Z. Phys. B - Condensed Matter 43, 253-268 (1981)

Condensed Matter
Zeitschrift für Physik B


# The Energy Injection into a Fluid by Stochastic Volume Forces and Random Stirring Forces

M. Lücke
Institut für Festkörperforschung der Kernforschungsanlage Jülich, Federal Republic of Germany, and Institute for Theoretical Physics, University of California, Santa Barbara, CA 93106, USA

Received October 22, 1980; revised version March 19, 1981

The wavenumber spectrum of the stationary energy injection rate into an incompressible fluid described by the Navier-Stokes equations is evaluated for some simple realizations of stochastic volume as well as stirring forces. A general relation between energy injection, fluid's response, and force correlations is derived which was previously shown to be particularly simple for Gaussian distributed forces with white noise frequency spectrum. For two kinds of such model volume forces the energy injection rates are calculated: Fluid volume elements of variable size around randomly chosen positions are forced in one model centralsymmetrically in the other one antisymmetrically under inversion with various force density profiles. The circumstances under which both models display an energy injection rate $\sim k^{-1}$ into a band $dk$ around the wavenumber $k$ are discussed. As a simple realization of stochastic stirring forces externally moved hard spheres immersed in the fluid are considered. The equation of motion and energy balance for the velocity field of the combined system is discussed. The spectral distribution of energy injection by stirring is shown to be that of a volume force model.

## I. Introduction

The velocity field of a fluid described by the Navier-Stokes equations [1] (NSE) decays to zero due to viscous dissipation of momentum and kinetic energy into molecular degrees of freedom unless both are resupplied. The energy- and momentum input into the velocity fluctuations of the fluid flows which are investigated [2-6] in nature and in laboratories to understand turbulence [2-8] stems from the action of surface forces if body forces like thermal buoyancy are absent: Either a finite mean velocity field exerts Reynolds stresses [2, 4] upon the fluctuating components thereby transferring energy to them or surface forces exerted by boundaries and obstacles like grids in a windtunnel do work on the fluid. However, the velocity fluctuations of many of these flows when viewed in the appropriate frame of reference are neither generated continuously nor isotropically but rather represent decaying anisotropic turbulence. Thus, following the work of Kraichnan [9], Wyld [10], and Edwards [11] a statistically defined external random volume force field $\mathbf{f}(\mathbf{r}, t)$ has commonly been introduced [12-18] into statistical theories of turbulence in order to provide for a statistically stationary isotropic energy injection.

Such a force field $\mathbf{f}(\mathbf{r}, t)$ appears explicitly in the momentum balance of the velocity field. Hence the NSE becomes a nonlinear Langevin equation describing an externally driven stochastic process. The probability distribution of $\mathbf{f}(\mathbf{r}, t)$ when chosen to be invariant under time and space translation and under arbitrary rotations enforces a statistically stationary, homogeneous, isotropic, fluctuating velocity field. The latter adjusts itself in a statistically stationary state to the forcing such as to guarantee dissipation of energy at precisely the rate it is put in by the forces. The often heard statement that the random forces supply the energy which is dissipated by viscous stresses is misinterpreted if the former are thought to be adjusted to the latter. The converse is true - e.g. in the case of Gaussian distributed forces with white frequency spectrum the energy input into

0340-224X/81/0043/0253/$03.20

any wavenumber band being solely determined by the external forces can be chosen arbitrarily. The velocity field amplitudes react according to the NSE in direct response to the external field and via the action of the nonlinear mode-coupling terms and thus are capable to set up an energy flow in Fourier space by which in every wavenumber band any injection is balanced against transfer and dissipation.

These statistical theories [9–20] of turbulence have to cope with the absence of a physically motivated guideline which restricts the choice of the driving force as, e.g., in stochastic processes simulating fluctuations around thermal equilibrium [21]. In contrast to turbulent velocity fluctuations the former obey detailed balance and a fluctuation dissipation theorem [21, 22]. Hence the statistics of the driving forces is prescribed [22–24] via potential conditions and Einstein relations to ensure a stationary probability distribution of the thermal fluctuations of the form $e^{-\beta H}$. Furthermore, is the physical concept for introducing fluctuating forces much better founded in this case: they represent the microscopic degrees of freedom which have been projected out [25] in the Langevin equation for the macrovariables. The relevance of thermal fluctuations upon turbulence has also been investigated lately [26, 27] but we will not add to the discussion [28] of internally generated force fluctuations in the externally unforced nonlinear NSE. Their energy supply to those wavenumbers relevant in turbulence is negligible in comparison with injection by external random forces since the latter has to sustain a velocity field on a macroscopic scale large enough to entail high Reynolds-numbers [2–8].

A physically well founded concept for the external forcing device, however, has not been developed so far. Many authors, including the present one, have made more or less vague allusions to stirring forces but energy injection via external random surface forces has, to the best of our knowledge, not been discussed explicitly. On the other hand is turbulence, generated nonrandomly in cylindrical tanks by rotating impellers, investigated experimentally [29].

Practically all random forcing devices introduced into theories and numerical simulations have been Gaussian distributed volume force fields with white frequency spectrum. Their rate of energy input into the velocity field was either chosen, for mathematical convenience, to be power-law distributed over wavenumbers or to display a step function behavior restricting the energy injection to low-wavenumber bands. The latter choice is motivated by the Kolmogorov picture [30–32] of turbulence in which large-scale fluid motion (eddies) decay predominantly into smaller eddies of comparable size thereby transporting energy in a sequence of many eddy decay steps to the very small-scale motion where dissipation takes place. In this picture a band limited forcing is singled out by the following commonly adopted argumentation: It is reasonable to expect the statistical properties of the large-$k$ velocity fluctuations to be insensitive to details of the artificially introduced external forcing provided the latter generates directly only small-wavenumber modes such that large-$k$ modes receive their energy predominantly from those with smaller wavenumbers. However, a unanimously accepted solution to both problems invoked in the above argumentation, the validity of the Kolmogorov picture and the relation between forcing and the statistical dynamics of velocity fluctuations, has not yet been given (c.f. Fig. 1 of [33]) despite their central importance in the statistical theory of fully developed turbulence.

Since the net energy flow in $k$-space is dictated by the distribution of the external energy injection rates over wavenumber bands we feel it is worthwhile to investigate the latter in some detail. So in this work the wavenumber spectrum of the stationary energy injection rate is evaluated for some reasonable model realizations of external random volume as well as surface forces.

In Sect. II we list basic properties of the fluid's velocity field in the presence of an external random volume force field: The momentum and energy balances and the statistical description of the velocity field are reviewed. The correlation functions which represent energy injection, dissipation, and transfer rates and their role in the balance of energy flow through a wavenumber band are discussed together with their symmetry properties. A relation between energy injection, fluid response, and random force correlation derived in Appendix A for general forced stochastic processes is shown to reduce for Gaussian distributed forces with white noise spectrum to a particularly simple, well known form in which the fluid response drops out. The energy injection rate into volume element $d\mathbf{k}$ in Fourier space is then given by the spectrum $D(k)$ of the force field correlation.

In Sect. III $D(k)$ is evaluated for two kinds of model realizations of Gaussian distributed, white noise volume forces: Finite fluid volume elements of various sizes around random positions are forced in one model centralsymmetrically, in the other model antisymmetrically under inversion with different force density field profiles. The force field correlations $D(k)$ decompose naturally into self and coherent correlations. Their contributions to the energy injection are evaluated for various degrees of field coherence. We determine the circumstances (and their degree of universality) for which $D(k) \sim k^{-d}$ such that the energy injection rate $I(k)$ into wavenumber bands varies $\sim k^{-1}$. The special significance of an external injection rate $I(k) \sim k^{-1}$ is discussed by comparing it with the internal rate at which all Fourier velocity modes with smaller wavenumber than $k$ transfer energy into the band $dk$.

In Sect. IV random stirring by externally moved hard spheres immersed in the fluid is investigated. The resulting energy injection is shown to be equivalent to the one of one of the volume force models. Hence many results derived in Sect. III for volume forces apply to the stirring model as well. The last section summarizes the main results of this work.

## II. The Fluid Velocity Field in the Presence of Random Volume Forces

### *A. Balance of Fluid Momentum and Energy*

First the energy and momentum conservation laws of an incompressible fluid of density $\rho$ in the presence of an external field of volume forces are briefly reviewed. The balance equation for the momentum of the fluid particles enclosed in a volume $V(t)$ moving along with the fluid velocity field $\mathbf{u}(\mathbf{r},t)$ reads [34] upon neglection of thermal effects

$$\frac{d}{dt} \int_{V(t)} d\mathbf{r}\, \rho\, u_\alpha(\mathbf{r},t) = \int_{S(t)} \pi_{\alpha\beta}(\mathbf{r},t)\, dS_\beta(\mathbf{r},t) + \int_{V(t)} d\mathbf{r}\, f_\alpha(\mathbf{r},t). \tag{2.1}$$

The momentum of the fluid particles in $V(t)$ is changed by surface forces acting on the surface $S(t)$ of $V(t)$ and by the volume force acting within $V(t)$. Here $\pi(\mathbf{r},t) \cdot d\mathbf{S}(\mathbf{r},t)$ is the force exerted by the fluid upon the surface element $d\mathbf{S}(\mathbf{r},t)$ whose normal points out of $V(t)$ and $\mathbf{f}(\mathbf{r},t)$ is the density of the external force field. The fluid stress tensor [1]

$$\pi_{\alpha\beta}(\mathbf{r},t) = -\delta_{\alpha\beta}\, p(\mathbf{r},t) + \eta\,[\nabla_\alpha u_\beta(\mathbf{r},t) + \nabla_\beta u_\alpha(\mathbf{r},t)] \tag{2.2}$$

contains the pressure $p(\mathbf{r},t)$ and viscous frictional forces. The latter cause a dissipative momentum flux into microscopic motion proportional to velocity gradients and the viscosity $\eta$. Only via this transport coefficient are molecular degrees of freedom (e.g. transverse current density excitations) coupled to the macroscopic field. Corrections to the above momentum balance of the fluid continuum from microscopic motion are negligible even for (ordinary) fully turbulent velocity fields [7].

