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Periodically Forced Ferrofluid Pendulum: Effect of Polydispersity

By A. Leschhorn* and M. Lücke

Institut für Theoretische Physik, Universität des Saarlandes, D-66041 Saarbrücken, Germany

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Ferrofluid / Torsional Pendulum / Response Function

We investigate a torsional pendulum containing a ferrofluid that is forced periodically to undergo small-amplitude oscillations. A homogeneous magnetic field is applied perpendicular to the pendulum axis. We give an analytical formula for the ferrofluid-induced "selfenergy" in the pendulum's dynamic response function for monodisperse as well as for polydisperse ferrofluids.

1. Introduction

Real ferrofluids [1] contain magnetic particles of different sizes [2]. This polydispersity strongly influences the macroscopic magnetic properties of the ferrofluid. We investigate here the effect of polydispersity on the dynamic response of a ferrofluid pendulum.

A torsional pendulum containing a ferrofluid is forced periodically in a homogeneous magnetic field $\mathbf{H}_{\text{ext}} = H_{\text{ext}}\mathbf{e}_x$ that is applied perpendicular to the pendulum axis $\mathbf{e}_z$ (see Fig. 1). Such a ferrofluid pendulum is used for measuring the rotational viscosity [3]. The cylindrical ferrofluid container is here of sufficiently large length to be approximated as an infinite long cylinder. We consider rigid-body rotation of the ferrofluid with the time dependent angular velocity $\mathbf{\Omega} = \dot{\varphi}\mathbf{e}_z$ as can be realized with the set-up of [3]. The fields $\mathbf{H}$ and $\mathbf{M}$ inside the cylinder are spatially homogeneous and oscillating in time.

* Corresponding author. E-mail: andy@lusi.uni-sb.de

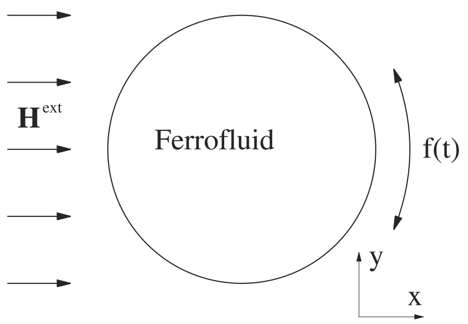

Fig. 1. Schematic plot of the system.

2. Equations

First, the Maxwell equations demand that the fields $\mathbf{H}$ and $\mathbf{M}$ within the ferrofluid are related to each other via

$$\mathbf{H} + N\mathbf{M} = \mathbf{H}^{\text{ext}} \tag{2.1}$$

with $N = 1/2$ for the infinitely long cylinder. Then we have the torque balance

$$\ddot{\varphi} = -\omega_0^2\varphi - \Gamma_0\dot{\varphi} - \frac{T}{\Theta} + f(t) \tag{2.2}$$

with the eigenfrequency ω_0 and the damping rate Γ_0 of the pendulum without field and the total moment of inertia Θ. The magnetic torque reads

$$T = -\mu_0 \int \mathrm{d}V(\mathbf{M} \times \mathbf{H})_z = -\mu_0 V(\mathbf{M} \times \mathbf{H}^{\text{ext}})_z\,, \tag{2.3}$$

and $f(t)$ is the external mechanical forcing.

Finally, we need an equation describing the magnetization dynamics. Here, we consider the polydisperse ferrofluid as a mixture of ideal monodisperse paramagnetic fluids. Then the resulting magnetization is given by $\mathbf{M} = \sum \mathbf{M}_j$, where $\mathbf{M}_j$ denotes the magnetization of the particles with diameter d_j. We assume that each $\mathbf{M}_j$ obeys a simple Debye relaxation dynamics described by

$$d_t\mathbf{M}_j - \boldsymbol{\Omega} \times \mathbf{M}_j = -\frac{1}{\tau_j}\left[\mathbf{M}_j - \mathbf{M}_j^{\text{eq}}(\mathbf{H})\right]. \tag{2.4}$$

