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Some properties of an eight-mode Lorenz model for convection in binary fluids

Guenter Ahlers
Department of Physics, University of California, Santa Barbara, California 93106

M. Lücke
Institut für Theoretische Physik, Universität des Saarlandes, D-6600 Saarbrücken, West Germany
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We examine a number of static and dynamic properties of an eight-mode model for convection in binary mixtures proposed recently by Cross [Phys. Lett. (to be published)]. We find that the model allows richer dynamics than is observed in experiments. We also find differences between the bifurcations and fixed points of the eight-mode model and the five-mode model proposed earlier by Veronis [J. Mar. Res. **23**, 1 (1965)]. The effect of nonlinearities on the transient dynamics is similar to recent experimental observations. The bifurcations, fixed points, and dynamics are influenced significantly by the addition of an inhomogeneous "field" term to the model.

Convection in a thin horizontal layer of a binary mixture heated from below has aroused renewed interest recently both among theorists[1] and experimentalists.[2–9] The origin of this activity may be found in perceived opportunities to observe interesting dynamics near a Hopf bifurcation, particularly near a codimension-two point where this bifurcation collides with a stationary bifurcation.[1] Veronis[10] introduced a Lorenz-like five-mode model for the amplitudes of the lowest Fourier components of the solutions to the equations of motion of a thermohaline system. For certain parameter ranges this model yields a Hopf bifurcation to standing waves (SW) in the convective flow. It has been used by several authors[7,11] to study the properties of this system. However, it is known from various theoretical analyses[12] that standing waves in this case are unstable with respect to traveling-wave (TW) transients. The instability or nonexistence of the limit cycle corresponding to SW states was indeed revealed recently by experiments,[4,6] which were unable to detect stable oscillations near the codimension-two point in spite of a careful search.[13] Other experimental work[5,14] has demonstrated the existence of interesting long-lived transients, consisting of the superposition of two TW's moving in opposite directions, which experience no nonlinear saturation, even very near the Hopf bifurcation, until rather large amplitudes are reached. These various experimental and theoretical results provoked Cross to propose a simple extension of the model of Veronis to eight modes.[15] This extended model permits TW transients, and its dynamics has a number of common features with experimental results.

We reexamined the eight-mode model, and found the following.

(1) The Hopf bifurcation, although it occurs at the same parameter values as that of the five-mode model, differs in that *four* complex eigenvalues (two identical complex-conjugate pairs) cross the imaginary axis rather than two. An inhomogeneous constant "field" term ξ, representing experimental imperfections, will, in general, remove this degeneracy.

(2) Beyond the Hopf bifurcation, the eight-mode model permits[15] TW transients, but also the evolution of SW states, or of modulated-traveling-wave (MTW) transients which may be regarded as superpositions, in proportions determined by the initial conditions, of pure TW and SW transients. The nature of the transients is also influenced by ξ.

(3) Random initial conditions will, in general, yield MTW transients. If the eight-mode model is relevant to the unique transients seen in experiments,[5] then these transients must be selected from among all the allowed ones by conditions which are beyond the scope of the model, e.g., the lateral boundaries.[16]

(4) When the nonlinearities become important during the approach toward the nonlinear fixed point, the model *does* select pure TW transients. During that phase of the dynamics the frequency drops quite rapidly, typically by a factor of 2 or 3. This sudden decrease in frequency, accompanied by the formation of pure right- or left-traveling waves, recently has been observed in experiments.[17]

(5) The nonlinear fixed point differs from that of the five-mode model in that it involves a common arbitrary phase ϕ of three variable pairs. A finite ξ uniquely selects a particular phase.

(6) For $\xi=0$, the nonlinear fixed point becomes unstable upon decreasing the Rayleigh number R, as a real eigenvalue describing the growth rate of the phase velocity $\dot{\phi}$ becomes positive. A finite ξ stabilizes the fixed point.

