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Influence of Magnetic Fields on Taylor Vortex Formation in Magnetic Fluids

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The flow of a magnetic fluid placed inside a small gap between concentric rotating cylinders is investigated for axial, radial and azimuthal magnetic fields. An equation of motion is derived phenomenologically to describe the hydrodynamics of magnetic fluids. Studied are the changes in the critical Taylor number T_c and wave number k_c which characterize the instability of primary circular Couette flow towards Taylor vortices. It is found that all above magnetic fields have a stabilizing effect on circular Couette flow and that k_c increases or decreases, depending on the direction of the magnetic field. Besides this, the influence of the magnetic fields on the correlation length ξ_0, the wave number of maximal growth k_m and the linear growth rate amplitude σ_0 is determined.

I. Introduction

Magnetic fluids exhibit a wealth of new phenomena [1] in comparison to ordinary fluids when magnetic fields are applied. This possibility to exert additional volume forces on a fluid is already exploited in many technical applications [2] and has stimulated also basic research [1].

Here we shall investigate the effect of magnetic fields on a pattern forming hydrodynamic instability that is well studied in ordinary, unmagnetic fluids: the appearance of Taylor vortices in the flow between two concentric cylinders of which the inner one is rotating (for a review see e.g.: [3, 4]). Experimental investigations of the changes in the first two flow instabilities in the rotating Couette problem have just started [5]. Berkovsky [6] recently studied the effects of a steady uniform magnetic field in axial direction on the formation of Taylor vortices and determined quantitatively the critical Reynolds number R_c by linear stability analysis. He observed a decrease of the critical wave number, but does not give quantitative values. Here we present explicit quantitative results on both, the critical Taylor number $T_c = 2R_c^2 \frac{\delta}{\delta+2}$ (δ = gap width d/radius of inner cylinder R_1) and the critical wave number k_c for three differently oriented steady magnetic fields (axial, radial and azimuthal). The analytical results of a linear stability analysis are compared with direct computer simulations. Besides this, growth rates, correlation length and wave numbers of maximum growth for Taylor numbers slightly larger than T_c are determined numerically for all three magnetic fields.

In modelling the time-dependent dynamics of a magnetic fluid, one has to use a macroscopic equation of motion. Up to now at least five derivations can be found in the literature [7–10, 12]. A crucial point in these models is the equation of magnetization. 1972 Shliomis [7, 11] proposed a relaxation equation of magnetization. He combined this one with an equation for conservation of momentum and another one for conservation of intrinsic angular momentum and used these equations successfully for stationary fluids to derive an expression for the additional rotational viscosity in magnetic fluids.

Berkovsky 1980 suggested a partial equilibrium model [8] which accounts for both a relaxation of magnetization by rotation of the particles (Brownian motion) and for a relaxation of magnetization within the particle (intrinsic superparamagnetism by Neel-fluctuations). But as pointed out by himself, a solution of the complete equation does not seem useful if the magnetization **M** is close to the equilibrium magnetization $\mathbf{M}_0$. This is normally the case since the characteristic hydrodynamic time scale t in general exceeds by far the typical Brownian time scale, $\tau_B \approx 10^{-6}$ sec, for relaxation of the magnetization. In this case he obtains an equation relating the deviation $\boldsymbol{\mu} \equiv \mathbf{M} - \mathbf{M}_0$ to the magentic field $\mathbf{H}$, it's change in time $\frac{d}{dt}\mathbf{H}$ and vorticity $\boldsymbol{\Omega} \equiv \nabla \times \mathbf{v}$.

Jansons [12] gives stress tensors for a dilute magnetic liquid in homogeneous and in nonuniform, space and time varying magnetic fields. He considers spheroidal particles at small Peclet numbers and arrives at lengthy and complicated formulas.

A fourth proposition has been given recently by Kroh and Felderhof [9]. In their model for 'electromagnetohydrodynamics of polar liquids and suspensions' they consider explicitly memory-effects in their constitutive equations for polarisation and magnetization. These memory-effects may be quite small and negligible if one is interested in times $t \gg \tau_B$ as we are here.

Finally, 'New constituitive equations for conducting magnetic fluids with internal rotation' have been derived from thermodynamical considerations by Shizawa and Tanahashi [10]. Assuming nonconductivity, these equations may also be used for the normal, nonconducting magnetic fields. In their paper they derive a constitutive equation of magnetization which is, up to some small difference in the definition of the magnetic relaxation time τ, identical to the one obtained by Berkovsky for steady magnetic fields. However, without stating this explicitly, these authors implicitly use the difference in time scales $\tau_B/t \ll 1$ when they assume that the magnetization is always parallel to an effective magnetic field which is determined in turn by the velocity field.

Considering only steady magnetic fields, it is shown in this paper that the same equation for $\boldsymbol{\mu}$ may be obtained starting directly from the well known phenomenological relaxation equation of magnetization proposed by Shliomis 1972 and considering $\tau_B/t \ll 1$. Because of the different view points of the previous authors and because of slight differences in their results we first give a concise derivation of a macroscopic equation of motion in Sect. II. Section III contains the stability analysis of primary Couette flow and in Sect. IV the influence of magnetic fields on parameters of the Ginzburg-Landau equation and on the wave number of maximal growth are determined.