For later use the momentum balance was presented in the integral form (2.1). The more familiar form of the Navier-Stokes equation for the rate of change of momentum density is obtained with the help of Reynold's transport theorem [34, 35]

$$\frac{d}{dt} \int_{V(t)} d\mathbf{r}\, \rho\, a(\mathbf{r},t) = \int_{V(t)} d\mathbf{r}\, \rho \frac{d\, a(\mathbf{r},t)}{dt} \tag{2.3}$$

valid for the density $a(\mathbf{r},t)$ of an arbitrary quantity of the fluid. Then (2.1) is divided by $V(t)$ and the limit $V(t) \to 0$ is taken such that all extensions of $V(t)$ shrink to zero. With Gauß's theorem one arrives [34] at the NSE

$$\rho \frac{\partial}{\partial t} u_\alpha(\mathbf{r},t) = -\rho\, \mathbf{u}(\mathbf{r},t) \cdot \nabla u_\alpha(\mathbf{r},t) + \nabla_\beta \pi_{\alpha\beta}(\mathbf{r},t) + f_\alpha(\mathbf{r},t). \tag{2.4}$$

Here, the relation $d/dt = \partial/\partial t + \mathbf{u}(\mathbf{r},t) \cdot \nabla$ between substantial and partial time derivative in a moving medium as well as the restriction [1]

$$\nabla \cdot \mathbf{u}(\mathbf{r},t) = 0 = \mathbf{k} \cdot \mathbf{u}(\mathbf{k},t) \tag{2.5}$$

imposed by incompressibility were employed. The Fourier transformed velocity field $\mathbf{u}(\mathbf{k},t)$ is defined by

$$\mathbf{u}(\mathbf{k},t) = \int d\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}}\, \mathbf{u}(\mathbf{r},t). \tag{2.6}$$

Since the velocity field is solenoidal the pressure $p(\mathbf{k},t) = -\frac{k_\alpha k_\beta}{k^2} \pi_{\alpha\beta}(\mathbf{k},t)$ and the longitudinal part of the force are determined from (2.4) by a nonlinear combination of velocity fields which is nonlocal both in **k**-space and in real space but local in time. The instantaneous global adjustment of pressure and velocity field is a consequence of the incompressibility assumption which implies sound propagation with infinite velocity [31]. Projecting out the longitudinal parts on the right side of (2.4) with the transverse projector

$$P_{\alpha\beta}(\mathbf{k}) = \delta_{\alpha\beta} - \frac{k_\alpha k_\beta}{k^2} \tag{2.7}$$

one finally obtains a rate equation which contains only solenoidal quantities

$$(\partial_t + \nu k^2)\, u_\alpha(\mathbf{k},t) = -i P_{\alpha\beta}(\mathbf{k})\, k_\gamma \int \frac{d\mathbf{q}}{(2\pi)^3} u_\beta(\mathbf{q},t)\, u_\gamma(\mathbf{k}-\mathbf{q},t) + \frac{1}{\rho} f_\alpha^\perp(\mathbf{k},t). \tag{2.8}$$

Here $\nu = \eta/\rho$ is the kinematic viscosity of the fluid and $f_\alpha^\perp(\mathbf{k}) = P_{\alpha\beta}(\mathbf{k})\, f_\beta(\mathbf{k})$ is the solenoidal part of the external force density. In this and the following section $\rho=1$ for convenience.

Neglecting thermal effects [34, 36] the rate of change of the total energy of particles in a volume $V(t)$ – i.e. kinetic energy of the macroscopic velocity field and internal energy contained in the microscopic degrees of freedom of the fluid – is given (arguments $\mathbf{r}, t$ suppressed) by the sum of two contributions [34]

$$\frac{d}{dt}\int_{V(t)} d\mathbf{r}[e_{\text{int}}+e_{\text{kin}}]=\int_{S(t)} u_\alpha \pi_{\alpha\beta}\, dS_\beta+\int_{V(t)} d\mathbf{r}(\mathbf{u}\cdot\mathbf{f}). \quad (2.9)$$

Fluid stresses do work on a surface element $d\mathbf{S}$ of $S(t)$ at a rate $u_\alpha \pi_{\alpha\beta}\, dS_\beta$ and in addition the external force does work on a volume element $d\mathbf{r}$ within $V(t)$ at a rate $(\mathbf{f}\cdot\mathbf{u})\, d\mathbf{r}$.

Again the local balance of total energy density $\frac{d}{dt}(e_{\text{int}}+e_{\text{kin}})$ follows from (2.9) by using Reynolds' transport theorem (2.3), dividing by $V(t)$, and taking the limit $V(t)\to 0$. A comparison with the rate of change of the kinetic energy density $e_{\text{kin}}(\mathbf{r},t) =\frac{1}{2}\rho u^2(\mathbf{r},t)$ which is obtained from (2.4) by scalar multiplication with $\mathbf{u}(\mathbf{r},t)$ shows [34] that the internal energy density of the fluid increases at a rate

$$\frac{d e_{\text{int}}(\mathbf{r},t)}{dt}=\varepsilon_{\text{diss}}(\mathbf{r},t)=\frac{\eta}{2}[\nabla_\alpha u_\beta(\mathbf{r},t)+\nabla_\beta u_\alpha(\mathbf{r},t)]^2 \geqq 0. \quad (2.10)$$

Hence the macroscopic velocity field dissipates kinetic energy into internal energy at a rate $\varepsilon_{\text{diss}}(\mathbf{r},t)$ while it receives energy per unit volume at the rate $\mathbf{f}(\mathbf{r},t)\cdot\mathbf{u}(\mathbf{r},t)$ through the action of the force field.

$$\frac{\partial}{\partial t} e_{\text{kin}}(\mathbf{r},t)=-\varepsilon_{\text{diss}}(\mathbf{r},t)+\mathbf{f}(\mathbf{r},t)\cdot\mathbf{u}(\mathbf{r},t)$$
$$+\nabla_\beta\{u_\alpha(\mathbf{r},t)[\pi_{\alpha\beta}(\mathbf{r},t)-\delta_{\alpha\beta}\, e_{\text{kin}}(\mathbf{r},t)]\}. \quad (2.11)$$

Since the internal energy of the molecular degrees of freedom is much larger than the kinetic energy of the macroscopic velocity field the former can be treated as a reservoir which is not changed (e.g. to a higher temperature) by irreversible transfer of kinetic energy into it.

### *B. Statistical Description*

Here the basic formulas for a statistical description of the velocity field in the presence of external random forces are given. Averages are taken over an ensemble of the external force field and indicated by $\langle\ldots\rangle$. To ensure independence of $\langle\ldots\rangle$ of the initial velocity field the latter is prepared in each realization of the ensemble at time $t_0=-\infty$. Preparation of $\mathbf{u}(\mathbf{r},t)$ at a finite $t_0$ requires an additional average over an ensemble of initial values $\mathbf{u}(\mathbf{r},t_0)$.

A statistically stationary, homogeneous, isotropic velocity field is enforced by an ensemble of random forces which is invariant under time and space translation and under arbitrary rotation. Then the mean values of all single fields vanish. Moreover, imply the above invariances

$$\langle f_\alpha^\perp(\mathbf{k},\omega)\, f_\beta^\perp(\mathbf{k}',\omega')\rangle=D(k,\omega)$$
$$\cdot P_{\alpha\beta}(\mathbf{k})(2\pi)^4\,\delta(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega') \quad (2.12)$$

$$\langle u_\alpha(\mathbf{k},\omega)\, u_\beta(\mathbf{k}',\omega')\rangle=C(k,\omega)$$
$$\cdot P_{\alpha\beta}(\mathbf{k})(2\pi)^4\,\delta(\mathbf{k}+\mathbf{k}')\,\delta(\omega+\omega') \quad (2.13)$$

where the autocorrelation functions

$$D(k,\omega)=\tfrac{1}{3}\int d\mathbf{r}\int dt\, e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\langle\mathbf{f}(\mathbf{r},t)\cdot\mathbf{f}(\mathbf{0},0)\rangle \quad (2.14)$$

$$C(k,\omega)=\tfrac{1}{2}\int d\mathbf{r}\int dt\, e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\langle\mathbf{u}(\mathbf{r},t)\cdot\mathbf{u}(\mathbf{0},0)\rangle \quad (2.15)$$

are real, positive, even in $\omega$ and dependent only on $k=|\mathbf{k}|$. The correlations of longitudinal and transversal force components were assumed to be the same in (2.14). However, the general case $D_T(k,\omega) \neq D_L(k,\omega)$ does not pose any problems. Since the longitudinal part of the force is eliminated from the NSE one only has to consider the statistics of the solenoidal part. For the commonly considered purely solenoidal forces the prefactor in (2.14) would be 1/2.

Equation (2.15) suggests to interpret $C(k,\omega)\, d\mathbf{k}\, d\omega$ as the field energy per unit mass in the volume element $d\mathbf{k}\, d\omega$ of $\mathbf{k},\omega$ space and hence $C(k,t=0)\, d\mathbf{k}$ as the field energy in $d\mathbf{k}$. A main object of interest of stationary isotropic turbulence is the energy spectrum [31, 32] which is defined in $d=3$ dimensions by

$$E(k)=\frac{4\pi k^2}{(2\pi)^3}\, C(k,t=0). \quad (2.16a)$$

It describes the spectral distribution of field energy (per unit mass) over wavenumber bands such that the integral over all bands

$$\int_0^\infty dk\, E(k)=\tfrac{1}{2}\langle u^2\rangle \quad (2.16b)$$

determines the average kinetic energy per unit mass.

In Hamiltonian systems evaluation of equal time correlations does not require solving equations of motion. However, since the stationary velocity field distribution of the randomly forced fluid is not known, except for the uninteresting case simulating thermal equilibrium one has to solve the equation of motion

$$2(-i\omega+\nu k^2)\, C(k,\omega)=C_3(k,\omega)+2\,C_{fu}(k,\omega) \qquad (2.17)$$

for the dynamic correlation function $C(k,\omega)$ in order to obtain $E(k)$ via a frequency integral. Equation (2.17) is obtained by Fourier transforming the NSE (2.8) with respect to $t$, multiplying with $\mathbf{u}(\mathbf{k}',\omega')$, averaging, and integrating over $\mathbf{k}',\omega'$. Both correlation functions

$$C_{fu}(k,\omega))=\tfrac{1}{2}\int dt \int d\mathbf{r}\, e^{i(\omega t-\mathbf{k}\cdot\mathbf{r})}\langle \mathbf{f}(\mathbf{r},t)\cdot\mathbf{u}(\mathbf{0},0)\rangle \qquad (2.18)$$

and $C_3(k,\omega)$ which is determined by triple velocity correlations depend, because of isotropy, only on $|\mathbf{k}|$. Their real (imaginary) parts are even (odd) in $\omega$ since the fields $\mathbf{u}(\mathbf{r},t)$ and $\mathbf{f}(\mathbf{r},t)$ are real.

Integrating (2.17) over frequencies all imaginary parts drop out and one obtains the local balance in $k$-space between the rates of energy dissipation $2\nu k^2\, C(k)$, energy transfer $C_3(k)$, and energy injection $2\,C_{fu}(k)$ in a volume element $d\mathbf{k}$

$$2\nu k^2\, C(k)=C_3(k)+2\,C_{fu}(k). \qquad (2.19)$$

That interpretation is deduced from the integral properties of the above equaltime correlations for which the arguments $t=0$ will be suppressed henceforth: The average energy dissipation rate (2.10) is given ($d=3$) by

$$\langle \varepsilon_{\mathrm{diss}}(\mathbf{r},t)\rangle=2\nu\int\frac{d\mathbf{k}}{(2\pi)^3}k^2\, C(k)=2\nu\int_0^\infty dk\, k^2\, E(k) \qquad (2.20)$$

and the average injection rate by

$$\langle \mathbf{f}(\mathbf{r},t)\cdot\mathbf{u}(\mathbf{r},t)\rangle=2\int\frac{d\mathbf{k}}{(2\pi)^3}\, C_{fu}(k)=2\int_0^\infty dk\, I(k). \qquad (2.21\,\mathrm{a})$$

The energy injection spectrum

$$I(k)=\frac{4\pi k^2}{(2\pi)^3}\, C_{fu}(k) \qquad (2.21\,\mathrm{b})$$

is the spectral distribution of the energy injection rate into the velocity field over wavenumber bands.

