Externally Modulated Rayleigh-Bénard Convection: Experiment and Theory

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Quantitative experimental results are presented on the response of a convecting fluid to periodic modulation of the imposed heat current. A theoretical model which is a generalization of the Lorenz equations is proposed to describe the results. It yields semiquantitative agreement with experiment in both linear and nonlinear regimes.

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External modulation of a nonlinear system can dramatically alter its behavior and can induce novel dynamic states, particularly near a point of instability. Examples are a pendulum or an electrical circuit under parametric modulation, or the more interesting case of a phase-separating fluid undergoing periodic spinodal decomposition.[1] An example from classical hydrodynamics is Rayleigh-Bénard convection under periodic external modulation. This problem has been studied theoretically,[2] with primary focus on modifications of the *threshold* of convection caused by modulation. There have been few experimental tests[3] of the calculations and little theoretical or experimental work on the *non*linear aspects of the problem, i.e., on phenomena occurring *above* threshold. The purpose of the present work is to report quantitative experiments and to present a theoretical model which is accurate enough to yield semiquantitative agreement with experiment yet simple enough to permit calculations in both linear and nonlinear regimes.

Consider a laterally infinite Rayleigh-Bénard system in which the temperature $T^l(t)$ of the lower plate is a periodic function of time with period $2\pi/\omega$ and the upper plate temperature T^u is held constant. We define a time-dependent (reduced) Rayleigh number

$$r(t) = r_0[T^l(t) - T^u]/(\bar{T}^l - T^u), \qquad (1)$$

where $r_0 = \bar{R}/R_c$, R_c is the critical Rayleigh number[4] in the absence of modulation, and $\bar{R}$ is the static Rayleigh number corresponding to the average temperature difference $\bar{T}^l - T^u$. [Times are measured in units of the vertical diffusion time d^2/κ, where d is the thickness of the fluid layer and κ its thermal diffusivity.] By a truncation[5] of the Oberbeck-Boussinesq (OB) equations of hydrodynamics which retains three spatial Fourier modes of the velocity and temperature fields, one arrives[6] at the following (Lorenz) equations for the amplitudes x, y, and z of these modes in the modulated case:

$$\tau\dot{x}(t) = -\sigma x(t) + \sigma y(t), \qquad (2a)$$

$$\tau\dot{y}(t) = -y(t) + [\tilde{r}(t) - z(t)]x(t), \qquad (2b)$$

$$\tau\dot{z}(t) = -\tfrac{8}{3}[z(t) - x(t)y(t)], \qquad (2c)$$

$$j^{\text{conv}}(t) = j(t) - j^{\text{cond}}(t) = g^{-1}z(t), \qquad (2d)$$

where σ is the Prandtl number,[4] τ and g are constants determined in the absence of modulation,[7] and $\tilde{r}(t)$ is a periodic function whose nth Fourier coefficient[8] is related to that of $r(t)$ by $\tilde{r}_n = 4\pi^2 r_n/(4\pi^2 - i\omega n)$. The total heat current at the lower plate $j(t)$ (normalized to its value at R_c in the absence of modulation) consists of a convection part $j^{\text{conv}}(t)$ calculated by Eq. (2d), and a conductive part $j^{\text{cond}}(t)$ which was determined experimentally in our work.[6]

The stability properties of the conductive state $x = y = z = 0$ are those of the parametrically driven oscillator

$$(\tau^2/\sigma)\ddot{x}(t) + \tau(\sigma+1)\sigma^{-1}\dot{x}(t) - [\tilde{r}(t) - 1]x(t) = 0 \qquad (3)$$

which follows from Eqs. (2) by linearization. Let us consider for simplicity a harmonic modulation in

(1), i.e.,[8]

$$r(t)=r_0\,\mathrm{Re}[1+\Delta\exp(-i\omega t)]. \tag{4}$$

The modulation stabilizes the conductive state $x=0$ by damping out small convective perturbations,[2] thus leading, in general, to an upward shift of the threshold value of the average[8] Rayleigh number r_0 at which convection sets in. This threshold can be calculated as a function of ω and Δ using Eq. (3), and the values agree closely[6] with those calculated exactly from the linearized OB equations without truncation in limiting cases[2] such as small Δ or ω. Figure 1 shows as dashed lines results obtained numerically[6] from (2a)–(2d) for the average convective current[8] j_0^{conv} as a function of r_0 for fixed ω and two values of Δ. The suppression of convection caused by modulation is seen by comparison with the dotted line which corresponds to $\Delta=0$ (no modulation).