The models of Veronis[10] and Cross[15] are based on permeable, free-slip horizontal boundary conditions. Analogous five- and eight-mode models derived[18] for impermeable boundaries yield partly similar results. We therefore pursue here the free-slip, permeable eight-mode model, which we prefer to write in the form[19]

$$\tau_0\dot{X}=-\sigma(X-Y-U)+\sigma\xi\ , \qquad (1a)$$

$$\tau_0\dot{Y}=-Y+(r-Z)X\ , \qquad (1b)$$

$$\tau_0\dot{U}=-L(U-\psi Y)+(r\psi-V)X\ , \qquad (1c)$$

$$\tau_0\dot{Z}=-b(Z-X_1Y_1-X_2Y_2)\ , \qquad (1d)$$

$$\tau_0\dot{V}=-bL(V-\psi Z)+b(X_1U_1+X_2U_2)\ , \qquad (1e)$$

where $X = X_1 + iX_2$, $Y = Y_1 + iY_2$, $U = U_1 + iU_2$, and $\xi = \xi_1 + i\xi_2$. In Eq. (1a) we have added a constant forcing field[20] $\sigma\xi$. The parameters are the Prandtl number σ, the Lewis number L, the separation ratio ψ,[1] and the reduced Rayleigh number $r = R/R_c^0$ ($R_c^0 = 27\pi^4/4$ is the critical Rayleigh number at $\psi = 0$). With free-slip, permeable horizontal boundaries the critical wave number for all ψ is $k_c = \pi/\sqrt{2}$. For $k = k_c$ one has $q_c^2 = 3\pi^2/2$, $\tau_0 = 1/q_c^2$, and $b = 4\pi^2\tau_0$. Setting $\xi = 0 = X_2 = Y_2 = U_2$ yields the five-mode model which after rescaling variables and parameters can be brought into the form of Veronis.[10] In terms of our variables, however, the z component w of the velocity field $\mathbf{v}$ is given by

$$w = q_c(Xe^{-ik_cx} + \text{c.c.})\sqrt{2}\sin\pi z \ . \tag{2a}$$

The deviations θ and c from the pure-conduction temperature and concentration profiles are

$$\theta = (R_c^0/q_c)(Ye^{-ik_cx} + \text{c.c.})\sqrt{2}\sin\pi z - (R_c^0/\pi\sqrt{2})Z\sqrt{2}\sin 2\pi z \ , \tag{2b}$$

and

$$c = (R_c^0/q_c)(Ue^{-ik_cx} + \text{c.c.})\sqrt{2}\sin\pi z - (R_c^0/\pi\sqrt{2})V\sqrt{2}\sin 2\pi z \ . \tag{2c}$$

The Nusselt number is given by $N = 1 + 2Z/r$. We have scaled length by the cell height d, time by d^2/κ where κ is the thermal diffusivity, temperature by $\nu\kappa/\beta_1 g d^3$, and concentration by $\nu\kappa/\beta_2 g d^3$. Here ν is the kinematic viscosity, g the gravitational acceleration, β_1 the thermal expansion coefficient at constant concentration, and β_2 the solutal expansion coefficient at constant temperature.

The pure conduction state of a fluid layer with free-slip, permeable boundary conditions loses stability via a Hopf bifurcation at[1,21]

$$r_{c0} = (1+L)(1+L+\sigma+L/\sigma)/(1+\sigma+\sigma\psi) \ , \tag{3}$$

provided

$$\psi < \psi_{\text{pc}} = -(1+\sigma)(1+\sigma+L^{-1}+\sigma/L+\sigma/L^2)^{-1} \ . \tag{4}$$

At r_{c0}, the Hopf frequency is given by

$$\tau_0\omega_{c0} = [-\psi(1+L)(\sigma+L)/(1+\sigma+\sigma\psi) - L^2]^{1/2} \ . \tag{5}$$

For $\xi = 0$ these properties are reproduced by the eight-mode and by the five-mode model. Since the fluid layer considered here is invariant under arbitrary translation in the x direction and reflection at the plane $x = 0$, the center manifold of the Hopf bifurcation is four-dimensional with four complex-conjugate eigenvalues crossing the imaginary axis in the form of two identical pairs. This is the case in the eight-mode model. In the five-mode model there is only one such pair since the dimension of the center manifold was reduced by a factor of 2 by ignoring the modes X_2, Y_2, and U_2. For $\xi \neq 0$ the Hopf bifurcation remains sharp but r_{c0} is reduced. The degeneracy of the eigenvalues is removed such that near r_{c0} the eight-mode model has *two* distinct frequencies with corresponding growth rates close to zero.