In stationary forced turbulence the transfer $C_3(k)$ is determined by the balance between external energy injection and dissipation in the considered volume element $d\mathbf{k}$ in Fourier space. A positive (negative) $C_3(k)$ measures [2, 4, 31, 32] the *net* rate at which the nonlinear interaction of different Fourier velocity modes has to transfer energy into (out of) $d\mathbf{k}$ in order to provide for the difference between dissipation and external injection. The nonlinear mode coupling process exchanges energy conservatively [31, 32, 37] so that the global energy balance between injection and dissipation is untouched by transfer in $\mathbf{k}$-space: $\int d\mathbf{k}\, C_3(k)=0$.

In three dimensions the balance equation (2.19) describes a net energy flux [2, 31, 32] from small wavenumbers to large ones if energy is injected at small $k$ since for $d=3$ the dissipation $\nu k^2\, E(k)$ works predominantly at large $k$. Hence, in a statistically stationary state internal nonlinear interactions transport energy to the large-$k$ dissipation sink. That is widely believed to be done in a universal manner [30–32] if energy source and sink are well separated in $k$-space. It still remains to be shown that the famous Kolmogorov $k^{-5/3}$ distribution of energy over wavenumber bands developes if to a wide range of wavenumbers energy is mostly (or only?) supplied by internal transfer. As a side remark it should be noted that for suitably chosen injection spectra energy flow in $\omega$-space has to be expected as well: There is dissipation $2\nu k^2\, C(k,\omega)$ and the energy injection rate $\mathrm{Re}C_{fu}(k,\omega)$ is positive as will be shown later on for Gaussian forces. It would be interesting to know whether the energy flow in $\omega$-space establishes also a universal power-law dominated frequency spectrum.

### C. Energy Injection by Random Volume Forces

Here the energy injection into the velocity field by random volume forces is worked out. Since the energy injection of the surface force model of Sect. IV can be mapped onto volume force injection we feel that the latter being somewhat generic for energy injection merits a thorough investigation.

The rate at which external forces do work on a system depends in general on the system's response (c.f. Appendix A for a discussion of the randomly forced fluid). In particular one has to consider the velocity response function $\chi(k,\omega)$ describing the change linear in $\tilde{h}$ of the velocity average in the presence of a small nonrandom external force $\tilde{h}_\alpha(\mathbf{r},t)$ added to the random force $f_\alpha$ in the NSE (2.4)

$$\left.\frac{\delta\langle u_\alpha(\mathbf{k},\omega)\rangle_{\tilde{h}}}{\delta\tilde{h}_\beta(\mathbf{k}',\omega')}\right|_{\tilde{h}=0}=\chi(k,\omega)$$
$$\cdot P_{\alpha\beta}(\mathbf{k})(2\pi)^4\,\delta(\mathbf{k}-\mathbf{k}')\,\delta(\omega-\omega'). \qquad (2.22)$$

For convenience the nonrandom field $\tilde{\mathbf{h}}$ is chosen to be purely solenoidal. In the presence of random forces $\mathbf{f}$ (2.22) is invariant under rotations and time- and space translations. Reality of the fields $\mathbf{u}(\mathbf{r},t)$ and $\tilde{\mathbf{h}}(\mathbf{r},t)$ implies the real (imaginary) part of $\chi(k,\omega)$ to be even (odd) in $\omega$ and causality demands $\chi(k,t<0)=0$.

The real part of $\chi(k,\omega)$ is positive which can be shown by exploiting as in Hamiltonian systems [38] the relation between the response and the average total rate of work done on the fluid by a small perturbation $\tilde{h}$. This work being positive

$$0 \leqq \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\omega}{2\pi} \tilde{h}_\alpha^*(\mathbf{k},\omega) \langle u_\alpha(\mathbf{k},\omega)\rangle_{\tilde{h}}$$
$$= \int \frac{d\mathbf{k}}{(2\pi)^3} \int \frac{d\omega}{2\pi} |\tilde{\mathbf{h}}(\mathbf{k},\omega)|^2 \operatorname{Re}\chi(k,\omega) \tag{2.23}$$

entails $\operatorname{Re}\chi(k,\omega)>0$ since the field $\tilde{\mathbf{h}}$ is arbitrary. In (2.23) the expansion $\langle u_\alpha(\mathbf{k},\omega)\rangle_{\tilde{h}} = \chi(k,\omega)\,\tilde{h}_\alpha(\mathbf{k},\omega) + O(\tilde{h}^2)$ of the average velocity in the presence of a small perturbation $\tilde{h}$ was used.

The energy injected into the velocity field by the random force field $\mathbf{f}(\mathbf{r},t)$ is determined (c.f. Appendix A) by its correlations and the response of the fluid to external perturbations. For Gaussian force statistics the relation is (A.13–A.14)

$$C_{fu}(k,\omega) = \chi^*(k,\omega)\, D(k,\omega) \tag{2.24}$$

For non-Gaussian statistics there appear (A.13–A.14) in addition higher-order force cumulants multiplied by higher-order response functions (A.11). Since the nonlinear velocity response reflecting nonlinear dependence of averages on (small) forces $\tilde{\mathbf{h}}$ can presumably be neglected in ordinary fluids relation (2.24) is a good approximation to $C_{fu}(k,\omega)$ also for non-Gaussian forces. Hence, with higher force cumulants not entering the expression for $C_{fu}(k,\omega)$ it is presumably sufficient to discuss the special case of Gaussian random forces which will be done in the following. Note that (2.24) implies $\operatorname{Re}C_{fu}(k,\omega)\geqq 0$: The energy injection into every Fourier mode of the velocity field is positive (or zero) since $\operatorname{Re}\chi(k,\omega)\geqq 0$ and $D(k,\omega)\geqq 0$. A proof that $\operatorname{Re}C_{fu}(k,\omega)\geqq 0$ holds also for general force statistics is still lacking.

For Gaussian forces with a white frequency spectrum ($D(k,\omega)=D(k)$) the frequency integral over (2.24) leads to the remarkable statement [11]

$$C_{fu}(k) = D(k) \tag{2.25}$$

that the energy injection rate into **k**-space is determined solely by the statistics of the force amplitudes. The equaltime response of the system [11, 14]

$$\chi(k,t=0) = 1 = \chi_{\text{lin}}(k,t=0) \tag{2.26}$$

being that of the linearized theory has dropped out since the instantaneous velocity field response is purely local [13] $\chi(r,t=0)=\delta(\mathbf{r})$ and shows neither amplification nor damping of the perturbation.

Thus the velocity field of the fluid driven with arbitrary forcing adjusts itself such as to ensure dissipation of the injected energy (2.25) which, depending only on force amplitudes, is controlled externally. That is most obvious in the spectrum

$$C_{\text{lin}}(k,\omega) = \frac{D(k)}{\omega^2 + (\nu k^2)^2} \tag{2.27}$$

of the linearized NSE. The average square of the velocity field amplitude and with it the dissipation $\nu k^2 C(k)$ is determined by the average squared amplitudes of the force field.

## III. Energy Injection Spectra of Some Volume Force Models

Here some model realizations of random volume forces are discussed and the spectra of energy injection into the velocity field are evaluated. Consider volume forces acting centralsymmetrically within surroundings, shortly called blobs, of $N_b$ random positions $\mathbf{R}_n(t)$ such that the resulting force density field is

$$\mathbf{f}(\mathbf{r},t) = \sum_{n=1}^{N_b} \mathbf{F}_n(t)\, \Delta_n(|\mathbf{r}-\mathbf{R}_n(t)|). \tag{3.1}$$

Here $\mathbf{F}_n(t)$ is the total force on blob $n$ since the volume integral over

$$\Delta_n(r) = c\left(\frac{r}{R_n^b}\right) \Big/ \int d\mathbf{r}\, c\left(\frac{r}{R_n^b}\right) \tag{3.2}$$

is normalized to unity. The above profile function of the force density field is defined via a dimensionless cutoff $c(x)$, e.g.,

$$c(x) = \Theta(1-x),\, e^{-x^2},\, e^{-x},\, (1+x^2)^{-\alpha}. \tag{3.3}$$

The force density rapidly drops to zero beyond the blob "radius" $R_n^b$. The exponent $2\alpha$ of the power-law cutoff has to be larger than 3 or, for general dimensions $d$, larger than $d$ to guarantee a finite blob "volume" $\int d\mathbf{r}\, c(r/R_n^b)$.

Only forces $\{\mathbf{F}_n(t)\}$ which are $\delta$-function correlated in time thus leading to a white frequency spectrum of the force density field

$$\langle \mathbf{f}(\mathbf{r},t)\cdot\mathbf{f}(\mathbf{r}',t')\rangle = \delta(t-t')\, D(\mathbf{r},\mathbf{r}') \tag{3.4}$$

will be considered in the rest of this paper. Such forces might be visualized by equidistant force pulse sequences $\mathbf{F}_n(t) = \sum_i \tilde{\mathbf{F}}_n(t_i)\,\delta(t-t_i)$. Their strengths being uncorrelated at different times are such that

$\frac{1}{\tau}\langle \tilde{F}_n^\alpha(t_i)\,\tilde{F}_m^\beta(t_j)\rangle = \delta_{ij}\frac{1}{\tau}\langle \tilde{F}_n^\alpha(t_i)\,\tilde{F}_m^\beta(t_i)\rangle$ is bounded in the limit of vanishing pulsseparation $\tau$. To avoid clumsiness we will henceforth call the quantity $\tilde{\mathbf{F}}(t)/\sqrt{\tau}$ "force" and use for it also the old symbol $\mathbf{F}(t)$. Then

$$D(\mathbf{r},\mathbf{r}')=\sum_{n,m}\langle \mathbf{F}_n\cdot\mathbf{F}_m\,\Delta_n(|\mathbf{r}-\mathbf{R}_n|)\,\Delta_m(|\mathbf{r}'-\mathbf{R}_m|)\rangle$$
$$=D(|\mathbf{r}-\mathbf{r}'|) \tag{3.5}$$

is given by an average over a translational and rotational invariant ensemble of forces $\{\mathbf{F}_n\}$, positions $\{\mathbf{R}_n\}$, and blob volumes $\{V_n^b\}$. Exploiting translational invariance one obtains for the wavenumber spectrum the manifestly positive expression

$$D(k)=\frac{1}{V}\sum_{n,m}\langle \mathbf{F}_n\cdot\mathbf{F}_m\,\Delta_n(k)\,\Delta_m(k)\,e^{-i\mathbf{k}\cdot(\mathbf{R}_n-\mathbf{R}_m)}\rangle \tag{3.6}$$

into which enter the blobpositions $\{\mathbf{R}_n\}$ only via phase factors. The Fourier transform $\Delta_n(k)$ being dimensionless depends only on $kR_n^b$

$$\Delta_n(k)=\Delta(kR_n^b). \tag{3.7}$$

Due to the normalization one has $\Delta_n(k=0)=\Delta(kR_n^b=0)=1$ so that

$$D(k=0)=\frac{1}{V}\sum_{n,m}\langle \mathbf{F}_n\cdot\mathbf{F}_m\rangle. \tag{3.8a}$$

Consequently the band injection rate $I(k)$ (2.21) increases for small wavenumbers like

$$I(k)\sim D(k=0)\,k^{d-1}. \tag{3.8b}$$

Here $D(k=0)$ is assumed to be finite. If the total force on the fluid $\sum_{n=1}^{N_b}\mathbf{F}_n$ not only vanishes on the average but is constructed to vanish also in every realization at every time one has $D(k=0)=0$ and residual correlations of the forces on different blobs: $\langle \mathbf{F}_n\cdot\mathbf{F}_m\rangle=-\langle F_n^2\rangle/(N_b-1)$. However any experimental realization of stochastic forces one might think of as being feasible, like e.g. random stirring, will presumably guarantee only the vanishing of the (time) averaged total force. Hence the above mentioned effects and similar ones arising if the total torque exerted on the fluid is required not only to vanish on the average but also at every instant will not be dealt with here.
Obviously $D(k)$ can be written as the sum of two functions

$$D(k)=D_{\text{self}}(k)+D_{\text{coh}}(k). \tag{3.9}$$

The second one comes from all terms $n\neq m$ in (3.6) and gives the correlation between force densities in different blobs. Directly measuring the coherence of the field $\mathbf{f}(\mathbf{r},t)\,D_{\text{cob}}(k)$ vanishes if the forces $\mathbf{F}_n,\mathbf{F}_m$ on different blobs are uncorrelated. Thus one expects the energy injection rate $D(k)$ to be dominated by the self correlations of the force density field.