We take the equilibrium magnetization to be given by a Langevin function

$$\mathbf{M}_j^{\text{eq}}(\mathbf{H}) = \chi_j(H)\mathbf{H} = w_j \mathcal{L}\left(\frac{\mu_0\pi M_{\text{mat}}}{6k_{\text{B}}T} d_j^3 H\right)\frac{\mathbf{H}}{H} \tag{2.5}$$

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with the saturation magnetization of the material M_{mat} and the magnetization distribution $w_j(d_j)$. Note that the magnetization Eq. (2.4) for the different particle sizes are coupled by the internal field $\mathbf{H} = \mathbf{H}^{\text{ext}} - N\mathbf{M}$. As relaxation rate we combine Brownian and Néel relaxation $\frac{1}{\tau_j} = \frac{1}{\tau_{\text{B}}^j} + \frac{1}{\tau_{\text{N}}^j}$. The relaxation times depend on the particle size by $\tau_{\text{B}}^j = \frac{\pi\eta}{2k_{\text{B}}T}(d_j + 2s)^3$ and $\tau_{\text{N}}^j = f_0^{-1}\exp\left(\frac{\pi K d_j^3}{6k_{\text{B}}T}\right)$ with η the viscosity, s the thickness of the nonmagnetic particle layer, and K the anisotropy constant.

Altogether we use the following system of equations:

$$\dot{\varphi} = \Omega \tag{2.6}$$

$$\dot{\Omega} = -\omega_0^2\varphi - \Gamma_0\Omega - \mu_0\frac{V}{\Theta}H^{\text{ext}}M_y + f(t) \tag{2.7}$$

$$\dot{M}_x^j = -\Omega M_y^j - \frac{1}{\tau_j}\left[M_x^j - \chi_j(H)\big(H^{\text{ext}} - NM_x\big)\right] \tag{2.8}$$

$$\dot{M}_y^j = \Omega M_x^j - \frac{1}{\tau_j}M_y^j - \frac{1}{\tau_j}N\chi_j(H)M_y\,. \tag{2.9}$$

3. Linear response analysis

For the equilibrium situation of the unforced pendulum at rest that we denote in the following by an index 0 one has $\varphi_0 = \Omega_0 = M_y^{j0} = 0$ and $M_x^{j0} = M_{\text{eq}}^j(H_0)$. Furthermore, $M_0 = \sum M_{\text{eq}}^j(H_0)$ with H_0 solving the equation $H_0 = H^{\text{ext}} - NM_0(H_0)$.

External forcing with small $|f|$ leads to small deviations of φ, of Ω, and of $\delta\mathbf{H} = \mathbf{H} - \mathbf{H}_0 = -N(\mathbf{M} - \mathbf{M}_0) = -N\delta\mathbf{M}$ from the above described equilibrium state. We expand each $\chi_j(H)$ up to linear order in $\delta\mathbf{H}$

$$\chi_j(|\mathbf{H}_0 + \delta\mathbf{H}|) = \chi_{j0} - \chi'_{j0}N\delta M_x + \mathcal{O}(\delta\mathbf{H})^2\,. \tag{3.1}$$

Here, $\chi_{j0} = \chi_j(H_0)$ and χ'_{j0} is the derivative of χ_{j0}. Then we get the linearized equations

$$\dot{\varphi} = \Omega \tag{3.2}$$

$$\dot{\Omega} = -\omega_0^2\varphi - \Gamma_0\Omega - \kappa y + f(t) \tag{3.3}$$

$$\dot{x}_j = -\frac{1}{\tau_j}x_j - \frac{1}{\tau_j}N\big(\chi_{j0} + \chi'_{j0}H_0\big)x \tag{3.4}$$

$$\dot{y}_j = \Omega x_j^0 - \frac{1}{\tau_j}y_j - \frac{1}{\tau_j}N\chi_{j0}y\,. \tag{3.5}$$

We have introduced the abbreviations $x_j = \delta M_x/M_0$, $x_j^0 = M_x^{j,0}/M_0$, $y_j = \delta M_y/M_0$ and $x = \sum_j x_j$, $y = \sum_j y_j$. The strength of the coupling constant

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between the mechanical degrees of freedom φ, Ω and the magnetic ones is $\kappa = \mu_0 H^{\text{ext}} M_0 V/\Theta$.