The above results hold for ideal, laterally infinite systems. In real cells, the dynamic thermal mismatch between the fluid and the sidewalls causes lateral heat currents.[9] Within our truncation procedure, they give rise to additive forcing[6] in the velocity equation (2a) yielding

$$\tau\dot{x}(t)=-\sigma x(t)+\sigma y(t)+\sigma\xi(t). \tag{2a$'$}$$

The forcing term $\xi(t)$ was introduced earlier[4] to describe the finite onset time for convection when the Rayleigh number is raised from below to above R_c. The Fourier coefficients[8] of $\xi(t)$ are related to those of the modulation $r(t)$ by $\xi_n=-i\omega n r_n \times f\psi(\omega n)$.[6] The function $\psi(\mu)$ varies smoothly from 1 to 0 for $0\le\mu<\infty$ (for the frequencies considered here, ψ is close to 1). The dynamic mismatch parameter f was calculated theoretically for stress-free horizontal boundaries[9] and the resulting value explained the onset experiments[4] semiquantitatively. The present experiments are more sensitive to the precise value of f, and moreover they involve an extrapolation of previous studies[4,9] to higher frequencies. Therefore, we determined f by fitting to the experimental data points shown in Fig. 1. The resulting value $f=0.04$ is within a factor of 3 of the earlier estimates,[4,9] which can be considered satisfactory agreement. This value is kept fixed in all other calculations, and thus there are no further adjustable parameters. The effect of $\xi(t)$ on the average convective current j_0^{conv} is shown by the solid curves in Fig. 1. While the ideal modulated system shows a sharp Hopf bifurcation from conduction to convection (dashed curves), the sidewall forcing makes the bifurcation imperfect and masks the upward threshold shifts predicted for the ideal system.

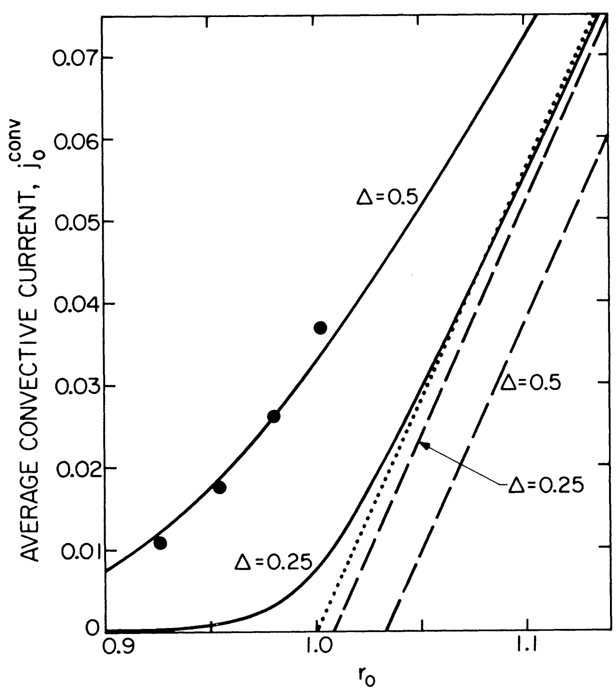

FIG. 1. Average convective current j_0^{conv} vs average Rayleigh number r_0 for $\omega=3.3$ and two different modulation amplitudes Δ. The dotted line is the unmodulated result ($\Delta=0$). The dashed lines are theoretical predictions for the ideal system with no sidewall forcing [Eqs. (2a)], and they show an upward shift of the convective threshold. The solid lines are for finite forcing [Eq. (2a)′] and they display an imperfect bifurcation from conduction to convection. The solid circles are the experimental data for $\omega=3.3$ and $\Delta=0.5$ from which the value $f=0.04$ of the mismatch parameter was determined.

The experiments were carried out in a cylindrical cell[10] of radius $4.78d$ and filled with normal ^{4}He ($\sigma=0.78$). A *sinusoidally* varying heat current[8] $j(t)=j_0+a\sin\omega t$ was applied to the lower plate, while the upper plate was held at constant temperature. The ensuing Rayleigh number was then measured as a function of time for different sets of j_0, a, ω.