### *A. Self Correlations*

Irrespective whether the forces or the positions whereto the forces are applied are coherent or not the energy injection rate has always the contribution

$$D_{\text{self}}(k)=n_b\langle F_n^2\,\Delta^2(kR_n^b)\rangle \tag{3.10}$$

stemming from the $N_b$ terms $n=m$ in (3.6) all being equal. Here $n_b=N_b/V$ denotes the number density of blobs whereto forces are applied. The blob index $n$ in (3.10) is arbitrary. Although (3.10) involves only an average over $\mathbf{F}_n$ and $V_n^b$ we continue to use here and in the following the same average symbol $\langle\ldots\rangle$ to avoid undue notational complications.

*1. Infinitesimal Blob Volumes – Minimal Self Correlations.* Maximal randomness obtains in our model for infinitesimal blob radii leading to $\delta$-function like force density distributions. In this case $R_n^b\to 0$ $D_{\text{self}}(k)=n_b\langle F_n^2\rangle$ is independent of $k$. In the absence of correlations between different blobs, $D_{\text{coh}}(k)=0$, the energy injection rate $D(k)$ caused by such forces is uniform in $k$-space and the band input rate $I(k)$ increases proportional to $k^{d-1}$.
The long-wavelength, low-frequency statistics of the velocity field generated by such volume forces has been investigated by renormalization group techniques [15, 18] and other methods [11, 16, 20].

*2. Finite Blobs of Equal Size.* The expression (3.10) for the spectrum simplifies if the distribution of blob volumes is sharply peaked: For the special ensemble where all blob radii are the same, $R_b$, one has ($q_b=kR_b$)

$$D_{\text{self}}(k)=n_b\langle F_b^2\rangle\,\Delta^2(q_b)$$
$$=n_b\langle F_b^2\rangle\left(\frac{3}{q_b^2}\right)\left[\frac{\sin q_b}{q_b}-\cos q_b\right]^2. \tag{3.11}$$

The second equality holds for a sharp cutoff in the force density profile in $d=3$ dimensions. However, that the energy injection rate is more or less restricted to wavenumbers $kR_b<1$ holds in arbitrary dimension for every finite force density profile with finite $R_b$: A finite density $\Delta(r=0)$ at the blob center implies $\Delta(q_b)$ to drop off for large $q_b$ whereas a force density profile diverging at the blob center (the extreme case of a $\delta$-function emerging for $R_b\to 0$ was discussed above) causes $\Delta(q_b\to\infty)$ to be larger and thus an enhanced energy tail at large wavenumbers.

However, even for the finite profiles (3.3) of the force density the confinement of $D_{\rm self}(k)$ to small wavenumbers is not sharp. A discontinuous profile of forces in real space has $k^{-4}$ tails in the energy injection rate $D(k)$ at large $k$. This corresponds to a rate $I(k)$ of energy injection into a wave number band $dk$ proportional to $k^{-2}$ with a straightforward generalization to other dimensions.

*3. Distribution of Blob Volumes.* As the most simple case consider a uniform distribution

$$P(V_b)=L^{-d}\left[1-\left(\frac{l}{L}\right)^d\right]^{-1}\cdot\begin{cases}1 & \text{for } l^d<V_b<L^d\\ 0 & \text{else}\end{cases} \qquad (3.12)$$

of blob volumes between $l^d$ and $L^d$. Since the situation $l\sim L$ does not differ drastically from the previously discussed sharp distribution let us concentrate in the following on a broad one with $l\ll L$. The average of (3.10) over blob volumes $V_b$ with the distribution (3.12)

$$D_{\rm self}(k)=d\,n_b\frac{1}{(kL)^d}\int_{kl}^{kL}dq_b\,q_b^{d-1}\,\Delta^2(q_b)\langle F_b^2\rangle;$$
$$q_b=kR_b \qquad (3.13)$$

represents a superposition of the injection spectrum (3.11) of single blob sizes. There are two obvious choices for its evaluation: *a)* The total force $\mathbf{F}_b$ on a blob is tied to the blob size, e.g. by $\langle F_b^2\rangle\sim V_b^2\langle f^2\rangle$, so that the force density, say at the blob centers, is roughly the same. *b)* The force $\mathbf{F}_b$ fluctuates independently of the blob size thus causing a stronger variation in the force density. Alternative *a)* leads to a faster decrease of $D_{\rm self}(k)$ at large $k$ since it allows, in contrast to case *b*, small blobs to feel only a small force. Using $\langle F_b^2\rangle\sim V_b^2$ in (3.13) this can readily be seen. Since however scenario *b)* has quite remarkable consequences we proceed to present them rather than dwelling longer on option *a)*.

As a consequence of the statistical independence of forces and blob sizes $\langle F_b^2\rangle$ appears as a constant in front of the above integral. Then the energy injection rate for wavenumbers small compared to the inverse of the largest blob size is again constant

$$D_{\rm self}(k\ll L^{-1})\simeq n_b\langle F_b^2\rangle. \qquad (3.14\text{a})$$

That follows from (3.13) with $\Delta(q_b\to 0)=1$. In addition one finds

$$D_{\rm self}(L^{-1}\ll k\ll l^{-1})\simeq D_{\rm self}(k=0)\,(kL)^{-d} \qquad (3.14\text{b})$$

and an even faster decay for $l^{-1}\ll k$. This result is quite universal holding whenever the fluid is forced in a wide distribution of blob volumes with Fourier transformable force density profiles. In such a case $\Delta(q\to 0)=1$ and $\Delta(q\to\infty)$ drops off fast enough so that one may replace the integral in (3.13) between $kl\ll 1$ and $kL\gg 1$ by $\int_0^\infty dq\,q^{d-1}\,\Delta^2(q)\simeq\int dr\,\Delta^2(r)$.

The energy injection rate $I(k)$ (2.21a) into wavenumber bands increases, according to (3.13), at small $k\ll L^{-1}$ proportional to $k^{d-1}$. In the wavenumber range $L^{-1}\ll k\ll l^{-1}$, on the other hand, $I(k)$ displays the power law $k^{-1}$ which has been shown [16, 17, 20, 39] to enforce a $k^{-5/3}$ distribution of energy over wavenumber bands. The significance of an energy injection rate $I(k)\sim k^{-1}$ will be dealt with in section III, *D*.

## *B. Coherent Correlations*

Here we investigate the relative importance of the contribution

$$D_{\rm coh}(k)=\frac{1}{V}\sum_{\substack{n,m\\(n\neq m)}}\langle\mathbf{F}_n\cdot\mathbf{F}_m\,e^{-i\mathbf{k}\cdot(\mathbf{R}_n-\mathbf{R}_m)}\,\Delta_n(k)\,\Delta_m(k)\rangle \qquad (3.15)$$

to the energy injection rate which stems from coherent correlations of the force field. $D_{\rm coh}(k)$ vanishes for totally random forces $\{\mathbf{F}_n\}$ even if the blob centers are maximally correlated whereas it is finite for finite force correlations even if the positions $\{\mathbf{R}_n\}$ are completely random. Thus in our model coherence of the field $\mathbf{f}(\mathbf{r},t)$ is caused mainly by correlations between the forces $\mathbf{F}_n,\mathbf{F}_m$ on different blobs and, to a lesser extent, by correlations between blob positions.

For the sake of simplicity let the blob sizes be the same within a particular realization (experiment) of the ensemble. Furthermore, let the coherence of the forces $\{\mathbf{F}_n\}$ (within a particular realization) be characterized by a correlation length $\kappa^{-1}$ measuring the distance up to which forces $\mathbf{F}_n,\mathbf{F}_m$ applied to blobs $n,m$ are correlated, e.g. aligned. We thus replace

$$\mathbf{F}_n\cdot\mathbf{F}_m\to F_n^2\,G(|\mathbf{R}_n-\mathbf{R}_m|\,\kappa) \qquad (3.16)$$

in (3.15). The dimensionless correlation function $G(r\kappa)$ could be a Gaussian $e^{-r^2\kappa^2/2}$ or $(1+r\kappa)^{-1}\cdot e^{-r\kappa}$ or a similar expression with $G(0)=1$; $G(1)\simeq\frac{1}{2}$, $G(\infty)=0$. Let furthermore blob sizes $\{R_n^b\}$ and force amplitudes $\{F_n^2\}$ be statistically independent of the position $\{\mathbf{R}_n\}$ then

$$D_{\rm coh}(k)=n_b\langle F_b^2\,\Delta^2(kR_b)\rangle\cdot\frac{1}{N_b}\sum_{\substack{n,m\\(n\neq m)}}\langle G(|\mathbf{R}_n-\mathbf{R}_m|\,\kappa)\,e^{-i\mathbf{k}\cdot(\mathbf{R}_n-\mathbf{R}_m)}\rangle$$
$$=D_{\rm self}(k)[S(k;\kappa)-1]. \qquad (3.17)$$

is a product of $D_{\text{self}}(k)$ with a factor [...] measuring the coherence of the force density field. The function

$$S(k;\,\kappa)=\int \frac{d\mathbf{q}}{(2\pi)^3}\,G_\kappa(|\mathbf{k}-\mathbf{q}|)\,S(q) \tag{3.18}$$

is a convolution of the Fourier transform of the force correlation $G(r\,\kappa)$ with the structure factor [40, 41] of the blob positions

$$S(k)=\frac{1}{N_b}\sum_{n,m}\langle e^{-i\mathbf{k}\cdot(\mathbf{R}_n-\mathbf{R}_m)}\rangle \tag{3.19}$$

which measures correlations of the latter. $S(k)$ contains a contribution $n_b(2\pi)^3\,\delta(\mathbf{k})$ in addition to a function which is nonsingular for $k\to 0$ and approaches 1 for $k\to\infty$. Isotropy causes this decay of $S(k)-1$ for $k$ large compared to the inverse of the average interblob distance $r_0$: Even the maximally attainable correlations between blob centers lying in every realization on a perfect lattice with only its orientation being different approach zero for large $kr_0$ [H40]. This universal behavior of $S(k)$ enforces also $S(k;\,\kappa)-1$ (3.20) and hence $D_{\text{coh}}(k)$ to drop to zero for $k\to\infty$ since also $G_\kappa(k\to\infty)\to 0$ because of $G(r\kappa=0)=1$. Thus, for large $k$, the coherent contribution (3.17) to the energy injection rate (3.9) is always negligible compared with $D_{\text{self}}(k)$ irrespective of blob center configurations or coherence of the forces $\{\mathbf{F}_n\}$. An investigation [42] of $D_{\text{coh}}(k)$ for, e.g., randomly oriented and positioned grids (arrays) of several blobs with forces aligned within an array shows that also at smaller wavenumbers coherent correlations do not play a decisive role. Hence the energy injection is dominated by $D_{\text{self}}(k)$.