For periodic forcing $f(t) = \hat{f} e^{-i\omega t}$ we look for solutions in the form

$$\begin{pmatrix} \varphi(t) \\ \Omega(t) \\ x_j(t) \\ y_j(t) \end{pmatrix} = \begin{pmatrix} \hat{\varphi} \\ \hat{\Omega} \\ \hat{x}_j \\ \hat{y}_j \end{pmatrix} e^{-i\omega t} . \tag{3.6}$$

Inserting the ansatz (3.6) into the linearized Eqs. (3.2)–(3.5) yields

$$\hat{\Omega} = -i\omega\hat{\varphi} \tag{3.7}$$

$$\hat{x} = 0 = \hat{x}_j \tag{3.8}$$

$$\hat{y}_j = -\left[\frac{i\omega\tau_j}{1-i\omega\tau_j} x_j^0 - \frac{N\chi_{j0}}{1-i\omega\tau_j}\frac{\omega}{\kappa}\Sigma\right]\hat{\varphi} \tag{3.9}$$

$$\hat{y} = -\frac{\omega}{\kappa}\Sigma\hat{\varphi} \tag{3.10}$$

and

$$\hat{\varphi} = G\hat{f} = \left[\omega_0^2 - \omega^2 - i\omega\Gamma_0 - \omega\Sigma\right]^{-1}\hat{f} . \tag{3.11}$$

The ferrofluid-induced 'selfenergy' $\Sigma(\omega)$ in the expression for the dynamical response function $G(\omega)$ of the torsional pendulum is

$$\Sigma(\omega) = i\kappa\left(1 + N\sum_j \frac{\chi_{j0}}{1-i\omega\tau_j}\right)^{-1} \sum_j \frac{\tau_j x_j^0}{1-i\omega\tau_j} . \tag{3.12}$$

Its imaginary part changes the damping rate Γ_0 of the pendulum for $\kappa = 0$, *i.e.*, in zero field. The real part shifts the resonance frequency of the pendulum. In the special case of a monodisperse ferrofluid one has

$$\Sigma(\omega) = \frac{i\kappa\tau}{1 - i\omega\tau + N\chi_0} . \tag{3.13}$$

4. Results

We evaluated the linear response function $G(\omega) = \hat{\varphi}(\omega)/\hat{f}$ of the pendulum's angular deviation amplitude $\hat{\varphi}(\omega)$ to the applied forcing amplitude $\hat{f}$ and the selfenergy $\Sigma(\omega)$ for some experimental parameters from [3]: $\omega_0/2\pi = 32.7$ Hz, $\Gamma_0 = 0.178$ Hz, $V/\Theta = 20$ m/kg. The cylinder is filled with the ferrofluid APG 933 of FERROTEC. Therefore, we used in Eq. (3.13) an experimental $\tau = 0.6$ ms and the experimental $M_{\text{eq}}(H)$ shown in Fig. 2. These

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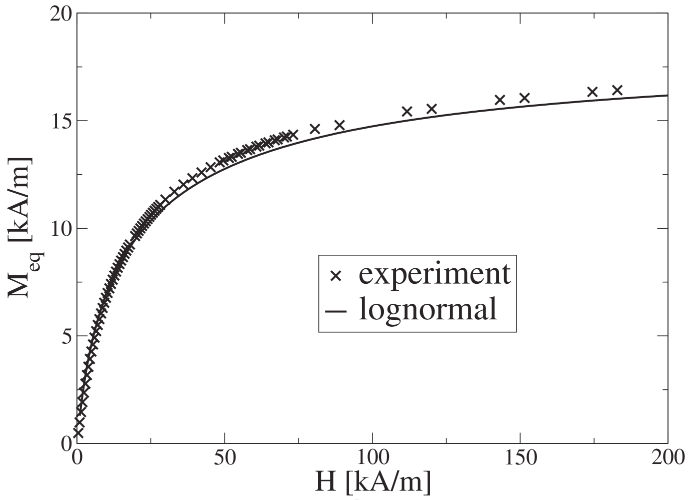

Fig. 2. ×: Experimental equilibrium magnetization $M_{eq}(H)$ used as input for the monodisperse calculations; full line: fit with lognormal contribution.