In order to compare the experimental $r(t)$ with the theory, it is necessary to "invert" the model,[6] i.e., to find the function $r(t)$ which leads to the experimentally imposed total current $j(t)$ via Eqs. (2). The inversion also requires a knowledge of the conductive current $j^{\mathrm{cond}}(t)$ which appears in Eq. (2d). Here, we determined $j^{\mathrm{cond}}(t)$ for a given $r(t)$ by separate experiments in the absence of convection.[6] The inversion required typically five to ten iterations. Results for a typical run are compared with experiment in Table I which displays the amplitudes and phases of the Fourier coefficients[8] of

TABLE I. Amplitudes and phases of the Fourier coefficients (Ref. 8) $|r_n|\exp^{i\phi_n}$ of $r(t)$, where a current $j(t) = 1.066 + 1.000 \sin \omega t$ was applied at the bottom plate.

	ω	r_0	$\|r_1\|$	ϕ_1	$10^3\|r_2\|$	ϕ_2	$10^3\|r_3\|$	ϕ_3	$10^3\|r_4\|$	ϕ_4	$10^3\|r_5\|$	ϕ_5
Exp.	9.8	1.044	0.1095	3.20	0.11	1.1	...	...	...	...	...	...
Th.	9.8	1.045	0.1105	3.20	0.05	3.1	...	...	...	...	...	...
Exp.	6.5	1.046	0.1565	2.98	0.55	1.0	0.03	−0.4	...	...	...	...
Th.	6.5	1.050	0.1530	2.99	0.30	2.2	0.03	1.1	...	...	...	...
Exp.	3.9	1.028	0.2375	2.70	4.0	−0.1	1.1	4.0	0.30	1.7	0.05	5.1
Th.	3.9	1.033	0.2355	2.71	3.7	0.2	1.1	4.4	0.25	2.2	0.05	6.1

$r(t)$ for a given *sinusoidal* $j(t)$. Although the Rayleigh number $r(t)$ is dominated by its fundamental Fourier component, the small contributions from the higher harmonics are well reproduced by the theory. Alternatively, we may compare the experimental and theoretical convective current[11] $j^{\text{conv}}(t)$, as is done in Fig. 2 for another typical sinusoidal $j(t)$. Note that $j^{\text{conv}}(t)$ differs dramatically in magnitude, phase, and harmonic content from the total imposed current $j(t)$. Nonetheless, there is good agreement between experiment and theory.

The Lorenz model (2a)–(2c) has the symmetry $x \leftrightarrow -x, y \leftrightarrow -y, z \leftrightarrow z$ which reflects the equivalence of the two turning directions of the convective rolls in a laterally infinite system. With the forcing term $\xi(t)$ [Eq. (2a)′], the two directions are no longer equivalent. In one of the two periodic states the velocities are suppressed by the sidewall forcing, while in the other they are enhanced. With increasing modulation amplitudes the forcing increases, and a transition from the unfavored to the favored convecting state can be induced. This transition, marked by an increase in j_0^{conv}, was indeed found experimentally at parameter values close to those predicted by the model. Details, as well as further experimental data, will be published elsewhere.[6]

In conclusion, we have presented the first quantitative experimental data on the response of a convecting fluid to periodic external modulation of the heat current near threshold and a theoretical model to describe these data. Although this model involves an uncontrolled truncation of the hydrodynamic equations, it gives a good account of both linear and *non*linear aspects of the experiments. Our specific results are: (i) the threshold shifts due to modulation in ideal (laterally infinite) systems [Eq. (3)] agree well with those obtained from exact solutions of the hydrodynamic equations in limiting cases; (ii) under modulation, the dynamic sidewall heating induces lateral heat flow into or out of the fluid, causing additional forcing of the velocity field. This effect masks the ideal threshold behavior and, in general, leads to significant rounding of the bifurcation. The extent of the rounding is controlled by a mismatch parameter which has been determined from the data and is of the same order of magnitude as a previous theoretical esti-