### C. Eddy Generating Forces

Here we present a force model with an antisymmetric profile leading to eddy generation. That causes the energy injection $D(k)$ to vanish for long wavelengths thus having a maximum at finite $k$ in contrast to the central-symmetric model. Consider a force density field (time arguments suppressed)

$$\mathbf{f}(\mathbf{r})=\sum_{n=1}^{N_b}\mathbf{f}_n(\mathbf{r}-\mathbf{R}_n) \tag{3.20}$$

described by a superposition of $N_b$ contributions of the form

$$\mathbf{f}_n(\mathbf{r})=\mathbf{F}_n\times\frac{\mathbf{r}}{R_n^b}\,\Delta_n(r) \tag{3.21}$$

which is antisymmetric under inversion. A typical force density profile around a center $\mathbf{R}_n$ as sketched in Fig. 1 for the equator plane orthogonal to $\mathbf{F}_n$ has maxima at a distance $R_n^b$ from the center.

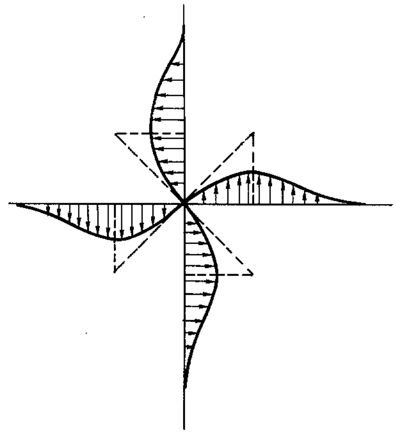

**Fig. 1.** Force density profile (arrows) of the eddy generating model in a plane orthogonal to $\mathbf{F}_n$ through the center $\mathbf{R}_n$. Full (dashed) lines denote smooth (sharp) cut off

Again the forces $\{\mathbf{F}_n(t)\}$ are chosen to be $\delta$-correlated in time. Then one finds in close analogy to (3.5–3.6)

$$D(k)=$$

$$P_{\alpha\beta}(\mathbf{k})\frac{1}{V}\sum_{n,m}\langle(\mathbf{F}_n)_\alpha(\mathbf{F}_m)_\beta\,\Delta'(q_n)\,\Delta'(q_m)\,e^{-i\mathbf{k}\cdot(\mathbf{R}_n-\mathbf{R}_m)}\rangle \tag{3.22}$$

to be an average over a translationally and rotationally invariant ensemble of $\{\mathbf{F}_n\}$, $\{\mathbf{R}_n\}$, and blob volumes $\{V_n^b\}$. The projector (2.7) reflects the transversality of

$$\mathbf{f}_n(\mathbf{k})=i\,\frac{\mathbf{F}_n\times\mathbf{k}}{k}\,\Delta'(q_n)=\mathbf{f}_n^*(-\mathbf{k}) \tag{3.23}$$

and $\Delta'(q_n)$ denotes the derivative of the Fourier transformed cutoff function with respect to the normalized wavenumber $q_n=kR_n^b$.
In contrast to the previously discussed central-symmetric model the force $\int d\mathbf{r}\,\mathbf{f}_n(\mathbf{r}-\mathbf{R}_n)$ on blob $n$ and with it the total force vanishes automatically at any time due to the antisymmetry of (3.21). In order to translate that unambiguously into the statement $\mathbf{f}_n(\mathbf{k}=0)=0$ one has to require that $\Delta'(q_n)$ vanishes at $q_n=0$. Hence power-law profiles $\Delta(r)$ have to decay sufficiently fast for large $r$ to compensate the extra $r$-power from the derivative $\Delta'(q\to 0)$.
Here we discuss only the self-correlation contribution $(n=m)$

$$D_{\text{self}}(k)=n_b\,\frac{d-1}{d}\,\langle F_n^2\,\Delta'^2(q_n)\rangle \tag{3.24}$$

which one obtains after replacing $(\mathbf{F}_n)_\alpha(\mathbf{F}_n)_\beta$ in the average by $\delta_{\alpha\beta}\,F_n^2/d$. The coherent contribution $D_{\text{coh}}(k)$ $(n\neq m)$ to the energy injection can be estimated as in Sect. III, B.

*1. Blobs of Equal Size.* For blobs of equal size $R_b$ in $d=3$ dimensions the energy injection has the contribution

$$D_{\text{self}}(k)=\tfrac{2}{3}n_b\langle F_b^2\rangle\,\Delta'^2(q_b)=\tfrac{2}{3}n_b\langle F_b^2\rangle\,4q_b^2(1+q_b^2)^{-6} \tag{3.25}$$

where the second equality holds for the exponential cutoff $c(x)=e^{-x}$ as a representative example. For all profiles (3.3) the bulk of the energy injection (3.25) is centered between $q_b=kR_b\simeq 0.5$ and $q_b\simeq 5$. Some of the cutoffs induce powerlaw tails in the energy injection spectrum at large wavenumbers. For all of them $D_{\text{self}}(k)$ increases proportional to $k^2$ at small $k$.

Hence the model of eddy generating forces allows to mimic forces originating from microscopic particle current excitations: If one keeps $n_b\langle F_b^2\rangle R_b^2$ fixed while letting $R_b\to 0$, e.g. by increasing the blob number density $n_b$, one can obtain the energy injection spectrum $D(k)=2(\nu/\rho)\,k_B T k^2$ which enforces a velocity field simulating microscopic fluctuations around thermal equilibrium.

*2. Distribution of Blob Volumes.* For the uniform distribution (3.12) of blob volumes one finds (c.f. Sect. III, A.3)

$$D_{\text{self}}(k)=(d-1)\,n_b\langle F_b^2\rangle\frac{1}{(kL)^d}\int_{kl}^{kL}dq_b\,q_b^{d-1}\,\Delta'^2(q_b). \tag{3.26}$$

That yields

$$D_{\text{self}}(k)=(d-1)\,n_b\langle F_b^2\rangle\cdot\begin{cases}\text{const}\,(kL)^2 & \text{for } kL\ll 1\\ \text{const}\,(kL)^{-d} & \text{for } L^{-1}\ll k\ll l^{-1}\end{cases} \tag{3.27}$$

showing for wavenumbers $kL\ll 1$ an increase $\sim k^2$ since $\Delta'(q\to 0)\sim q$ and a decrease $\sim k^{-d}$ in the wave number range $L^{-1}\ll k\ll l^{-1}$. The const is given by $\int_0^\infty dq\,q^{d-1}\,\Delta'^2(q)$. The corrections $\int_0^{kl}dq\ldots$ and $\int_{kL}^\infty dq\ldots$ to it are small for $kl\ll 1\ll kL$ since $\Delta'(q\to 0)\sim q$ and $\Delta'(q\to\infty)$ falls off rapidly enough to guarantee the existence of its Fourier transform. Hence, also in this eddy generating model the energy injection is distributed proportional to $k^{-1}$ over wavenumber bands provided the forces act over a wide range of scales $l\ll L$.

### D. Miscellaneous Comments

If around every blob center position $\mathbf{R}_n$ both type of forces act, those with central-symmetric density profile (3.1) and in addition eddy generating forces (3.21), then the total energy injection $D(k)$ is just a sum of the two contributions (3.6, 3.22). The cross terms vanish due to the different symmetries of $\mathbf{f}_n(\mathbf{r}-\mathbf{R}_n)$ under inversion. The integral of $D_{\text{self}}(k)$ over all wavevectors shows that the total energy injected by both kinds of forces is roughly of the order of $n_b\langle F^2/V_b\rangle$. The energy injection into the velocity field is largest for force density profiles with a sharp cutoff.

A band limited energy injection rate

$$I(k)=I_0\,\theta(k-k_1)\,\theta(k_2-k) \tag{3.28}$$

into the band $\Delta k=k_2-k_1$ follows, in $d=3$ dimensions, from force field correlations

$$D(r)=I_0\cdot\Delta k\,\frac{\mathrm{Si}(k_2 r)-\mathrm{Si}(k_1 r)}{\Delta k\cdot r}. \tag{3.29}$$

Such oscillatory correlations determined by the sine integral function $\mathrm{Si}(x)$ seem to be artificial and cannot be produced within our models.

The last point of this section is a comment on the special role of an external energy injection $I(k)\sim k^{-1}$ in fully developed turbulence. To that end we compare $I(k)$ with the rate $T_<(k)$ of energy input by internal transfer into the band $dk$ from all Fourier velocity modes with wavenumbers smaller than $k$. The latter input rate, if identified by [43]

$$\int_0^k dp\int_{k-p}^k dq\,S(k|p,q) \tag{3.30}$$

obeys the powerlaw $k^{-1}$ whenever the scaling behavior

$$S(ak|ap,aq)=a^{-3}\,S(k|p,q) \tag{3.31}$$

holds. Such a scaling is argued [32] to be consistent with a Kolmogorov picture of turbulence with an energy distribution $E(k)\sim k^{-5/3}$. Moreover was (3.31) found to hold [32] within the EDQNM closure approximation for unforced turbulence with a Kolmogorov energy spectrum. In addition $T_<(k)\sim k^{-1}$ obtains in the local cacade models [44] with band limited injection yielding $E(k)\sim k^{-5/3}$. Thus, if $T_<(k)\sim k^{-1}$ characterizes freely decaying or stationary turbulence with band limited injection then an external input rate $I(k)\sim k^{-1}$ mimics just the internal energy influx into the band $dk$ which causes $E(k)\sim k^{-5/3}$. Note that all our model forces when applied to a wide distribution of blobvolumes, entail energy supply rates $I(k)\sim k^{-1}$ to the band $dk$ around $k$.

By a self-consistency argument one furthermore concludes that external injection spectra $I(k)$ falling off faster than $k^{-1}$ lead to an energy spectrum

$E(k)\sim k^{-5/3}$ since in such a situation the supply rate $T(k)$ by internal transfer dominates. On the other hand, if $I(k)\sim k^{-\alpha}$ with $\alpha<1$ one might expect [16, 17, 20] $E(k)\sim k^{-1-\frac{2}{3}\alpha}$.

## IV. Stirring by Randomly Moved Hard Spheres

In this section we investigate the energy injection caused by moving macroscopic hard spheres in a fluid by external forces. This model provides a geometrical simplification of forces exerted on a fluid by moving randomly grids or other obstacles.

