Periodically Forced Ferrofluid Pendulum: Effect of Polydispersity

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We investigate a torsional pendulum containing a ferrofluid that is forced periodically to undergo small-amplitude oscillations. A homogeneous magnetic field is applied perpendicular to the pendulum axis. We give an analytical formula for the ferrofluid-induced “selfenergy” in the pendulum’s dynamic response function for monodisperse as well as for polydisperse ferrofluids.

1. Introduction

Real ferrofluids [1] contain magnetic particles of different sizes [2]. This polydispersity strongly influences the macroscopic magnetic properties of the ferrofluid. We investigate here the effect of polydispersity on the dynamic response of a ferrofluid pendulum.

A torsional pendulum containing a ferrofluid is forced periodically in a homogeneous magnetic field $H_{\text{ext}} = H_{\text{ext}} \hat{e}$, that is applied perpendicular to the pendulum axis $\hat{e}$ (see Fig. 1). Such a ferrofluid pendulum is used for measuring the rotational viscosity [3]. The cylindrical ferrofluid container is here of sufficiently large length to be approximated as an infinite long cylinder. We consider rigid-body rotation of the ferrofluid with the time dependent angular velocity $\Omega = \dot{\phi} \hat{e}$, as can be realized with the set-up of [3]. The fields $H$ and $M$ inside the cylinder are spatially homogeneous and oscillating in time.

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Fig. 1. Schematic plot of the system.

2. Equations

First, the Maxwell equations demand that the fields $\mathbf{H}$ and $\mathbf{M}$ within the ferrofluid are related to each other via

$$\mathbf{H} + N \mathbf{M} = \mathbf{H}^{\text{ext}}$$  \hspace{1cm} (2.1)

with $N = 1/2$ for the infinitely long cylinder. Then we have the torque balance

$$\dot{\omega} = -\omega_0^2 \omega - \Gamma_0 \dot{\omega} - \frac{T}{\Theta} + f(t)$$  \hspace{1cm} (2.2)

with the eigenfrequency $\omega_0$ and the damping rate $\Gamma_0$ of the pendulum without field and the total moment of inertia $\Theta$. The magnetic torque reads

$$T = -\mu_0 \int \mathbf{d}V (\mathbf{M} \times \mathbf{H}) = -\mu_0 \mathbf{V}(\mathbf{M} \times \mathbf{H}^{\text{ext}}) \cdot \mathbf{z},$$  \hspace{1cm} (2.3)

and $f(t)$ is the external mechanical forcing.

Finally, we need an equation describing the magnetization dynamics. Here, we consider the polydisperse ferrofluid as a mixture of ideal monodisperse paramagnetic fluids. Then the resulting magnetization is given by $\mathbf{M} = \sum \mathbf{M}_j$, where $\mathbf{M}_i$ denotes the magnetization of the particles with diameter $d_j$. We assume that each $\mathbf{M}_j$ obeys a simple Debye relaxation dynamics described by

$$d_j \mathbf{M}_j - \mathbf{\Omega} \times \mathbf{M}_j = -\frac{1}{\tau_j} [\mathbf{M}_j - \mathbf{M}_j^{\text{eq}}(\mathbf{H})].$$  \hspace{1cm} (2.4)

We take the equilibrium magnetization to be given by a Langevin function

$$\mathbf{M}_j^{\text{eq}}(\mathbf{H}) = \chi_j(H) \mathbf{H} = w_j L \left( \frac{\mu_0 \pi M_{\text{sat}} d_j}{6k_B T} \right) \frac{\mathbf{H}}{H} \mathbf{H}.$$  \hspace{1cm} (2.5)
with the saturation magnetization of the material $M_{sat}$ and the magnetization distribution $w_j(d_j)$. Note that the magnetization Eq. (2.4) for the different particle sizes are coupled by the internal field $H = H^{ext} - NM$. As relaxation rate we combine Brownian and Néel relaxation $	au = \frac{1}{\tau_B} + \frac{1}{\tau_N}$. The relaxation times depend on the particle size by $\tau_B = \frac{(d_j + 2s)^3}{\eta}$ and $\tau_N = f_0^{-1} \exp \left( \frac{\varepsilon_{He} d_j}{2k_B T} \right)$ with $\eta$ the viscosity, $s$ the thickness of the nonmagnetic particle layer, and $K$ the anisotropy constant.