II. Ferrohydrodynamic Equation of Motion

A magnetic fluid is a suspension of single domain magnetic particles. Typical particle sizes are 10 nm. They are coated by some surfactant to hinder agglomeration and thus stabilize the suspension. Details on magnetic fluids may be found in the book of Rosensweig [1].

The magnetic fluid considered here is assumed to be incompressible, nonconducting, to have a constant temperature and a homogeneous distribution of magnetic particles. The last assumption may be justified by an estimate of the drift velocity caused by a magnetic field gradient or a gravitational field. This drift velocity is so small [see e.g. 12] that slippage can be neglected on normal experimental time scales. In accord with many experimental findings it is assumed that the particles do not exhibit intrinsic superparamagnetism. Thus their magnetic moment is rigidly locked to the particle, caused by anisotropy effects. Within the continuum model, the internal angular momentum can serve as a new macroscopic characteristic of the rotations of the small particles. Its volume density $\mathbf{S}$, in the case of small concentrations of identical spherical particles, can be written as $\mathbf{S} = I\,\boldsymbol{\omega}$, where $I = n I_0$ is the sum of the moments of inertia of the spheres in a unit volume and $\boldsymbol{\omega}$ is their mean ordered angular velocity.

Now Cauchy's equation of motion is given by

$$\rho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \nabla \cdot \underset{\sim}{\mathbf{T}} = \nabla \cdot \underset{\sim}{\mathbf{T}}^{(v)} + \nabla \cdot \underset{\sim}{\mathbf{T}}^{(em)} \tag{1}$$

where $\frac{\mathrm{d}}{\mathrm{d}t} \cdot = \frac{\partial}{\partial t} \cdot + \mathbf{v} \cdot \nabla \cdot$.

A fluid with internal angular momentum shows an asymmetry of the viscous stress tensor [1, 11, 13]:

$$\underset{\sim}{\mathbf{T}}^{(v)} = -p_0 \underset{\sim}{\mathbf{I}} + \eta((\nabla \mathbf{v}) + (\nabla \mathbf{v})^T) + 2\zeta\, \bar{\varepsilon} \cdot \left(\frac{\boldsymbol{\Omega}}{2} - \boldsymbol{\omega}\right) \tag{2}$$

$\bar{\varepsilon}$ is the alternating unit tensor $\varepsilon_{ijk}\, \mathbf{e}_i \mathbf{e}_j \mathbf{e}_k$. The last term describes a coupling of the internal spin rate to the fluid motion, given here by the vorticity $\boldsymbol{\Omega} = \nabla \times \mathbf{v}$. The coupling should be proportional to the difference of these two rates of rotation. For dilute suspensions and spherical particles the coupling constant ζ which has been termed vortex viscosity is related to the viscosity η and the volume concentration ϕ of the suspended particles by $\frac{3}{2}\eta\phi$ [1, 7]. Since we consider only stationary magnetic fields, electric fields are neglected:

$$\underset{\sim}{\mathbf{T}}^{(em)} = \underset{\sim}{\mathbf{T}}^{(m)} = \{-\tfrac{1}{2}\mu_0 H^2 \underset{\sim}{\mathbf{I}} + \mathbf{B}\mathbf{H}\}, \tag{3}$$

$$\nabla \cdot \mathbf{B} = 0, \qquad \mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}), \qquad \nabla \times \mathbf{H} = 0, \tag{4}$$

The conservation law of angular momentum can be expressed by (we use the nesting convention of Chapman and Cowley [14] for multiple products of tensors)

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{S} = -\bar{\varepsilon}:\underset{\sim}{\mathbf{T}} = \mu_0\,\mathbf{M}\times\mathbf{H} - \frac{1}{\tau_s}\left(\mathbf{S} - I\,\frac{\boldsymbol{\Omega}}{2}\right) \tag{5}$$

where $\frac{1}{\tau_s} = \frac{4\zeta}{I}$ is the viscous relaxation time being of the order of magnitude of 10^{-11} s.

For a wide range of flows the time in which significant changes in the velocity take place, and which is therefore the time scale of interest, is larger than 10^{-6} s. To give an example, the natural time scale in the Couette-Taylor-experiment we are interested in here is the diffusion time for velocity. This time is found to be of the order $\frac{d^2}{2\pi\nu}$ [15], ν being the kinematic viscosity and d the gap width. Inserting reasonable values ($d \sim 1$ cm, $\nu = 3\cdot 10^{-6}$ m²/s for hydrocarbon based magnetic fluids) one obtains 5 s. Also the time step of 2–$3\cdot 10^{-3}\, d^{2/2}\pi\nu$ that we used in our numerical investigations is much larger than 10^{-6} s and therefore surely larger than τ_s. Thus one can make an adiabatic approximation and neglect the inertial term in equation (5) which results in

$$\mathbf{S} = I\,\frac{\boldsymbol{\Omega}}{2} + \tau_s\,\mu_0\,\mathbf{M}\times\mathbf{H} \tag{6}$$

with the total stress tensor becoming symmetric.

The 13 equations (1), (4) and (6) do not form a complete set to solve for all the 16 unknowns **S**, **v**, **H**, **M**, **B**, p_0. The additional equation used here is the phenomenological relaxation equation for magnetism, as proposed by Shliomis [7].