Consider a fluid containing $N_s$ hard spheres at positions $\{\mathbf{R}_n(t)\}$. For the sake of simplicity they have the same mass density $\rho$ as the fluid. External forces $\{\mathbf{F}_n(t)\}$ act upon the spheres leading to a velocity field

$$\mathbf{u}(\mathbf{r},t)=\begin{cases}\mathbf{u}_{\text{fluid}}(\mathbf{r},t) & \text{for } \mathbf{r}\in\text{fluid}\\ \mathbf{U}_n(t) & \text{for } \mathbf{r}\in n\text{-th sphere}\end{cases} \tag{4.1a}$$

which is by definition spatially uniform within the volume occupied by a sphere. A schematic sketch is shown in the lower part of Fig. 2. The no-slip boundary condition requires the fluid velocity at the surface $S_n(t)$ of the $n$'th sphere to be equal to the sphere's velocity

$$\mathbf{u}(\mathbf{r},t)=\mathbf{u}_{\text{fluid}}(\mathbf{r},t)=\mathbf{U}_n(t) \quad \text{for } \mathbf{r}\in S_n(t). \tag{4.1b}$$

Here the external forces are supposed to enforce zero angular velocity for all spheres in order to avoid unnecessary complications.

We want to describe the statistical dynamics of the system - fluid and hard spheres - in terms of the above velocity field. First its equation of motion is discussed.

### *A. Momentum and Energy Balance of the Field*

Considering the momentum change of an infinitesimal volumeelement in the fluid or of the volume $V_n$ of the $n$'th sphere, respectively, one deduces from (2.1)

$$\rho\frac{du_\alpha(\mathbf{r},t)}{dt}=\begin{cases}\nabla_\beta\,\pi_{\alpha\beta}(\mathbf{r},t) & \mathbf{r}\in\text{fluid} \qquad (4.2\text{a})\\ \dfrac{1}{V_n}F_n^\alpha(t)+\dfrac{1}{V_n}\displaystyle\int_{S_n(t)}\pi_{\alpha\beta}(\mathbf{r}',t)\,dS_\beta(\mathbf{r}',t) & \mathbf{r}\in V_n. \qquad (4.2\text{b})\end{cases}$$

Whereas the momentum balance (4.2a) inside the fluid is just the ordinary NSE the momentum change $M_n\,\dot{U}_n^\alpha(t)$ of the $n$'th sphere (4.2b) is the result of the external force $F_n^\alpha(t)$ and of the force $\int_{S_n(t)}\pi_{\alpha\beta}\,dS_\beta$ which the fluid exerts on the surface $S_n(t)$ of the sphere. The latter is determined by the fluid's stress tensor $\pi_{\alpha\beta}$ being a functional of the velocity field. The rate of change of the momentum density field is spatially uniform within $V_n$. Note that $\mathbf{u}(\mathbf{r},t)$ (4.1) is not a pure Eulerian field: "$\mathbf{r}\in n$'th sphere" in (4.1) means that at all points $\mathbf{r}$ defined via $|\mathbf{r}-\mathbf{R}_n(t)|\leqq R_n^s$ with respect to the moving sphere center $\mathbf{R}_n(t)$ the velocity is $\mathbf{U}_n(t)$, and furthermore all these points experience the same acceleration (4.2b).

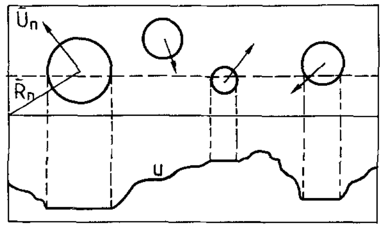


**Fig. 2.** Sketch of a velocity field component along the dashed line in a fluid with hard spheres

The rate of change of the velocity field's energy density follows from (4.2)

$$\frac{\rho}{2}\frac{du^2(\mathbf{r},t)}{dt}=\begin{cases}-\varepsilon_{\text{diss}}(\mathbf{r},t)+\nabla_\beta(u_\alpha(\mathbf{r},t)\,\pi_{\alpha\beta}\mathbf{r},t)) & \mathbf{r}\in\text{fluid} \qquad (4.3\text{a})\\ \dfrac{1}{V_n}\mathbf{U}_n(t)\cdot\mathbf{F}_n(t)+\dfrac{1}{V_n}U_n^\alpha(t)\displaystyle\int_{S_n}\pi_{\alpha\beta}\,dS_\beta & \mathbf{r}\in V_n \qquad (4.3\text{b})\end{cases}$$

if one uses the relation $u_\alpha\nabla_\beta\pi_{\alpha\beta}=\nabla_\beta(u_\alpha\pi_{\alpha\beta})-\varepsilon_{\text{diss}}$ (2.14).

### *B. Energy Injection*

Let, for the sake of simplicity, the sphere's motion be such that $U_n^2(t)=\text{const}$, which is true anyhow on the average. Then the external force $\mathbf{F}_n(t)$ injects energy into the system at a rate

$$\mathbf{U}_n(t)\cdot\mathbf{F}_n(t)=-U_n^\alpha(t)\cdot\int_{S_n(t)}\pi_{\alpha\beta}(\mathbf{r}',t)\,dS_\beta(\mathbf{r}',t). \tag{4.4}$$

Note however, that (4.4) really describes the energy injection into the fluid: Integrating (4.3a) over the fluid's volume $V_{fl}$ the rate (4.4) appears as a source term in the balance of the fluid kinetic energy if one rewrites

$$\int_{V_{fl}}\nabla_\beta(u_\alpha\pi_{\alpha\beta})=-\sum_n U_n^\alpha\int_{S_n}\pi_{\alpha\beta}\,dS_\beta+\int_{S_\infty}u_\alpha\pi_{\alpha\beta}\,dS_\beta \tag{4.5}$$

into integrals over the sphere surfaces and over the external surface $S_\infty$ enclosing the system.

Relations (4.3, 4.4) allow for two equivalent interpretations: *(i)* The surfaces $\{S_n(t)\}$ of the hard spheres do work on the fluid velocity field at a rate $-\sum_n U_n^\alpha(t) \int_{S_n(t)} \pi_{\alpha\beta}\, dS_\beta$. *(ii)* Energy is injected into the velocity field $\mathbf{u}(\mathbf{r},t)$ per unit volume at a rate

$$\mathbf{u}(\mathbf{r},t)\cdot\mathbf{f}(\mathbf{r},t)=\sum_n \mathbf{U}_n(t)\cdot\begin{cases}\frac{1}{V_n}\mathbf{F}_n(t) & \text{for } \mathbf{r}\in V_n\\ 0 & \text{else}\end{cases} \tag{4.6}$$

which is uniformly distributed over $V_n$ and vanishes outside. The force density field $\mathbf{f}(\mathbf{r},t)$ can be read off (4.3b, 4.6) to be

$$\mathbf{f}(\mathbf{r},t)=\sum_n \mathbf{F}_n(t)\,\Delta_n(|\mathbf{r}-\mathbf{R}_n(t)|) \tag{4.7}$$

and $\Delta_n$ denotes the sharp cutoff function (3.2, 3.3). We will adopt interpretation *(ii)* to determine the spectral distribution of the average energy injection – into the velocity field $\mathbf{u}(\mathbf{r},t)$ to be precise. An evaluation of statistical properties of the rate of work $[-u_\alpha \pi_{\alpha\beta}\, dS_\beta/dS]_{\mathbf{r}\in S_n}$ done per *unit area* of the spheres on the *fluid velocity field* $\mathbf{u}_{fl}$ is not only too difficult but also unnecessary if the statistical dynamics of the system is described by the velocity field $\mathbf{u}(\mathbf{r},t)$.

### C. Spectral Distribution of the Energy Injection

In the following the spectral density $\frac{1}{2}\int d\mathbf{r}\, e^{-i\mathbf{k}\cdot\mathbf{r}} \langle f(\mathbf{r})\cdot\mathbf{u}(0)\rangle$ of the energy injection rate $\langle \mathbf{f}(\mathbf{r})\cdot\mathbf{u}(\mathbf{r})\rangle$ will be related to the correlation function of the external force density field. For a Gaussian distributed field $\mathbf{f}(\mathbf{r},t)$ this relation is simplest: For other field distributions there appear (c.f. Appendix A) higher-order response functions which, presumably, are small anyhow. So let us construct for the $N_s$ hard spheres an ensemble of initial conditions $\{\mathbf{R}_n(t_0)\}_1$, $\{\mathbf{R}_n(t_0)\}_2, \ldots$ and force histories $\{\mathbf{F}_n(\tau);\, t_0<\tau<t\}_1$, $\{\mathbf{F}_n(\tau);\, t_0<\tau<t\}_2, \ldots$ such that the resulting force density field $\mathbf{f}(\mathbf{r},t)$ (4.7) is Gaussian distributed over the realizations $1,2,\ldots$ of this ensemble. In addition, to keep things simple, we consider only force density fields which are uncorrelated in time by applying forces such that $\langle \mathbf{F}_n(t)\cdot\mathbf{F}_m(t')\rangle = D_{nm}\,\delta(t-t')$.

One might visualize the construction in a Gedanken experiment as follows: Choose a particular realization of initial positions $\{\mathbf{R}_n(t_0)\}$ for the hard spheres, apply forces $\{\mathbf{F}_n(t_0\leqq\tau\leqq t)\}$ to them, record the trajectories $\{\mathbf{R}_n(t_0\leqq\tau\leqq t)\}$ and hence the field $\mathbf{f}(\mathbf{r},t_0\leqq\tau\leqq t)$ (4.7). Then repeat the whole procedure aiming at a Gaussian distribution of $\mathbf{f}$ over the resulting realizations. Since $\mathbf{f}$ (4.7) is the sum of $N_s$ ($N_s\to\infty$) contributions which can be varied almost independently from each other we believe that by the law of large numbers it should be possible to construct distributions of $\mathbf{f}(\mathbf{r},t)$ meeting both requirements: Gaussian statistics and white noise spectrum.

In that case one can read the equation of motion (4.2) for the field $\mathbf{u}(\mathbf{r},t)$ as describing a stochastic process driven by a *preprogrammed* force field $\mathbf{f}(\mathbf{r},t)$ with Gaussian white noise statistics. The field $\mathbf{u}(\mathbf{r},t)$ reacts to the driving field $\mathbf{f}(\mathbf{r},t)$ according to (4.2)

$$\rho\frac{d\mathbf{u}(\mathbf{r},t)}{dt}-\mathbf{f}_{\text{int}}(\mathbf{r},t)=\mathbf{f}(\mathbf{r},t). \tag{4.8}$$

However, determining the statistics of this reaction might be even more difficult than for (2.4) since the left side of (4.8) depends nonlocally in time on the velocity field $\mathbf{u}$: the force density field

$$f_{\text{int}}^\alpha(\mathbf{r},t)=\begin{cases}\nabla_\beta\, \pi_{\alpha\beta}(\mathbf{r},t) & \text{for } \mathbf{r}\notin V_n;\ n=1,\ldots,N_s\\ \frac{1}{V_n}\int_{S_n(t)} \pi_{\alpha\beta}(\mathbf{r}',t)\, dS_\beta(\mathbf{r}',t) & \text{for } \mathbf{r}\in V_n;\ n=1,\ldots,N_s\end{cases} \tag{4.9}$$

generated internally by the velocity field is not only a functional, nonlocal in space, of the instantaneous field $\mathbf{u}(\mathbf{r},t)$ but it depends in addition on a set of unique positions $\{\mathbf{R}_n(t)\}$ (sphere centers) since the instantaneous velocity field $\mathbf{u}(\mathbf{r},t)$ alone does not allow to identify unambiguously the spheres $1,\ldots,N_s$ by spherical regions of uniform $\mathbf{u}$. Hence $\{\mathbf{R}_n(t)\}$ have to be thought of as being determined by $\{\mathbf{R}_n(t_0)\}$ and the complete time evolution of the velocity field $\mathbf{u}$ from time $t_0$ to $t$. And thus the left-hand side of (4.8) is not only a functional of the instantaneous velocity field but also of its history.