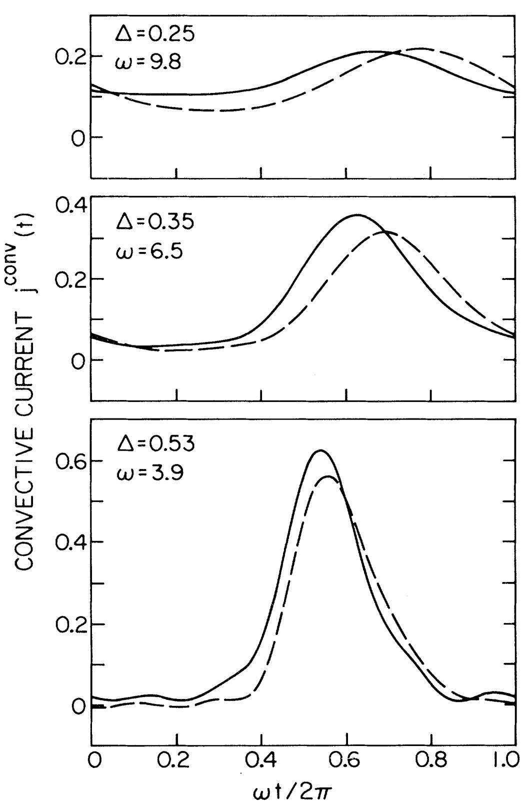

FIG. 2. The time dependence of the convective current $j^{\text{conv}}(t)$ over one period of three frequencies ω and modulation amplitudes Δ, and a sinusoidal imposed current $j(t) = 1.301 + 1.235 \sin(\omega t)$. The solid lines are experimental results and the dashed lines come from iteration of Eqs. (2). Note that the shape of $j^{\text{conv}}(t)$ differs dramatically from that of the imposed current.

mate[9] and an independent experimental determination;[4] (iii) the convective current above threshold can be calculated numerically in a straightforward manner for both ideal and real systems; (iv) quantitative experiments have been carried out, and the convective current $j^{\text{conv}}(t)$ has been measured for various modulations of the total current; (v) comparison between experiment and theory for $j^{\text{conv}}(t)$ yields semiquantitative agreement.

After this work was completed, we became aware of recent theoretical results of Roppo, Davis, and Rosenblat[12] who showed that in a modulated laterally infinite system a hexagon pattern will be stable near threshold. In that case, the mode truncation leading to Eqs. (2) is no longer applicable. This effect suggests interesting future experimental studies. However, for the present range of parameters, recent flow visualization experiments have shown that flow in the form of cylindrical rolls is stable in the presence of modulation.[13] We conclude that the hexagonal pattern[12] is suppressed by the dynamic forcing from the sidewalls, and that our model is applicable to a wide range of finite systems.

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[7]In a laterally infinite system with stress-free horizontal boundaries (Ref. 4) $R_c = 27\pi^4/4$, $\tau = 2/3\pi^2$, $g = \frac{1}{2}$, and Eqs. (2) reproduce the usual Lorenz model (Ref. 5) in the absence of modulation ($r_n = 0$ for $|n| \geq 1$). For that case, the model yields the exact threshold R_c, and the exact slope g^{-1} of j^{conv} vs $(r-1)$ above R_c. In the more realistic case of rigid horizontal boundaries, the appropriate values are (Ref. 4) $R_c = 1708$, $\tau = 0.7534 \times [(\sigma + 0.5117)/(\sigma + 1)]2/3\pi^2$, $g = 0.6994 - 0.0047\sigma^{-1} + 0.0083\sigma^{-2}$. In a laterally finite system, the parameters R_c, τ, and especially g are modified, and they depend on the specific flow pattern which exists in the system (Ref. 4). In the analysis presented here, we used experimental values of these parameters, determined in the absence of modulation.

[8]Fourier coefficients will be defined by $A(t) = \sum_{n=-\infty}^{\infty} A_n \exp(-i\omega n t)$, where $A(t)$ is any one of the functions $r(t)$, $\tilde{r}(t)$, $j(t)$, $j^{\text{conv}}(t)$, $\xi(t)$. The Fourier coefficient r_1 is related to Δ in Eq. (4) by $r_1 = r_0\Delta/2$. The coefficient j_1 is related to the amplitude a of the sinusoidal current used in the experiment by $j_1 = ia/2$.

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