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{M} = \frac{\mathbf{S}}{I}\times\mathbf{M} - \frac{1}{\tau}(\mathbf{M} - \mathbf{M}_0). \tag{7}$$

The last term describes a relaxation of the magnetization **M** to the equilibrium magnetization $\mathbf{M}_0$ in a coordinate system Σ' which is rotating with angular velocity $\boldsymbol{\omega} = \frac{\mathbf{S}}{I}$ with respect to a fixed system Σ. Thus in the fixed frame of reference one obtains the equation for magnetization cited above.

The equilibrium magnetization $\mathbf{M}_0$ can be determined by a Langevin formula. The relaxation time τ is determined by the Brownian relaxation time τ_B which is of the order of $\sim 1\cdot 10^{-6}$ s. Shliomis successfully used this equation to calculate the additional anisotropic rotational viscosity in stationary flows. This additional rotational viscosity for magnetic fluids appears as a result of applied magnetic fields: a hindrance of the free rotation of the particles shows up if the rotation axis and the magnetic field are not oriented parallel. Since Shliomis considered stationary fluids only, he assumed the magnetization to be stationary too and neglected the term $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{M}$. But for the Couette-Taylor-System and all other time-dependent flows showing a typical time scale t of more than $\tau_B \sim 10^{-6}$ s one can neglect the term $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{M}$ even if one is interested in time-dependent flow behavior because of adiabaticity. Since $\Omega\tau_B \ll 1$ for colloidal suspensions [11] (even in high speed magnetic seals this is satisfied) and the deviation $\boldsymbol{\mu} = \mathbf{M} - \mathbf{M}_0$ caused by the flow field is small, if $t \gg \tau_B$, one finds by inserting equation (6) in (7), neglecting $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{M}$ and expanding to first order in $\boldsymbol{\mu}$ and $\Omega\tau_B$:

$$\boldsymbol{\mu} = \tau_\perp \frac{M_0}{2H}(\boldsymbol{\Omega}\times\mathbf{H}). \tag{8}$$

Thus $\boldsymbol{\mu}$ is orthogonal to **H**; the parallel component is (to first order) not influenced by the velocity and vanishes identically. $\tau_\perp$ is an abbreviation for $\tau_B(1 + \frac{\tau_S\,\tau_B}{I}\,\mu_0 M_0 H_0)^{-1}$. Up to some differences in the definition of $\tau_\perp$ this is the equation which is obtained by Shizawa et al. and the one given by Berkovsky for steady magnetic fields.

Combining the equations given above one obtains the ferrohydrodynamic equation of motion

$$\rho\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} = \nabla p + \eta\,\nabla^2\mathbf{v} + \mu_0(\boldsymbol{\mu}\cdot\nabla)\mathbf{H} + \tfrac{1}{2}\mu_0\,\nabla\times(\boldsymbol{\mu}\times\mathbf{H}) \tag{9}$$

$$\begin{aligned} &= -\nabla p + \eta\,\nabla^2\mathbf{v} + c'(|\mathbf{H}|)\{(\mathbf{H}\times[\boldsymbol{\Omega}\times\mathbf{H}]\times\nabla|\mathbf{H}|\} \\ &\quad + c(|\mathbf{h}|)\{\nabla\times(\mathbf{H}\times[\mathbf{H}\times\boldsymbol{\Omega}]) \\ &\qquad - 2([\mathbf{H}\times\boldsymbol{\Omega}]\cdot\nabla)\mathbf{H}\} \end{aligned} \tag{10a}$$

which may also be written in a sometimes more useful form (with $\mathbf{F} \equiv \mathrm{rot}\,\mathbf{v}\times\mathbf{H} = \boldsymbol{\Omega}\times\mathbf{H}$):

$$\begin{aligned} \rho\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v} &= \nabla p + \eta\,\nabla^2\mathbf{v} + c'(|\mathbf{H}|)\{(\mathbf{H}\times\mathbf{F})\times\nabla|\mathbf{H}|\} \\ &\quad + c(|\mathbf{h}|)\{\mathbf{F}\,\nabla\cdot\mathbf{H} - \mathbf{H}\times(\nabla\times\mathbf{F}) - \mathbf{H}\,\nabla\cdot\mathbf{F}\}, \end{aligned} \tag{10b}$$

$c(|\mathbf{H}|)$ denotes $\frac{1}{4}\tau_\perp\mu_0\frac{M_0}{H}$ and c' is its derivative with respect to $|\mathbf{H}|$. The pressure p differs from p_0 by a term resulting from $(\mathbf{M}_0\cdot\nabla)\,\mathbf{H}$ which may be rewritten as $\nabla p'$. Some-times additional magnetic pressure terms which account for dipole interactions and magnetostriction are added to the normal pressure. Since we consider noninteracting dipoles and a homogeneous incompressible fluid, these terms are neglected here (see a detailed discussion of these terms in Rosensweig, Sect. 4.3 [1]). Besides this, magnetic force density terms which take the form of a gradient of a pressure have no influence on the incompressible magnetic fluids. Note that special care has to be given to the boundary conditions.