For our purposes that does not cause any problem since the relation (A.13, A.14)

$$C_{uf}(k)=\chi(k)\,D(k) \tag{4.10}$$

between the energy injection $C_{uf}(k)=C_{fu}(k)$ (2.21), the equaltime velocity field response $\chi(k)$ (c.f. 2.22), and the correlation spectrum $D(k)$ of the random force field is independent of these details of the mapping between the fields $\mathbf{u}(\mathbf{r},t)$ and $\mathbf{f}(\mathbf{r},t)$. What matters for (4.10) to hold is the Gaussian statistics of $\mathbf{f}$ and its white frequency spectrum.

The instantaneous response $\chi(k)$ of the velocity field in our model, however, cannot be assumed to be purely local, $\chi(r)=\delta(\mathbf{r})$ (2.26), as for the NSE (2.4). In addition to the local response of the velocity field to an external force field one expects here a nonlocal instantaneous reaction

$$\chi(r, t=0) = \alpha\,\delta(\mathbf{r}) + \chi_{\text{nonlocal}}(r, t=0) \tag{4.11}$$

since the hard spheres can respond in an instantaneous, nonlocal way to an external force. However, $\chi_{\text{nonlocal}}(r, t=0)$ will drop off sharply in $r$ beyond, say, the average sphere size $\langle R_s \rangle$ since perturbing the velocity field outside a considered sphere does not cause the sphere to respond instantaneously. Furthermore, the strength $\alpha$ of the local response will exceed the nonlocal one since the inertia of the hard sphere's mass keeps the velocity response of the latter low. Thus the Fourier spectrum

$$\chi(k) = \alpha + \chi_{\text{nonlocal}}(k) \tag{4.12}$$

is dominated by the local contribution $\alpha$. Therefore the energy injection rate $C_{uf}(k)$ into the velocity field is again determined by the squared Fourier amplitudes

$$D(k) = \frac{1}{V} \sum_{n,m} \langle \mathbf{F}_n \cdot \mathbf{F}_m\, \Delta_n(k)\, \Delta_m(k)\, e^{-i\mathbf{k}\cdot(\mathbf{R}_n - \mathbf{R}_m)} \rangle \tag{4.13}$$

of the force density field (4.7).

Thus the stirring force model's energy injection spectrum is equivalent to that one of a volume force model: The results of Sect. III, A, B for the sharp cutoff hold also for the spectrum (4.13) and their discussion can be taken over from Sect. III replacing the word blob by the word sphere. For example, causes stirring with grids of spheres the coherent contribution to the energy injection which was investigated in Sect. III, B for the case of arrays of blobs. Furthermore, there will be a $k^{-4}$-injection tail in $D(k)$ (c.f. 3.11) for $k \cdot R_s \gg 1$ if the fluid is stirred by spheres of one size $R_s$. In the presence of spheres of all sizes between $l$ and $L$ the energy injection into the bands of wavenumbers $L^{-1} \ll k \ll l^{-1}$, on the other hand, will drop off like $k^{-1}$ as before (3.14).

## V. Summary

We have investigated the statistical properties of energy injection by random forces into a fluid whose velocity field evolves according to the NSE. The wavenumber spectrum of the energy injection rate was evaluated for some model realizations of random volume forces and for random stirring forces realized by externally moved hard spheres immersed in the fluid. We derived a general relation between the energy injection, the fluid's response, and the force correlations which is most simple for Gaussian distributed force fields with white noise [11]. In that case the energy injection into wavenumber space is given by $D(k)$, the spectral strength of the force field autocorrelation function.

The energy injection into the velocity field by randomly stirring with spheres was shown to be equivalent to a volume force model with a force density field being constant (in space) within spherical blobs around randomly chosen positions $\{\mathbf{R}_n(t)\}$ and vanishing elsewhere. To that end we considered the velocity field of the fluid plus spheres and described the action of external forces $\{\mathbf{F}_n(t)\}$ upon the spheres by a force density field with the above described profile around the sphere centers. In order to explicitly show the equivalence of the energy injection spectrum to that of a Gaussian distributed force density field it was assumed that an ensemble of external force histories can be chosen such that the resulting force density field is Gaussian distributed over the realizations of the ensemble.

The energy injection spectrum $D(k)$ of volume force fields was evaluated for: *i)* spherically symmetric force density profiles around random blob positions $\{\mathbf{R}_n(t)\}$ with sharp and smooth cutoffs – the results for the former being also applicable to stirring with hard spheres; *ii)* eddy generating forces which are antisymmetric under inversion at blob centers with force density profiles cut off sharply and smoothly. In both cases the self-correlations $D_{\text{self}}(k)$ of the force density field within the same blob dominate the energy injection $D(k)$. Coherent correlations $D_{\text{coh}}(k)$ between the forces acting in different blobs contribute to the energy injection at most for small wavenumbers. Even for randomly positioned grids of a few blobs (or spheres in the case of the stirring model) with a grid spacing $r_0$ and forces aligned within a grid $D_{\text{coh}}(k)$ can be neglected in comparison with $D_{\text{self}}(k)$ above a few inverse interblob distances.

Whereas the central symmetric model leads to $D(k \to 0) = \text{const}$ the eddy generating force model yields $D(k \to 0) \sim k^2$. The latter allows to simulate the energy injection from microscopic fluctuations around thermal equilibrium. Both force models entail energy injection into velocity field amplitudes with wavenumbers large compared to the inverse blob (sphere) size, the functional form of the injection rate being determined by the square of the Fourier-transformed force density profile. Since the sharp cutoff causes a pronounced large-$k$ tail stirring a fluid with solid obstacles ("paddles") leads to forcing at large wavenumbers.

Both model volume forces yield an energy injection $I(k) \sim k^{-1}$ into wavenumber bands between $L^{-1}$ and $l^{-1}$ when the forces act within a wide range of blob sizes $l \leq R_b \leq L$. That result is independent of the form of the force density profile provided the latter is Fourier transformable. It also holds for random stirring with spheres (or obstacles) of many sizes.

Discussions with H. Horner are gratefully acknowledged.

## Appendix

*Energy Injection, Response, and Force Correlations in Randomly Forced Systems*

In order to facilitate the investigation of the above relation for general and Gaussian random force statistics a condensed notation is introduced first within which the NSE (2.8) read

$$L(1\bar{1})\,\phi(\bar{1})+W(1\bar{1}\bar{2})\,\phi(\bar{1})\,\phi(\bar{2})=\xi(1)+\tilde{h}(1). \tag{A.1}$$

Here $1=\mathbf{k},\omega,\alpha$ and $\phi(1)$ denotes the velocity field, $\xi(1)$ the random force field. A small nonrandom external volume force field $\tilde{h}(1)$ has been added in (A.1) to the NSE in order to study the fluid's response to it. Barred arguments like $\bar{1}$ imply integration and summation.

We do not pause to write down to vertices $L(12)$ and $W(123)$ [16] since they can be read off from (2.8) and since they are not needed anyhow: The form of the relation (A.13) between $\langle\phi(1)\,\xi(2)\rangle$, i.e. the correlation of field $\phi(1)$ and random force $\xi(2)$, and the random force correlation function $\langle\xi(1)\,\xi(2)\rangle$ does not depend on the form of the left side of (A.1). It holds for generalized Langevin equations

$$\psi(1)=\xi(1)+\tilde{h}(1) \tag{A.2}$$

where

$$\psi=\psi[\phi] \tag{A.3}$$

is *any* functional of the field $\phi$ whose time evolution is governed by the stochastic process (A.2) driven by the random force $\xi(1)$. To be concrete we refer in the following to the NSE (A.1). It should however be kept in mind that the general case (A.2, A.3) is included as well.

Correlation functions of the fields $\phi$ and $\xi$ are determined by a functional integral like

$$\langle\phi(1)\ldots\phi(n)\rangle=\int d[\xi]\,\phi(1)\ldots\phi(n)\,P[\xi] \tag{A.4}$$

over the force field. The probability distribution $P[\xi]$ of the random force field $\xi$ is here only required to show all the symmetries and invariances discussed in Sect. II B. The fields $\phi$ in (A.4) have to be thought of as solving the NSE in the presence of the external field $\xi$ (and $\tilde{h}$) thus being functionals of $\xi$ (and $\tilde{h}$). Later on we will read the NSE (A.1) also the other way around interpreting the field $\xi$ as a functional of $\phi$ (and $\tilde{h}$) as described by (A.1).

Let us introduce in analogy to the work of Martin et al. [14] an auxiliary field $\tilde{\phi}(1)$ by functional Fourier transform [45, 46] of the random force distribution

$$P[\xi]=\int d[\tilde{\phi}]\,\tilde{P}[\tilde{\phi}]\,e^{-i\tilde{\phi}^*(\bar{1})\,\xi(\bar{1})}. \tag{A.5}$$

That proved to be very helpful [17] in other problems [47] as well. Note that $\tilde{\phi}^*(\bar{1})\,\xi(\bar{1})=\int d\mathbf{r}\int dt\,\tilde{u}_\alpha(\mathbf{r},t)\,f_\alpha(\mathbf{r},t)$ is real since $\tilde{u}$ is taken to be real.