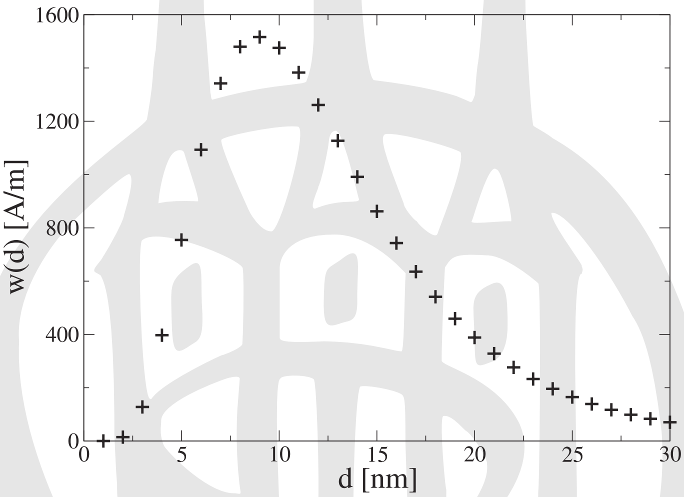

Fig. 3. lognormal contribution $w(d_i)$ ($d_1 = 1$ nm … $d_{30} = 30$ nm) used as input for the polydisperse calculations.

monodisperse results were compared with the expression (3.12) for the polydisperse case for the typical parameter values $M_{mat} = 456\,kA/m$, $\eta = 0.5$ Pa s, $s = 2$ nm, $K = 44\,kJ/m^3$ and $f_0 = 10^9$ Hz. The contributions $w(d_j)$ that enter into the formulas (2.5) for the susceptibilities χ_j are given by a lognormal

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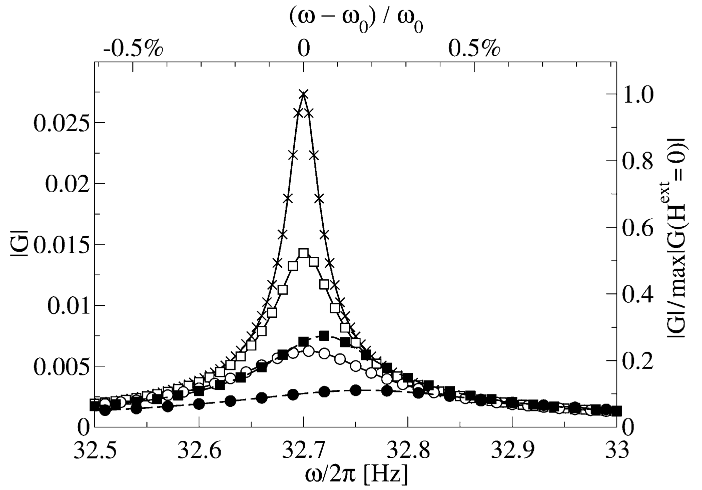

Fig. 4. $|G|$ near the resonance ω_0; $\times$: $H^{\mathrm{ext}} = 0\,\mathrm{kA/m}$, squares: $H^{\mathrm{ext}} = 5\,\mathrm{kA/m}$, circles: $H^{\mathrm{ext}} = 10\,\mathrm{kA/m}$; filled symbols: polydisperse.