These equations describe the change of velocity as a function of ρ, η, c, c', $\mathbf{v}$ and the local magnetic field $\mathbf{H}$. In many cases this local field can be approximated by the externally applied field $\mathbf{H}_0$. Field perturbations caused by the magnetic fluid can be neglected when the value of the equilibrium susceptibility $\chi = M_0/H_0$ is small. This is frequently observed in practice. Then Eqs. (4) reduce to $\nabla \times \mathbf{H}_0 = 0$ and $\nabla \cdot \mathbf{H}_0 = 0$. (If one wants to account for perturbation of $\mathbf{H}_0$, at least the demagnetizing field and the Lorentz field have to be considered. Both fields depend on the magnetization and counterpart each other, thus the net effect will be small; for more details see the electrical analogue [16].)

All additional terms are proportional to $\mathbf{H} \times \operatorname{rot} \mathbf{v}$. This agrees with normal perceptions since one would expect a hindrance of the free rotation only in this case. If one specializes this equation for stationary one-dimensional Couette flow one obtains exactly the terms found by Shliomis [11] describing the additional rotational viscosity of ferrofluids as it should be.

Summarising, Eq. (10) describes the influence of a nonuniform magnetic field on the dynamics of a magnetic fluid. The basic assumption is that the time t representing the time scale for dynamical changes of the flow is larger than 10^{-6} s.

III. Stability Analysis of Couette Flow

Now we are going to use this equation to explore the stability boundaries of primary Couette flow for a magnetic fluid between two infinite, concentric cylinders of radius R_1 and R_2. The inner cylinder is rotating with $\boldsymbol{\Omega}_1 = \frac{V_1}{R_1}$ and the outer one is at rest. Three differently oriented, axisymmetric magnetic fields are applied: an axial field $\mathbf{H} = \hat{H}\mathbf{e}_z$, an azimuthal $\mathbf{H} = \frac{\hat{H}}{r}\mathbf{e}_\varphi$ and a radial one $\mathbf{H} = \frac{\hat{H}}{r}\mathbf{e}_r$. Since both, the primary Couette flow and the Taylor vortices are axisymmetric, we investigate only the axisymmetric problem. Besides this, we consider the narrow gap approximation which means that the gap width $d = R_2 - R_1$ is assumed to be small in comparison to the radii. So, in lowest order δ^0 of the gap parameter $\delta = \frac{d}{R_1} \ll 1$, one finds the deviations $\mathbf{u} = (u, v, w)$ and p from the basic Couette flow having only an azimuthal component $V_0(x) = 1 - x$ to satisfy the linearized equations for the axial magnetic field:

$$\begin{aligned} &\partial_t u - (1+S)(\partial_x^2 + \partial_z^2)\, u + \partial_x p = 2TV_0\, v \\ &\partial_t v - (1+S)(\partial_x^2 + \partial_z^2)\, v + S\,\partial_x^2 v = u \\ &\partial_t w - (1+S)(\partial_x^2 + \partial_z^2)\, w + \partial_z p = 0 \end{aligned} \tag{11}$$

azimuthal magnetic field:

$$\begin{aligned} &\partial_t u - (\partial_x^2 + \partial_z^2)\, u + \partial_x p = 2TV_0\, v \\ &\partial_t v - (1+S)(\partial_x^2 + \partial_z^2)\, v = u \\ &\partial_t w - (\partial_x^2 + \partial_z^2)\, w + \partial_z p = 0 \end{aligned} \tag{12}$$

radial magnetic field:

$$\begin{aligned} &\partial_t u - (1+S)(\partial_x^2 + \partial_z^2)\, u + \partial_x p = 2TV_0 v \\ &\partial_t v - (1+S)(\partial_x^2 + \partial_z^2)\, v + S\,\partial_z^2 v = u \\ &\partial_t w - (1+S)(\partial_x^2 + \partial_z^2)\, w + \partial_z p = 0 \end{aligned} \tag{13}$$

incompressibility condition:

$$\partial_x u + \partial_z w = 0. \tag{14}$$

Consistent with the narrow gap approximation, the magnetic field strength is considered to be constant across the gap: changes of this value are of the order δ. Thus the third term on the right hand side of (10) which is proportional to $\nabla|\mathbf{H}|$ has been neglected. Likewise, considering the fourth term, spatial derivatives are only taken of the unit vectors describing the orientation of the magnetic field. In the narrow gap limit the Taylor number T reduces to $R^2\,\delta$, the Reynolds number R being defined by $R_1\,\Omega_1\,d/\nu$ as usual. Besides linearizing, we have introduced $x = (r - R_1)/d$. Length, time, pressure, azimuthal, radial and axial velocity are scaled in units of d, d^2/ν, $\rho\,(\nu/d)^2$, $R_1\,\Omega_1$, and ν/d, respectively. The additional rotational viscosity $\nu_r \equiv c(|\mathbf{H}|)\cdot H^2/\rho$ being an effect of an applied field is given in units of the suspension viscosity ν without any magnetic field: $S \equiv \nu_r(|\mathbf{H}|)/\nu$. So here all the information on the magnetic field strength and on the specific properties of the magnetic fluid is collected in this parameter S. Up to now the possible range of S is not quite clear: values up to 100 are reported for some commercially available magnetic fluids [24]. But since these data show a strong shear rate dependence (which should not be the case if ν_r is defined the way we did here following Shliomis), they are attributed to interactions of the magnetic particles like clustering and chain formation. Rosensweig [17] reports values of 1.15 in concentrated ferrofluids being nearly shear rate independent. Much smaller values are found by other experimentalists (e.g. 0.15 [18]). So we investigated the range $0 \leqq S \ll 1$ for all three magnetic fields.

