$ to the situation shown in fig. 3: Fixed points III, IV have moved away and cycle I coexists with cycle II. Steady Taylor vortex flow

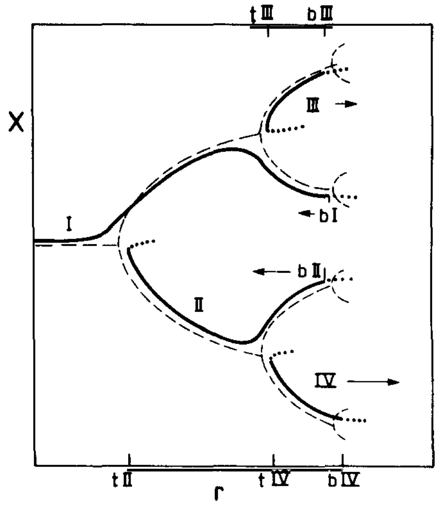

Fig. 2. Schematic bifurcation diagram for $N = 4$ and infinitesimal Δ compared to that of the unmodulated map (dashed lines). Full lines (dots) denote different stable (unstable) fixed points of $f^{(4)}(x)$. The four members (only one is shown) of each four-cycle are located in pairs around the dashed lines. Secondary four-cycles are born via tangent bifurcations (t). Each termination point b is the beginning of a sequence of period-doubling bifurcations. Arrows indicate change upon increasing Δ.

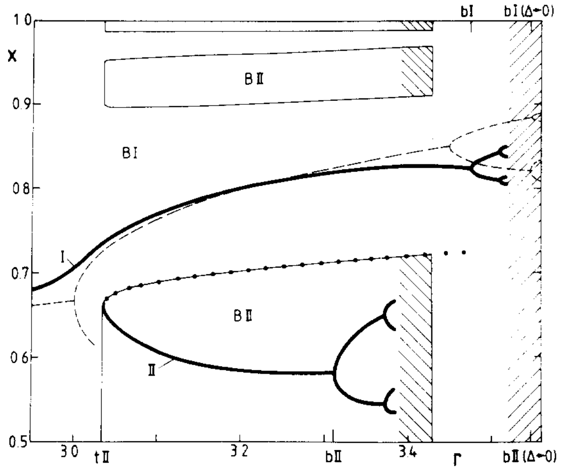

Fig. 3. Bifurcation diagram and basins of attraction for modulation "time" $T = 4$ and amplitude $\Delta = 0.05$. Thick lines denote stable fixed points I, II of $f^{(4)}(x)$ or of its iterates. BI and BII are the basins of attraction of the two four-cycles, of their period-doubling cascades, and of the ensuing chaos (shaded areas) with imbedded limit cycles. The basins are symmetric around $x = 1/2$. Their boundaries are the unstable fixed point $\mathrm{II_u}$ (dots) and its preimages. Near $x = 1$ BI and BII alternate on scales too fine for this figure. Dashed lines refer to $\Delta = 0$.

between concentric cylinders of finite length seems to undergo bifurcations [10] similar in structure to those of fig. 3 near $r = 3$: Increasing the Reynolds number r smoothly the flow is characterized by a basic fixed point I, i.e. a particular number of rolls (determined by the annulus length) with a particular handedness of the spiralling flow in each roll (determined by top and bottom boundaries). Above a critical value tII one can also prepare a different stable flow II e.g. with a different number of rolls or with different handedness of the rolls. The state II vanishes catastrophically when r is decreased below tII and state I reappears. In formal analogy to our case this situation was explained qualitatively [10,11] in terms of the unfolding of the bifurcation exhibited by an ideal system, $\Delta = 0$, (with ideal top and bottom boundary conditions) by a perturbation (which incorporates by a fraction $\Delta \ll 1$ the realistic boundary conditions).

Our system has not only dynamically similar states (cycle I and II) but it shows for larger r also coexistence of the primary state I with the higher bifurcations of II and the chaos (including inbedded limit cycles) generated beyond the periodic doubling cascade II: Initial fluctuations out of the shaded region of BII lead to chaotic orbits while all $x_0 \in \mathrm{BI}$ are attracted for corresponding r to the fixed point I of $f^{(4)}(x)$. Similarly coexistence of different types of chaos is possible for other parameters. The large r behavior seems to have a qualitative similarity with the Taylor-vortex experiment where the steady vortex state coexists with wavy, time-dependent vortex states [12].

The sharp transition at $r_\mathrm{T} \approx 3.43$ where chaos II disappears and cycle I attracts again the interval (0, 1) occurs when the image $f^{(4)}(e)$ of a particular extremum e [which is easily read off the graph of $f^{(4)}(x)$] falls upon the unstable fixed point $\mathrm{II_u}$ separating the basins. Then, for $r > r_\mathrm{T}$ the points of BII are iterated to its outside, i.e. into BI. This situation at r_T is analogous to that of the unmodulated map at $r = 4$ [7].

References

[1] S.H. Davis, Ann. Rev. Fluid Mech. 8 (1976) 57.
[2] N. Minorsky, Nonlinear oscillations (Krieger, Huntigton, 1974).
[3] G. Ahlers, P.C. Hohenberg and M. Lücke, Bull. Am. Phys. Soc. 26 (1981) 1271; to be published.
[4] L.D. Landau and E.M. Lifshitz, Mechanics (Pergamon Press, Oxford, 1960).
[5] S.H. Davis and S. Rosenblat, Studies Appl. Math. 57 (1977) 59.
[6] A. Ito, Progr. Theor. Phys. 61 (1979) 815.
[7] R.M. May, Nature 261 (1976) 459;
P. Collet and J.P. Eckmann, Iterated maps on the interval as dynamical systems (Birkhäuser, Basel, 1980).
[8] M.J. Feigenbaum, J. Stat. Phys. 19 (1978) 25.
[9] E.A. Coddington and N. Levinson, Theory of ordinary differential equations (McGraw-Hill, New York, 1955).
[10] T.B. Benjamin and T. Mullin, Proc. Roy. Soc. A377 (1981) 221.
[11] D.G. Schaeffer, Math. Proc. Camb. Phil. Soc. 87 (1980) 307.
[12] T. Mullin and T.B. Benjamin, Nature 288 (1980) 567.