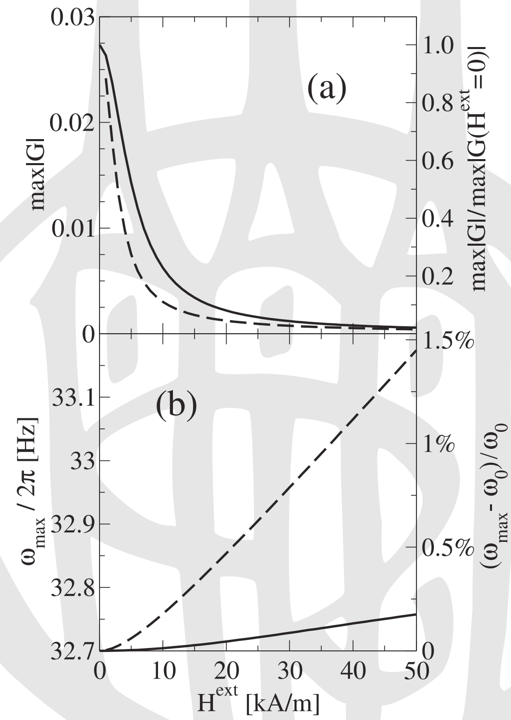

Fig. 5. Maximum value $\max|G|$ (a) and peak position $\omega_{\max}$ (b) as a function of external field H^{ext}; full line: monodisperse, dashed line: polydisperse.

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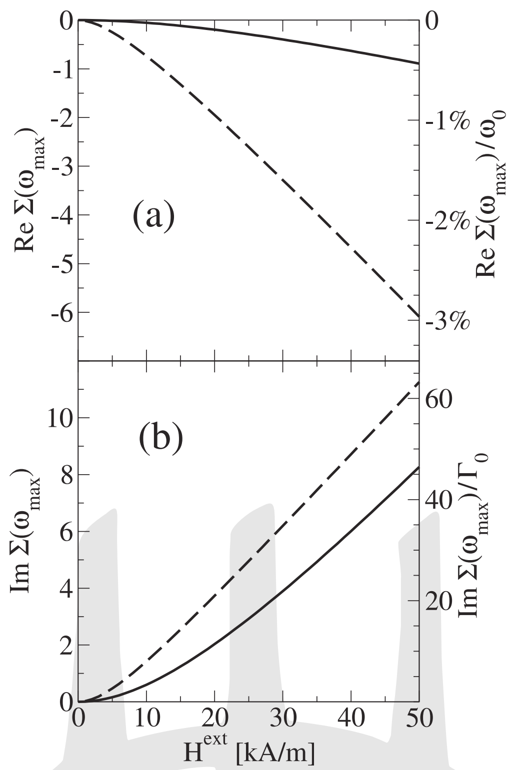

Fig. 6. Re(Σ) (a) and Im(Σ) (b) at $\omega = \omega_{max}$; full line: monodisperse, dashed line: polydisperse.

distribution [2]:

$$w(d_j) = M_{sat} \frac{g(d_j)d_j}{\sum_{k=1}^{30} g(d_k)d_k}$$

$$\text{with} \quad g(d_j) = \frac{1}{\sqrt{2\pi}d_j \ln\sigma} \exp\left(-\frac{\ln^2(d_j/d_0)}{2\ln^2\sigma}\right). \tag{4.1}$$

Fitting the experimental $M_{eq}(H)$ with a sum of Langevin functions (2.5) yields $M_{sat} = 18\,149$ A/m, $d_0 = 7$ nm and $\sigma = 1.47$ (see Fig. 2). We used here 30 different particle sizes from $d_1 = 1$ nm to $d_{30} = 30$ nm (see Fig. 3).

The calculations show the additional damping rate caused by the interaction between ferrofluid and external field. An increasing magnetic field leads to smaller amplitudes; in polydisperse ferrofluids the amplitude decreases faster [Figs. 4 and 5(a)]. Furthermore, one can see a shift of the peak position to higher frequencies ω_{max}, which is stronger in polydisperse ferrofluids [Figs. 4, 5(b) and 6(a)].

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