Appropriate boundary conditions have to be added to these equations. The natural boundary conditions for normal fluids are no-slip-conditions, which are u, v, $w = 0$ at $x = 0$, 1. This is not exactly true for magnetic fluids, as Brenner pointed out [19]. Due to wall effects the viscous translational and rotational motions of a particle are inseparably coupled when the particle is near a bounding wall. Although this in principle gives rise to a suspension scale slip velocity, these effects may be neglected here because they are very small.

Here we first impose idealized conditions as proposed by Kuhlmann [20]: no slip in azimuthal ($v=0$), but free slip in axial direction ($\partial_x w=0$). Since there are no axial friction forces along the cylinder walls, the effect of free slip is to enhance the onset of Taylor vortices: $T_c=1695$, $k_c=3.12$ are obtained for rigid boundaries and $T_c=654$, $k_c=2.23$ for idealized boundary conditions. The advantage of applying idealized boundary conditions is that the axisymmetric Taylor vortex field can be decomposed in trigonometric normal modes which simplifies the calculations considerably.

$$\mathbf{u}=\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}\hat{\mathbf{u}}(n,m,t)\sin(n\pi x)\cos(mkz)\right.$$
$$\left.+\mathbf{e}_\varphi\,\hat{v}(n,o,t)\sin(n\pi x)\right). \tag{12}$$

Truncating this expansion at $m=1$ and using an exponential time dependence $e^{\sigma t}$ together with the linearized equations (11)–(14), one is left with a generalized eigenvalue problem as outlined in [20]. The solvability condition for $\sigma=0$ yields the marginal stability curve in the (k, T) plane. Since the trigonometric functions in radial direction are not eigenfunctions of the differential equations, they have to be considered as basis functions. It is found that the expansion converges rapidly with the number of harmonics N in x – direction, in accordance with [20]. For $S=0$ and $N=1$, all three magnetic fields yield the values for a normal liquid:

$$k_c=\frac{\pi}{\sqrt{2}},\qquad T_c=\frac{27}{4}\pi^4 \tag{15}$$

as it should be. Results obtained for axial, azimuthal and radial magnetic fields and $N=1$ are given in terms of the reduced quantities $q_r:=\frac{q_c(S)}{q_c(S=0)}$ thus concentrating on the changes caused by the fields

$$\mathbf{e}_z\colon T_r(S)$$
$$=\{4(1+S)+[\sqrt{1+8(1+S)}-1]\}^2$$
$$\cdot\{4+[\sqrt{1+8(1+S)}-1]\}/(108[\sqrt{1+8(1+S)}-1])$$
$$k_r(S)=\left\{\frac{\sqrt{1+8(1+S)}-1}{2(1+S)}\right\}^{\frac{1}{2}}, \tag{16}$$

$$\mathbf{e}_\varphi\colon T_r(S)=(1+S)$$
$$k_r(S)=1, \tag{17}$$

$$\mathbf{e}_r\colon T_r(S)=\frac{1+S}{108}\left\{(1+S)\left[\sqrt{1+\frac{8}{1+S}}-1\right]+4\right\}^2$$
$$\cdot\left\{1+\frac{4}{\left[\sqrt{1+\frac{8}{1+S}}-1\right]}\right\}$$
$$k_r(S)=\left\{\frac{1+S}{2}\left[\sqrt{1+\frac{8}{1+S}}-1\right]\right\}^{\frac{1}{2}}. \tag{18}$$

These formulae correspond to the lines given in Fig. 1 and Fig. 2; the results for $N=3$ and $N=5$ (axial field) are indicated by small markers and are nearly indistinguishable from the first order approximation in harmonics. The lines in Fig. 1 clearly indicate that in any case a magnetic field stabilizes the Couette flow. This effect is found to be strongest for the radial field and weakest for the azimuthal field. The reduced wave number may be shifted to larger or smaller values, depending on the orientation of the magnetic field, as indicated by the lines in Fig. 2.

In order to investigate the no-slip-case, the full ferrohydrodynamic equation of motion has been solved numerically. Again it is assumed that the variation of the magnetic field strength across the gap is negligible. Thus terms like $\nabla|\mathbf{H}|$ and $\frac{\partial}{\partial r}|\mathbf{H}|$ are neglected in (10). We choose the radius ratio R_1/R_2 equal to 0.95. For this value many experiments have been done for ordinary fluids and the corresponding $\delta=0.0526$ is still small enough to allow for a comparison with results obtained in narrow gap theory.