Altogether we use the following system of equations:

$$
\dot{\varphi} = \Omega \quad (2.6)
$$

$$
\dot{\Omega} = -\omega^2 \varphi - \Gamma_0 \Omega - \mu_0 \frac{1}{\Theta} H^{ext} M_x + f(t) \quad (2.7)
$$

$$
\dot{M}_i^x = -\Omega M_i^y - \frac{1}{\tau_j} \left[ M_i^x - \chi_j(H)(H^{ext} - N M_i) \right] \quad (2.8)
$$

$$
\dot{M}_i^y = \Omega M_i^x - \frac{1}{\tau_j} M_i^y - \frac{1}{\tau_j} N \chi_j(H) M_i \quad (2.9)
$$

### 3. Linear response analysis

For the equilibrium situation of the unforced pendulum at rest that we denote in the following by an index 0 one has $\varphi_0 = \Omega_0 = M_{i0}^y = 0$ and $M_j^y = M_j^y(H_0)$. Furthermore, $M_0 = \sum M_j^y(H_0)$ with $H_0$ solving the equation $H_0 = H^{ext} - N M_0(H_0)$.

External forcing with small $|f|$ leads to small deviations of $\varphi$, of $\Omega$, and of $\delta H = H - H_0 = -N(M - M_0) = -N \delta M$ from the above described equilibrium state. We expand each $\chi_j(H)$ up to linear order in $\delta H$:

$$
\chi_j(H_0 + \delta H) = \chi_j(H_0) + \chi_j'(H_0) \delta M + \Theta (\delta H)^2 \quad (3.1)
$$

Here, $\chi_j(H_0) = \chi_j(H_0)$ and $\chi_j'$ is the derivative of $\chi_j(H_0)$. Then we get the linearized equations

$$
\dot{\varphi} = \Omega \quad (3.2)
$$

$$
\dot{\Omega} = -\omega^2 \varphi - \Gamma_0 \Omega - \kappa y + f(t) \quad (3.3)
$$

$$
\dot{x}_j = -\frac{1}{\tau_j} x_j - \frac{1}{\tau_j} N (\chi_j + \chi_j' H_0) x \quad (3.4)
$$

$$
\dot{y}_j = \Omega x_j^0 - \frac{1}{\tau_j} y_j - \frac{1}{\tau_j} N \chi_{j0} y \quad (3.5)
$$

We have introduced the abbreviations $x_j = \delta M_j / M_0$, $x_j^0 = M_j^{y0} / M_0$, $y_j = \delta M_j / M_0$ and $x = \sum x_j$, $y = \sum y_j$. The strength of the coupling constant...
between the mechanical degrees of freedom $\varphi$, $\Omega$ and the magnetic ones is

$$\kappa = \mu_0 H^m M_0 V/\Theta .$$

For periodic forcing $f(t) = \hat{f} e^{-i\omega t}$ we look for solutions in the form

$$\begin{pmatrix} \varphi(t) \\ \Omega(t) \\ x_j(t) \\ y_j(t) \end{pmatrix} = \begin{pmatrix} \hat{\varphi} \\ \hat{\Omega} \\ \hat{x}_j \\ \hat{y}_j \end{pmatrix} e^{-i\omega t} .$$

Inserting the ansatz (3.6) into the linearized Eqs. (3.2)–(3.5) yields

$$\hat{\Omega} = -i\omega \hat{\varphi}$$

$$\hat{x} = 0 = \hat{x}_j$$

$$\hat{z}_j = -\left[ \frac{i\omega \tau_j}{1 - i\omega \tau_j} x_j^0 - \frac{N \chi_{\omega} \omega}{1 - i\omega \tau_j \kappa} \right] \hat{\varphi}$$

$$\hat{\gamma} = -\frac{\omega}{\kappa} \Sigma \hat{\varphi}$$

and

$$\hat{\varphi} = G \hat{f} = \left[ \omega_0^2 - \omega^2 - i\omega \Gamma_0 - \omega \Sigma \right]^{-1} \hat{f} .$$