Since $\xi(2)\,P[\xi]$ is the Fourier transform of $-i\delta\tilde{P}[\phi]/\delta\tilde{\phi}^*(2)$ one finds for the correlation function $C_{uf}(12)$ of the velocity field $\phi(1)$ and the random field $\xi(2)$

$$C_{uf}(12)=\langle\phi(1)\,\xi(2)\rangle=-i\left\langle\phi(1)\frac{\delta\ln\tilde{P}[\tilde{\phi}]}{\delta\tilde{\phi}^*(2)}\right\rangle. \tag{A.6}$$

Here the average of a functional $F[\phi,\tilde{\phi}]$ is defined by

$$\langle F[\phi,\tilde{\phi}]\rangle=\int d[\xi]\,d[\tilde{\phi}]\,F[\phi,\tilde{\phi}]\,\tilde{P}[\tilde{\phi}]\,e^{-i\tilde{\phi}^*(\bar{1})\,\xi(\bar{1})}. \tag{A.7}$$

The expression (A.6) for $C_{uf}(12)$ is simple only if the random force distribution $P[\xi]$ is Gaussian. Then the cumulant expansion of

$$\ln\tilde{P}[\tilde{\phi}]=\ln\int d\left[\frac{\xi}{2\pi}\right]e^{i\tilde{\phi}^*(\bar{1})\,\xi(\bar{1})}\,P[\xi] \tag{A.8}$$

terminates with the second order cumulant $D(12)$ of the random force ($\langle\xi(1)\rangle=0$)

$$\langle\xi(1)\,\xi(2)\rangle=D(12)=D(k_1,\omega_1)\,P_{\alpha_1\alpha_2}(\mathbf{k}_1)\,2\pi)^4\cdot\delta(\mathbf{k}_1+\mathbf{k}_2)\,\delta(\omega_1+\omega_2). \tag{A.9}$$

Hence, for non-Gaussian forces the correlation function

$$\langle\phi(1)\,\xi(2)\rangle=-i\sum_{n=2}\frac{i^n}{n!}\left\langle\phi(1)\frac{\delta}{\delta\tilde{\phi}^*(2)}\tilde{\phi}^*(\bar{1})\ldots\tilde{\phi}^*(\bar{n})\right\rangle\cdot\langle\xi(\bar{1})\ldots\xi(\bar{n})\rangle_c \tag{A.10}$$

contains higher order cumulants of the random forces and correlations of the velocity field $\phi(1)$ with higher powers of the response field. These are higher-order response functions [46, 47]. One finds that $\langle\phi(1)\,\tilde{\phi}^*(1')\ldots\tilde{\phi}^*(n')\rangle$ is the $n$-th Taylor coefficient in a functional expansion of the average velocity field $\langle\phi(1)\rangle_{\tilde{h}}$ with respect to the non-random force field $\tilde{h}$:

$$\left.\frac{\delta^n\langle\phi(1)\rangle_{\tilde{h}}}{\delta\tilde{h}(1')\ldots\delta\tilde{h}(n')}\right|_{\tilde{h}=0}=i^n\langle\phi(1)\,\tilde{\phi}^*(1')\ldots\tilde{\phi}^*(n')\rangle. \tag{A.11}$$

This is most easily verified by a "change of variables" in (A.8) from $\xi$ to $\phi$ as determined by the NSE (A.1): The integration measure $d[\xi] = d[\phi]\, J[\phi]$ is transformed with the Jacobean $J[\phi] = \partial[\xi]/\partial[\phi]$ of the mapping (A.1). Hence,

$$\langle F[\phi,\tilde\phi]\rangle_{\tilde h} = \int d[\phi]\, d[\tilde\phi]\, F[\phi,\tilde\phi]\, J[\phi]\, \tilde P[\tilde\phi]\, e^{-i\phi^*(\bar 1)\xi(\bar 1)} \quad \text{(A.12)}$$

where for $\xi$ in the exponential the NSE $\xi(1) = L(1\bar 1)\,\phi(\bar 1) + W(1\bar 1\bar 2)\,\phi(\bar 1)\,\phi(\bar 2) - \tilde h(1)$ has to be inserted. Taking a functional derivative of (A.12) with respect to $\tilde h(1)$ generates a field $i\tilde\phi^*(1)$ under the integral. That leads to the relation (A.11).

Inserting (A.11) into (A.10) one finally obtains for the correlation $\langle\phi(1)\,\xi(2)\rangle$ of velocity field $\phi$ and random force $\xi$ the general relation

$$C_{uf}(12) = \sum_{n=1} \frac{1}{n!} \left.\frac{\delta^n \langle\phi(1)\rangle_{\tilde h}}{\delta\tilde h(\bar 1)\ldots\delta\tilde h(\bar n)}\right|_{\tilde h=0} \langle\xi(\bar 1)\ldots\xi(\bar n)\,\xi(2)\rangle_c. \quad \text{(A.13)}$$

Its spectral function $C_{uf}(k,\omega) = C^*_{fu}(k,\omega)$ (2.18) is thus determined for Gaussian force statistics by a product of the (linear) response function $\chi(k,\omega)$ (2.22) with the random force correlation function $D(k,\omega)$ (A.9, 2.14). For general statistics there appear in addition $n$-th order response functions $\chi^{(n)}$ and $(n+1)$-th order force cumulants $D^{(n+1)}$ with $n \geqq 2$

$$C_{uf}(k,\omega) = \chi(k,\omega)\, D(k,\omega) + \sum_{n=2} \frac{1}{n!} [\chi^{(n)*}\, D^{(n+1)}]_{k,\omega}. \quad \text{(A.14)}$$

## References


1. Landau, L.D., Lifshitz, E.M.: Fluid Mechanics. Oxford: Pergamon Press 1975
2. Monin, A.S., Yaglom, A.M.: Statistical Fluid Mechanics. Vol. I and II. Cambridge: MIT 1975
3. For a review with many references see the articles. In: Turbulence. Bradshaw, P. (ed.). Topics in Applied Physics. Vol. 12. Berlin, Heidelberg, New York: Springer Verlag 1978
4. Hinze, J.O.: Turbulence. Maidenhead: McGraw Hill 1975
5. Tritton, D.J.: Physical fluid dynamics. New York: van Nostrand Reinhold 1977
6. Rotta, J.C.: Turbulente Strömungen. Stuttgart: B.G. Teubner 1972
7. Tennekes, H., Lumley, J.L.: A first Course in Turbulence. Cambridge: MIT Press 1972
8. Batchelor, G.K.: The theory of homogeneous turbulence. New York: Cambridge University Press 1953
9. Kraichnan, R.H.: Phys. Rev. **109**, 1407 (1958)
10. Wyld, H.W.: Ann. Phys. (N.Y.) **14**, 143 (1961)
11. Edwards, S.F.: J. Fluid Mech. **18**, 239 (1964)
12. Kraichnan, R.H.: J. Fluid Mech. **5**, 497 (1959)
    Lewis, R.H., Kraichnan, R.H.: Commun. Pure Appl. Math. **15**, 397 (1962)
    Lee, L.L.: Ann. Phys. (N.Y.) **32**, 292 (1965)
    Herring, J.R.: Phys. Fluids **8**, 2219 (1965)
    Hosokawa, I.: J. Phys. Soc. Jpn. **25**, 271 (1968)
    Edwards, S.F., McComb, W.D.: J. Phys. A **2**, 157 (1969)
    Nelkin, M.: Phys. Rev. A **9**, 388 (1974)
    Phythian, R.: J. Phys. A **8**, 1423 (1975)
    Garrido, L., San Miguel, M.: Prog. Theor. Phys. **59**, 40 (1978)
    Horner, H., Lipowsky, R.: Z. Phys. B – Condensed Matter **33**, 223 (1979)
    Lilly, D.K.: Phys. Fluids **12**, Suppl. II, 240 (1969)
    Fyfe, D., Montgomery, D., Joyce, G.: J. Plasma Phys. **17**, 369 (1977)
13. Novikov, E.A.: Sov. Phys. JETP **20**, 1290 (1965)
14. Martin, P.C., Siggia, E.D., Rose, H.A.: Phys. Rev. A **8**, 423 (1973)
15. Forster, D., Nelson, D.R., Stephen, M.J.: Phys. Rev. A **16**, 732 (1977)
    Enz, C.P.: Physica **94** A, 20 (1978)
16. Lücke, M.: Phys. Rev. A **18**, 282 (1978)
17. Dominicis, C.De, Martin, P.C.: Phys. Rev. A **19**, 419 (1979)
18. Pouquet, A., Fournier, J.D., Sulem, P.L.: J. Phys. Paris **39** L, 199 (1979)
19. Pouquet, A., Lesieur, M., André, J.C., Basdevant, C.: J. Fluid Mech. **72**, 305 (1975)
    Frisch, U., Lesieur, M., Sulem, P.: Phys. Rev. Lett. **37**, 895 (1976)
20. Fournier, J.D., Frisch, U.: Phys. Rev. A **17**, 747 (1978)
21. Hohenberg, P.C., Halperin, B.I.: Rev. Mod. Phys. **49**, 435 (1977)
22. Deker, U., Haake, F.: Phys. Rev. A **11**, 2043 (1975)
23. Graham, R., Haken, H.: Z. Phys. **243**, 289 (1971)
24. Haken, H.: Synergetics, An Introduction. 2nd ed. Berlin, Heidelberg, New York: Springer 1978
25. Mori, H., Fujisaka, H.: Prog. Theor. Phys. **49**, 764 (1973)
26. Hosokawa, I.: J. Stat. Phys. **15**, 87 (1976)
27. Ruelle, D.: Phys. Lett. **72** A, 81 (1979)
28. Fox, R.F.: J. Math. Phys. **19**, 1993 (1978); Phys. Rep. **48**, 179 (1978)
29. Molen, K. van der, Maanen, H.R.E. van: Chem. Sci. Eng. **33**, 1161 (1978)
30. Kolmogorov, A.N.: C.R. Acad. Sci. USSR **30**, 301 (1941)
31. Orszag, S.A.: In: Fluid Dynamics. Balian, R., Peube, J.L. (eds.). New York: Gordon and Breach 1977
32. Rose, H.A., Sulem, P.L.: J. Phys. **39**, 441 (1978)
33. Martin, P.C., Dominicis, C. de: Suppl. Prog. Theor. Phys. **64**, 108 (1978)
34. See e.g. Batchelor, G.K.: An Introduction to Fluid Dynamics. London: Cambridge University Press 1967; Curle, N., Davies, H.J.: Modern Fluid Dynamics. Vol. 1. London: van Nostrand 1968
35. Aris, R.: Vectors, Tensors and the Basic Equations of Fluid Mechanics. Englewood Cliffs: Prentice-Hall 1962
36. Chandrasekhar, S.: Hydrodynamic and Hydromagnetic Stability. London: Oxford University Press 1961
37. Kraichnan, R.H.: Phys. Fluids **10**, 1417 (1967)
38. Martin, P.C.: In: Problème à $N$ corps. Witt, D. de, Balian, R. (ed.). New York: Gordon and Breach 1968
    Forster, D.: Hydrodynamic Fluctuations, Broken Symmetry, and Correlation Functions. Reading: W.A. Benjamin 1975
39. Lücke, M., Zippelius, A.: Z. Phys. B – Condensed Matter **31**, 201 (1978)
    Lücke, M.: Suppl. Prog. Theor. Phys. **64**, 124 (1978)
40. Hove, L. van: Phys. Rev. **95**, 249 (1954)
41. Egelstaff, P.A.: An Introduction to the Liquid State. London: Academic Press 1967
42. Lücke, M.: (unpublished)
43. Here the notation of [32] is used

44. Desnianskii, V.N., Novikov, E.A.: Prikl. Math. Mekh. (transl. J. Appl. Math. Mech.) **38**, 507 (1974)
Bell, T.L., Nelkin, M.: Phys. Fluids **20**, 345 (1977)
45. Janssen, H.K.: Z. Phys. B - Condensed Matter **23**, 377 (1976)
46. Phythian, R.: J. Phys. A **10**, 777 (1977)
47. See e.g. Bausch, R., Janssen, H.K., Wagner, H.: Z. Phys. B - Condensed Matter **24**, 113 (1976)
Dominicis, C. de, Peliti, L.: Phys. Rev. B **18**, 353 (1978)

M. Lücke
Institut für Festkörperforschung
Kernforschungsanlage Jülich GmbH
Postfach 1913
D-5170 Jülich 1
Federal Republic of Germany