The primitive equations which follow from (10) are solved by a finite difference method on a marker-and-cell-(=MAC)-grid consisting of 20 cells in radial and 40 cells in axial direction [15]. No-slip boundary conditions are employed at the cylinders and periodic ones in axial direction. A basic wavelength can be selected by fixing the periodicity length appropriately. To assure $\nabla\cdot\nabla$ ($\mathbf{v}$ is scaled in units of V_1) to be less than 10^{-7} in the end, the artificial compressibility method following Chorin [21] is used. The three codes developed so far are very efficient: 99.8% of the arithmetics is done within inner DO-Loops and vectorizes on the Cray-XMP at Jülich, MFLOP-rates of more than 110 have been obtained for the total code. As one should expect, the stationary state for a given Reynolds number does not depend on whether one starts the time-dependent calculation with primary Couette flow or from rest. No Taylor vortices develop if the axial flow component w is identical to zero everywhere within the gap which may also be

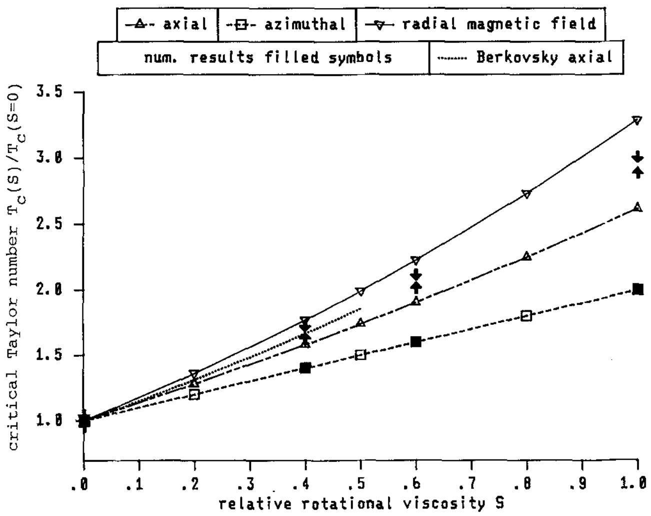

Fig. 1. Reduced critical Taylor number $T_c(S)/T_c(S=0)$ as a function of the relative rotational viscosity $S=\nu_r/\nu$ which is caused by three differently oriented magnetic fields. Lines with open symbols represent the results of linear stability analysis using idealized boundary conditions. The full squares and the arrows pointing up and down show the results obtained from numerical simulation for azimuthal, axial and radial fields. The dashed line corresponds to data [6] for the axial magnetic field. In any case a magnetic fluid stabilizes the primary circular Couette flow

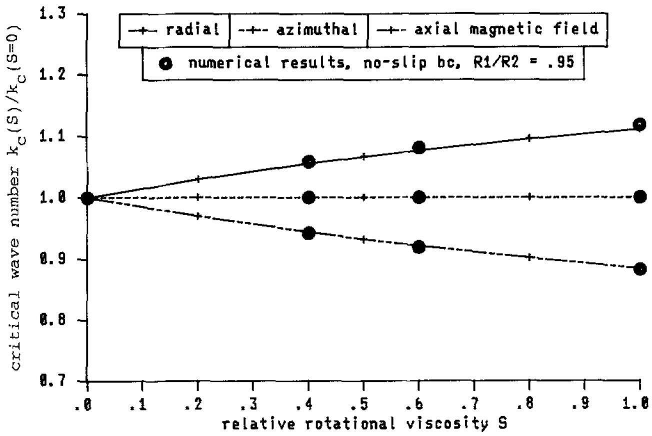

Fig. 2. Reduced critical wave number $k_c(S)/k_c(S=0)$ as a function of the relative rotational viscosity $S=\nu_r/\nu$. Lines give the results from linear stability analysis in first order, crosses represent higher order calculations and circles the results obtained from numerical simulation for each of the three differently oriented magnetic fields. Differences are very small. Depending on the orientation of the magnetic field, the critical wave number may be enlarged or reduced

seen from the analytical differential equations. One has to disturb the system thus simulating experimental noise or thermal fluctuations and it has been found that a normalized velocity disturbance of 10^{-11} at only one point in the w-velocity field is sufficient.

In determining critical wave numbers and Reynolds numbers by direct computer simulation one is faced with the critical slowing down near the marginal stability curve. This difficulty can be bypassed and the marginal curve is obtained from the time-dependent response of the flow to instantaneous changes in the Reynolds number R to supercritical and subcritical values. The critical numbers R_c and λ_c representing the minimum of this curve are $R_c=183.777$ and $\lambda_c=2.012$ for $\delta=0.0526$ and no applied magnetic field. The differences to the corresponding values obtained by linear stability analysis for ordinary fluids ($R_c=184.99$, $\lambda_c=2.0087$ [3]) are less than 1% and 0.2%, respectively, and thus are quite small. A second method using the square-root-dependence of the maximal radial velocity $u_{\max}$ on $\varepsilon'=\dfrac{R-R_c}{R_c}$ as found in lowest order of $\varepsilon'^{\frac{1}{2}}$ (see e.g. [3]) yields nearly identical values for the marginal curve. But since deviations from the lowest-order-($=\varepsilon'^{\frac{1}{2}}$)-predictions by higher orders in ($\varepsilon'^{\frac{1}{2}}$) show up even for ε' as small as 0.01 (see also [22]), the method reported first is used to determine the critical numbers and thus investigate the influence of the three magnetic fields on the formation of Taylor vortices.