The fixed points of (1) are given by

$$N_1N_2(|X|^4 + \alpha|X|^2 + \beta)X = \xi \ , \tag{6a}$$

$$Y = rN_1X \ , \tag{6b}$$

$$U = r\psi L(1+L)N_1N_2X \ , \tag{6c}$$

$$Z = rN_1|X|^2 \ , \tag{6d}$$

$$V = r\psi N_1N_2(1+L+L^2+|X|^2)|X|^2 \ , \tag{6e}$$

with $N_1 = (1+|X|^2)^{-1}$, $N_2 = (L^2+|X|^2)^{-1}$, $\alpha = 1 + L^2 - r$, and $\beta = L^2(1-r) - r\psi L(1+L)$. According to (6b) and (6c) the complex fixed-point solutions X, Y, U all have the same phase ϕ. This phase is arbitrary for $\xi = 0$ [cf. (6a)]. Thus, Eq. (1) is invariant under simultaneous phase changes of X, Y, U reflecting the invariance of the convective pattern Eq. (2) as a whole against translations along the x axis. Consequently, there is a continuous family of fixed points labeled by ϕ such that X, Y, U lie on concentric circles in the complex plane with $|X|$ given by the positive roots of the bracket in (6a). The special cases $\phi = 0$ and $\phi = \pi$ yield the fixed points of the five-mode model. Without symmetry-breaking imperfections ($\xi = 0$), initial conditions determine the final ϕ, i.e., the relative position of the stationary convective structure after TW transients have decayed. Because of the above invariances the approach to the final stationary state is very slow as illustrated in Fig. 3 of Ref. 15. For $\xi > 0$, ϕ is fixed by the phase of ξ and the slow transients are truncated. Thus, we do not expect them to be related to the experimentally observed traveling waves[2,8] in the nonlinear state.

For $\psi > \psi_{\text{pc}}$ and $\xi = 0$, the pure conduction state loses stability at

$$r_{\text{cs}} = [1 + \psi(1+1/L)]^{-1} \tag{7}$$

when a pair of degenerate real eigenvalues becomes positive.[1,21] For

$$\psi < \psi_t = -(1+L^{-1}+L^{-2}+L^{-3})^{-1} \tag{8}$$

this bifurcation is backwards, and for larger ψ it is forward into the fixed points given by Eqs. (6) with $\xi = 0$. For $\psi < \psi_t$, there is a saddle node at

$$r_s = 1 - L^2 - 2L(1+L)\psi + 2(1+L)L(\psi^2 + \psi - \psi/L)^{1/2} \ . \tag{9}$$

For the five-mode model, the upper stationary convecting branch is stable for all $r > r_s$, and ceases to exist for $r < r_s$. As pointed out by Cross,[15] this situation is changed in the eight-mode model where for $\psi < \psi_{\text{pc}}$ and $\xi = 0$ the convecting branch is stable only for $r > r' \cong r_{c0} > r_s$. Within our numerical accuracy of 10^{-6} one of two real eigenvalues becomes positive at $r' = r_{c0}$ while the other one remains zero because of rotational invariance. For $r < r'$ the positive eigenvalue equals $\ddot{\phi}/\dot{\phi}$. Thus, there is an acceleration of the phase, starting near the fixed point $\dot{\phi} = 0$, and a TW, e.g., $|X|\cos[kx - \phi(t)]$, begins to develop by an increase of its frequency (or velocity) from a vanishingly small initial value. For $\xi > 0$, the two real eigenvalues form a complex-conjugate pair. Thus, a finite-frequency instability occurs. The convecting branch is stabilized, i.e., $r' < r_{c0}$.