The ferrofluid-induced ‘selfenergy’ $\Sigma(\omega)$ in the expression for the dynamical response function $G(\omega)$ of the torsional pendulum is

$$\Sigma(\omega) = i\kappa \left( 1 + N \sum_j \frac{\chi_{\omega}}{1 - i\omega \tau_j} \right)^{-1} \sum_j \frac{\tau_j x_j^0}{1 - i\omega \tau_j} .$$

Its imaginary part changes the damping rate $\Gamma_0$ of the pendulum for $\kappa = 0$, i.e., in zero field. The real part shifts the resonance frequency of the pendulum. In the special case of a monodisperse ferrofluid one has

$$\Sigma(\omega) = \frac{i\kappa \tau}{1 - i\omega \tau + N \chi_{\omega}} .$$

4. Results

We evaluated the linear response function $G(\omega) = \hat{\varphi}(\omega)/\hat{f}$ of the pendulum’s angular deviation amplitude $\hat{\varphi}(\omega)$ to the applied forcing amplitude $\hat{f}$ and the selfenergy $\Sigma(\omega)$ for some experimental parameters from [3]: $\omega_0/2\pi = 32.7$ Hz, $\Gamma_0 = 0.178$ Hz, $V/\Theta = 20$ m/kg. The cylinder is filled with the ferrofluid APG 933 of FERROTEC. Therefore, we used in Eq. (3.13) an experimental $\tau = 0.6$ ms and the experimental $M_0(H)$ shown in Fig. 2. These
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Fig. 2. ×: Experimental equilibrium magnetization $M_{eq}(H)$ used as input for the monodisperse calculations; full line: fit with lognormal contribution.

Fig. 3. Lognormal contribution $w(d_i)$ ($d_i = 1$ nm ... $d_{30} = 30$ nm) used as input for the polydisperse calculations.

Monodisperse results were compared with the expression (3.12) for the polydisperse case for the typical parameter values $M_{max} = 456$ kA/m, $\eta = 0.5$ Pa s, $s = 2$ nm, $K = 44$ kJ/m$^3$ and $f_0 = 10^9$ Hz. The contributions $w(d_i)$ that enter into the formulas (2.5) for the susceptibilities $\chi_i$ are given by a lognormal
Fig. 4. $|G|$ near the resonance $\omega_0$: $x$: $H^{\text{ext}} = 0$ kA/m, squares: $H^{\text{ext}} = 5$ kA/m, circles: $H^{\text{ext}} = 10$ kA/m; filled symbols: polydisperse.

Fig. 5. Maximum value $\max |G|$ (a) and peak position $\omega^{\text{max}}$ (b) as a function of external field $H^{\text{ext}}$; full line: monodisperse, dashed line: polydisperse.
distribution [2]:
\[ w(d_j) = M_{\text{sat}} \frac{g(d_j) d_j}{\sum_{k=1}^{\infty} g(d_k) d_k} \]
with
\[ g(d_j) = \frac{1}{\sqrt{4\pi d_j \ln \sigma}} \exp \left( \frac{-\ln^2 (d_j / d_0)}{2 \ln^2 \sigma} \right). \] (4.1)

Fitting the experimental \( M_{\text{sat}}(H) \) with a sum of Langevin functions (2.5) yields \( M_{\text{sat}} = 18149 \) A/m, \( d_0 = 7 \) nm and \( \sigma = 1.47 \) (see Fig. 2). We used here 30 different particle sizes from \( d_1 = 1 \) nm to \( d_{30} = 30 \) nm (see Fig. 3).

The calculations show the additional damping rate caused by the interaction between ferrofluid and external field. An increasing magnetic field leads to smaller amplitudes; in polydisperse ferrofluids the amplitude decreases faster [Figs. 4 and 5(a)]. Furthermore, one can see a shift of the peak position to higher frequencies \( \omega_{\text{max}} \), which is stronger in polydisperse ferrofluids [Figs. 4, 5(b) and 6(a)].
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References