The results obtained are given in reduced numbers by the fat markers in Fig. 1 and Fig. 2. There is an excellent agreement in the critical wave numbers $k_r(S)$ resulting from the two different calculations although both different boundary conditions and different methods of calculation have been used. As shown in Fig. 1, the agreement in reduced Taylor numbers is also very good for the case of an azimuthal magnetic field, but less good for the two other fields. The values for the radial field are reduced to the values indicated by arrows pointing down and those for the axial one are enlarged as shown by the arrows pointing up. The results for the axial magnetic field are in good agreement with the numerical findings of Berkovsky [6] represented by the dashed curve. He showed that for a homogeneous axial magnetic field the reduced critical Reynolds number can be approximated by $(1+0.725\cdot S)$ independent of the radius ratio of the cylinders. From his figures a decrease of the critical wave number with S may be seen too, but quantitative predictions are not given.

The quantitative changes in the reduced Taylor numbers T_r may be explained by considering the change of the reduced critical numbers $(T_{r,\text{no-slip}}-T_{r,\text{idealized}})/T_{r,\text{idealized}}$ as a function of the change in the reduced wavelength $\lambda_r(S)/\lambda_r(S=0)-1$ as is done in Fig. 3. If one, by applying an axial field, enlarges λ_c, one also enlarges the length scale on which a difference in the boundary conditions is present. Thus in the numerical simulation, where no-slip boundary

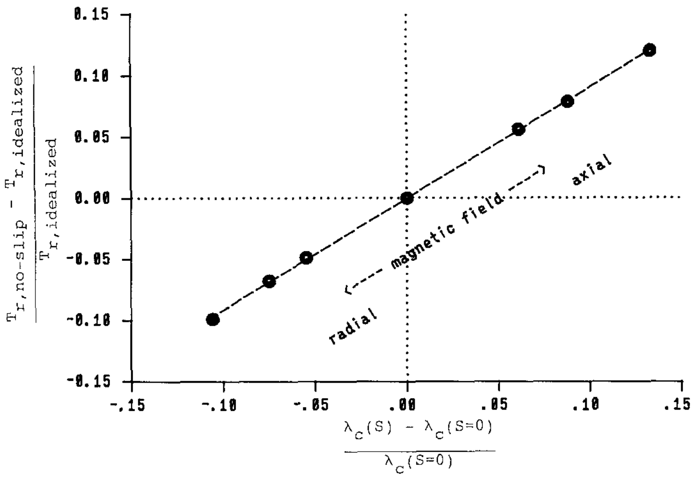

Fig. 3. Relative deviation of the reduced critical Taylor number $T_{r,\text{no-slip}}$, obtained by simulation, from the one obtained in linear stability analysis, as a function of the relative deviation of the wavelength $\lambda_c(S)$ from $\lambda_c(S=0)$. One can move along the line by applying axial and radial magnetic fields. A relationship is indicated clearly

conditions are imposed, additional axial forces arise for a single Taylor vortex pair. These forces obviously enlarge the critical Taylor numbers and consequently the already existing difference in Taylor numbers. Therefore, besides an increase caused by the axial magnetic field, one should expect an additional increase caused by the change in the critical wavelength. A similar argumentation holds in the case of the radial field resulting in a decrease of the reduced Taylor number. And this is exactly what is found and shown in Fig. 1.

IV. Parameters of the Amplitude Equation

Many features of hydrodynamic systems in the weakly nonlinear region just above a bifurcation can be described quantitatively by an amplitude equation of the Ginzburg-Landau form:

$$\sigma_0^{-1}\frac{\partial}{\partial t}A=\xi_0^2\frac{\partial^2}{\partial z^2}A+A(\varepsilon-|A|^2). \qquad (19)$$

The complex amplitude A is related to a stream function which contains the information on the velocity field. For the Taylor-Couette-system and nonmagnetic fluids the parameters σ_0 and ξ_0 representing the growth rate amplitude and the curvature of the marginal stability curve have been determined with high numerical accuracy by Dominguez-Lerma et al. [23]. Stimulated by their investigations, we explored the influence of magnetic fields on these values and on the wave number of maximal growth rate k_m if a magnetic fluid is placed inside a small gap of $\delta=0.0526$.

a) The curvature of the marginal stability curve is determined from a least-squares-parabola using data $(R_c(k), k)$, k deviating about 1% or less from the critical values k_c. Taking over the definitions $\xi_0^2=\frac{1}{2}\frac{\partial^2}{\partial k^2}\varepsilon_c\Big|_{k=k_c}$, ε_c defined as $\{T_c(k)-T_c(k_c)\}/T_c(k_c)$ [23], and scaling ξ_0 in units of d, the value obtained from the numerical data is 0.392 (± 0.005) for zero magnetic field strength. This is in reasonable agreement to 0.382 as reported by [23]. Within errors resulting from the numerical accuracy in determining R_c, only for the radial field a small influence could be observed resulting in a decrease from 0.395 to 0.347 as S changes to 1.

b) The wave number of maximal growth rate $k_m(\varepsilon)$ can be obtained as an extremum for a given ε from the Liapunov exponents $\sigma(\varepsilon, k)$ already calculated in Sect. III. For small ε the results are expressed in $\alpha=\lim_{\varepsilon\to 0}\frac{k_m(\varepsilon)-k_c}{\varepsilon k_c}$, following [23]. The limit $\varepsilon\to 0$ is approximated by finite differences and extrapolated to $\varepsilon=0$ by Richardson extrapolation. Without a magnetic field, the obtained value of 0.22 (± 0.04) agrees with 0.249 [23]. If a magnetic field is applied, this value changes to 0.35, 0.19 and 0.18 for axial, azimuthal and radial field, respectively. Although these changes seem to be quite large, one should be careful in interpretation: even such small errors as 0.05% in the determination of the wave number may result in relative errors as big as 30% in α because one has to evaluate the difference of two nearly identical numbers. Nevertheless, considering only the qualitative statements, it is interesting to observe that, if the critical wave number is reduced by the axial field, α increases and if k_c is increased by the radial field, α gets smaller.