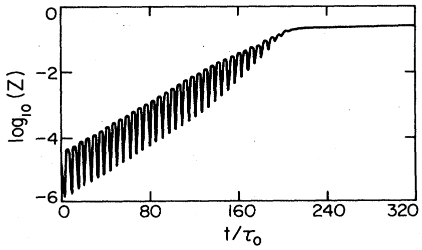

FIG. 1. The evolution of Z, on a logarithmic scale, as a function of time on a linear scale, for an MTW transient. The parameter values are $r=1.35$, $\psi=-0.25$, $\sigma=5$, $\xi=0$, and $L=0.01$. The overall exponential growth, and the oscillations which do not reach $Z=0$, are characteristic of MTW transients in the linear region. The smooth, nonoscillatory evolution at large t illustrates the selection of TW transients by the nonlinear terms.

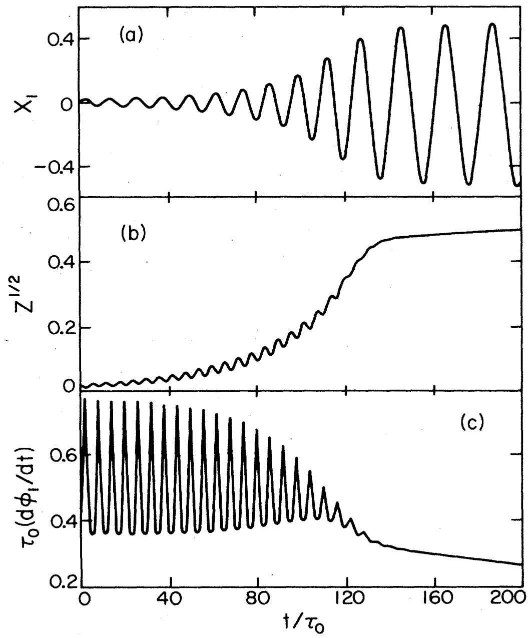

FIG. 2. (a) The time evolution of X_1, on linear scales, in the region of transition from linear to nonlinear behavior. The parameters are as for Fig. 1. The oscillations of X_1 are at a frequency ω, and ω decreases by a factor of 2 or so as the nonlinear region is entered near the right. (b) The time evolution as in (a), but for $Z^{1/2}$. For small t, Z oscillates with frequency 2ω. As the nonlinear region is entered, Z ceases to oscillate, while X continues to oscillate. (c) The time evolution, as in (a) and (b), but for the time derivative $\dot{\phi}$ of the phase ϕ of X. In the linear region $\dot{\phi}$ is modulated with frequency 2ω. This modulation is characteristic of MTW transients. In the nonlinear region a TW transient is selected in which $|X(t)|$ and $\omega(t)=\dot{\phi}(t)$ evolve smoothly without oscillations.

For $r>r_{c0}$, the small-amplitude dynamics of Eqs. (1) are easily understood when the nonlinearities are unimportant. In that case X_1, Y_1, and U_1 are uncoupled from X_2, Y_2, and U_2. Then the nature of the transients is determined by the initial phases and amplitudes and does not evolve in time. For example, a MTW transient maintains its initial proportion of SW and TW until nonlinear terms become important. We illustrate this in Fig. 1, where $Z(t)$ is shown for arbitrarily chosen initial conditions. As pointed out by Cross,[15] Z would evolve smoothly without oscillations for TW transients $(X\sim e^{(\lambda+i\omega)t})$. For SW transients $[X\sim e^{\lambda t}\cos(\omega t)]$, however, Z would oscillate as $\cos^2(\omega t)$ with frequency 2ω and vanish periodically. The case in Fig. 1 is clearly an intermediate one. There is overall exponential growth, oscillation with frequency 2ω, but the oscillation amplitude is too small for Z to ever vanish.