c) The linear growth rate defined by $\sigma_0=\lim_{\varepsilon\to 0}\frac{\sigma(\varepsilon, k)}{\varepsilon}$ is easily obtained from $\sigma(\varepsilon, k)$ already used in Sect. III. Since $\sigma(\varepsilon, k)$ is found to depend linear on ε near the marginal curve, σ_0 can be approximated by finite differences $\sigma(\varepsilon_i, k)/\varepsilon_i$. The errors are very small and the value obtained numerically, 13.03, is in excellent agreement with the high precision result [23] which is $\sigma_0=13.09$ $[\nu/d^2]$ for ordinary fluids. The magnetic field dependence is shown in Fig. 4. For all three magnetic fields a strong increase is observed. This increase is not related to the increase in Taylor numbers in a simple way since then the axial values should be larger than those of the azimuthal field and in addition near to the values of the radial field. Therefore a simple picture of a somehow delayed instability which develops the faster the more it is delayed does not fit here. However, we want to mention here that

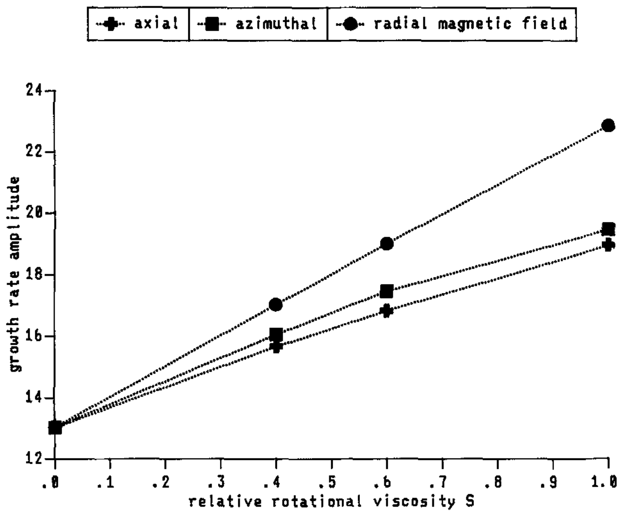

Fig. 4. Growth rate amplitude $\sigma_0[\nu/d^2]$ as a function of the relative rotational viscosity S. For all magnetic fields investigated σ_0 increases with S

the propagation velocity of a Taylor vortex front c_A is proportional to the product $\xi_0 \sigma_0$ [22]. Thus it is interesting to investigate c_A in magnetic fluids.

V. Conclusions

A crucial point in modelling the flow of a magnetic fluid is a macroscopic equation describing the deviation of the magnetization from its equilibrium value. Here we start from the well known and successfully used phenomenological relaxation equation of magnetization proposed by Shliomis [7]. Realizing that the time scale of interest in many flows is larger than 10^{-6} s, we directly derive an equation of magnetization for instationary flows. Combined with Cauchy's equation of motion and an equation for the intrinsic angular momentum, we arrive at a ferrohydrodynamic equation which describes the flow of an incompressible magnetic fluid in a nonuniform steady magnetic field.

This equation has been used to investigate the influence of magnetic fields on the formation of Taylor vortices of ferrofluids in the small gap between concentric rotating cylinders. Means of investigation have been linear stability analysis and direct computer simulation of the flow using finite differences. Summarizing the main results we find the following:

- all three investigated magnetic fields (axial, radial and azimuthal) have a stabilizing effect on the primary circular Couette flow.
- the critical wave number may be enlarged or reduced by applying a radial or an axial magnetic field, respectively. An azimuthal field does not change the wave number.
- the curvature of the marginal stability curve could be determined in accordance with the high precision numerical data of Dominguez-Lerma et al. [23]. A change of this value caused by a magnetic field could not be observed for axial and azimuthal ones; for radial fields there are some hints towards a small decrease.
- calculations of the growth rate amplitude σ_0 without a magnetic field are in excellent agreement (0.5%) with the data of Dominguez-Lerma et al. A magnetic field enlarges this value significantly.

One should stress here that a magnetic fluid offers the new possibility to influence the critical wave number by an external control parameter (magnetic field). This makes it favourable to use ferrofluids in studies of pattern selection mechanism.

The present work is intended as part of a Ph.D.-Thesis. In this context I wish to thank Prof. H. Müller-Krumbhaar and Prof. M. Lücke for suggesting this investigation and for their care.

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Note Added in Proof

After this work has been accepted for publication, we noticed a recent paper of Vislovich, A.N., et al.: J. Appl. Mech. Tech. Phys. (USA), **27**, 72 (1986). Starting from the model equations of ([8]), they obtain analogous results concerning the influence of magnetic fields on the critical Taylor number and wave number. Since they use Galerkin methods, this provides an additional evidence for the correctness of the effects presented