When the nonlinear terms become important, the dynamics changes dramatically. It is apparent from Fig. 1 that Z becomes nonoscillatory while X, Y, and U still oscillate in a way characteristic of TW $(X\sim e^{i\omega t})$. Furthermore, as Z changes from oscillatory to smooth behavior, the oscillation frequency of X drops, as can be seen in Figs. 2(a) and 2(b). The frequency evolution is illustrated in Fig. 2(c). During the initial linear MTW transient, the phase velocity $\dot{\phi}$ and $|X|$ are modulated at a frequency 2ω. In the nonlinear TW region, Z and $|X|$ evolve smoothly, $\dot{\phi}$ is nonoscillatory, smaller than its average value in the linear region by a factor of 2 or 3, and equal to the frequency ω of X observable in Fig. 2(a). The long-time evolution of $\dot{\phi}$, illustrated by Fig. 3 of Ref. 15, is an exponential decay on a time scale which is long compared to that of Fig. 2. The selection of pure TW's and the rather sudden decrease in ω by a factor of 2 or so at the time when the nonlinearities become important in the transients, has been observed in detail recently in experiments on alcohol-water mixtures.[17]

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[1]See, for instance, H. Brand, P. C. Hohenberg, and V. Steinberg, Phys. Rev. A **30**, 2548 (1984), and references therein.
[2]R. W. Walden, P. Kolodner, A. Passner, and C. M. Surko, Phys. Rev. Lett. **55**, 496 (1985).
[3]I. Rehberg and G. Ahlers, Phys. Rev. Lett. **55**, 500 (1985).
[4]G. Ahlers and I. Rehberg, Phys. Rev. Lett. **56**, 1373 (1986).
[5]P. Kolodner, A. Passner, C. M. Surko, and R. W. Walden, Phys. Rev. Lett. **56**, 2621 (1986).
[6]H. Gao and R. P. Behringer, Phys. Rev. A **34**, 697 (1986).
[7]I. Rehberg and G. Ahlers, in *Multiparameter Bifurcation Theory in Contemporary Mathematics,* edited by M. Golubitsky and J. Guckenheimer (Americal Mathematical Society, New York, 1986); and (unpublished).
[8]E. Moses and V. Steinberg, Phys. Rev. A **34**, 693 (1986).
[9]E. Moses and V. Steinberg (unpublished).
[10]G. Veronis, J. Mar. Res. **23**, 1 (1965); a similar truncation for

convection in a pure fluid was introduced earlier by E. N. Lorenz, J. Atmos. Sci. **20**, 130 (1963).

[11]M. G. Velarde and J. C. Antoranz, Phys. Lett. **80A**, 220 (1980); L. N. DaCosta, E. Knobloch, and N. O. Weiss, J. Fluid Mech. **25**, 109 (1981); D. R. Moore, J. Toomre, E. Knobloch, and N. O. Weiss, Nature **303**, 663 (1983); A. Agarwal, J. K. Bhattacharjee, and K. Banerjee, Phys. Rev. B **30**, 6458 (1984).

[12]C. S. Bretherton and E. A. Spiegel, Phys. Lett. **96A**, 152 (1983); P. Coullete, S. Fauve, and E. Tirapegui, J. Phys. (Paris) Lett. **46**, L787 (1985); E. Knobloch, Phys. Rev. A **34**, 1538 (1986), and references therein.

[13]The experimental situation is different for binary mixtures in a porous medium, where stable oscillations and the dynamics expected near the codimension-two point (see Ref. 1) have been observed (see Ref. 3).

[14]D. R. Caldwell, J. Fluid Mech. **64**, 347 (1974).

[15]M. C. Cross, Phys. Lett. (to be published).

[16]The effect of lateral boundaries on traveling waves is discussed by M. C. Cross (unpublished).

[17]R. Heinrichs, G. Ahlers, and D. S. Cannell (unpublished).

[18]S. J. Linz and M. Lücke (unpublished).

[19]The model of Ref. 15 follows from Eq. (1) by substituting $U \rightarrow -\psi U$, $V \rightarrow -\psi V$.

[20]G. Ahlers, P. C. Hohenberg, and M. Lücke, Phys. Rev. A **32**, 3493 (1985).

[21]D. T. J. Hurle and E. Jakeman, J. Fluid Mech. **47**, 667 (1971).