Flatness-Based Control: Kinematic Car

Tutorial 4: explicit path parametrization and open-loop controller

D. Kastelan
d.kastelan@lsr.uni-saarland.de
Chair of Systems Theory and Control Engineering
Saarland University

The kinematic car equations discussed up until now are singular in the zero velocity case $v = 0$:

$$v = \sqrt{\dot{y}_1^2 + \dot{y}_2^2} \tag{1a}$$
$$\frac{\tan\varphi}{l} = \frac{\ddot{y}_2\dot{y}_1 - \ddot{y}_1\dot{y}_2}{v^3} \tag{1b}$$
$$\dot{y}_1 = v\cos\theta \tag{1c}$$
$$\dot{y}_2 = v\sin\theta \tag{1d}$$

However, as discussed in §3.3.6 (Car: setpoint transitions) of the SR3 lecture notes, this problem may be treated by considering an explicit parametrization $y_2 = f(y_1)$, such that (1b) may be expressed as

$$\frac{\tan\varphi}{l} = \frac{\frac{\mathrm{d}^2 f}{\mathrm{d}y_1^2}}{\left(1 + \left(\frac{\mathrm{d}f}{\mathrm{d}y_1}\right)^2\right)^{3/2}}, \tag{3.39}$$

which remains well-defined for $v = 0$. An open-loop controller based on this parametrization is to be implemented in this tutorial that takes the car from rest at $(y_{1,A}, y_{2,A})$ with orientation θ_A to rest at $(y_{1,B}, y_{2,B})$ with orientation θ_B in time t_*.

Tasks

1. Like the discussion on the system's invariance to being written in terms of the arc length s, show that the kinematic car dynamic equations (1) are invariant to the transformation

$$\theta \mapsto \theta - \theta_A =: \tilde{\theta} \tag{2a}$$
$$y := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \mapsto \underbrace{\begin{pmatrix} \cos\theta_A & \sin\theta_A \\ -\sin\theta_A & \cos\theta_A \end{pmatrix}}_{R(\theta_A)} \begin{pmatrix} y_1 - y_{1,A} \\ y_2 - y_{2,A} \end{pmatrix} =: \tilde{y} \tag{2b}$$

for some initial planned position $y_A = (y_{1,A}, y_{2,A})^\top \in \mathbb{R}^2$ and orientation $\theta_A \in \mathbb{R}$. As a result, a trajectory may be planned in new coordinates $\tilde{\theta}$ and $\tilde{y} = (\tilde{y}_1, \tilde{y}_2)^\top$ with initial conditions $\tilde{\theta}_A = 0$ and $\tilde{y}_A = (0,0)^\top$.

Solution: Since $\dot{y}_A = (\dot{y}_{1,A}, \dot{y}_{2,A})^\top = (0,0)^\top$ and $\dot{R}(\theta_A) = 0$, transformation (2) yields

$$v = \sqrt{\dot{\tilde{y}}_1^2 + \dot{\tilde{y}}_2^2} \tag{3a}$$
$$\frac{\tan\varphi}{l} = \frac{\ddot{\tilde{y}}_2\dot{\tilde{y}}_1 - \ddot{\tilde{y}}_1\dot{\tilde{y}}_2}{v^3} \tag{3b}$$
$$\dot{\tilde{y}}_1 = v\cos\tilde{\theta} \tag{3c}$$
$$\dot{\tilde{y}}_2 = v\sin\tilde{\theta}, \tag{3d}$$

which is identical to (1), but in the new variables $(\tilde{y}_1, \tilde{y}_2, \tilde{\theta})$. This result demonstrates the system's invariance to the transformation (2) and should not be surprising as the kinematic equations should not depend on the location and orientation of the reference frame (so long as it is static).

2. Write the inverse transformation $(\tilde{y}, \tilde{\theta}) \mapsto (y, \theta)$ for (2).

   Solution: Since $R(\theta_A)$ is a rotation matrix (i.e., $\det R(\theta_A) = +1$), $R^{-1}(\theta_A) = R^\top(\theta_A)$ and the inverse of (2) with $y_A = (y_{1,A}, y_{2,A})^\top$ is

$$\begin{aligned} \theta &= \tilde{\theta} + \theta_A \\ y &= R^\top(\theta_A)\tilde{y} + y_A. \end{aligned}$$

3. Implement a polynomial trajectory generator $x \mapsto p_z(x) = \sum_{i=0}^{n_z} a_{z,i} x^i$ to realize rest-to-rest transitions of the car given boundary conditions $p_z(x), \frac{\mathrm{d}}{\mathrm{d}x} p_z(x), \ldots, \frac{\mathrm{d}}{\mathrm{d}x}^{(n_z-1)/2} p_z(x)$ at $x = 0$ and $x = 1$. To compute (along with their appropriate derivatives) are references

$$t \mapsto p_{\tilde{y}_{1,r}}\left(\frac{t}{t_*}\right) =: \tilde{y}_{1,r} \quad \text{with } n_{\tilde{y}_1} = 3,$$

   and

$$\tilde{y}_{1,r} \mapsto p_{\tilde{y}_2}\left(\frac{\tilde{y}_{1,r}}{\tilde{y}_{1,B}}\right) =: \tilde{y}_{r,2} \quad \text{with } n_{\tilde{y}_2} = 5.$$

   Use either the approach in §3.3.6 or modify the existing polynomial reference generator `PolyRef.m` for this task. Note that $p_{\tilde{y}_{1,r}}(0) = p_{\tilde{y}_{2,r}}(0) = 0$ corresponding to $\tilde{y}_A = (0, 0)^\top$ from Task 2 simplify calculations.

   Solution: See provided file `KinematicCarReference.m` that implements the polynomial coefficient calculations from §3.3.6.

4. Implement an open-loop controller for the car by computing (v, φ) using (1a) and (3.39) in terms of the reference trajectories for $\tilde{y}_{1,r}$ and $\tilde{y}_{2,r}$. Compute the corresponding reference variables $(y_{1,r}, y_{2,r}, \theta_r)$ in the system's original coordinates using the inverse transformation from Task 3. For this task, use the `KinematicCarReference.m` from Tutorial 2 or 3 as a template.

   Solution: From $\tilde{y}_{2,r} = f(\tilde{y}_{1,r})$, $\dot{\tilde{y}}_{2,r} = \dot{\tilde{y}}_{1,r} \frac{\mathrm{d}f}{\mathrm{d}\tilde{y}_{1,r}} \Rightarrow \frac{\mathrm{d}f}{\mathrm{d}\tilde{y}_{1,r}} = \frac{\dot{\tilde{y}}_{2,r}}{\dot{\tilde{y}}_{1,r}}$, such that from (3a)

$$v_r = \dot{\tilde{y}}_{1,r}\sqrt{1 + \left(\frac{\dot{\tilde{y}}_{2,r}}{\dot{\tilde{y}}_{1,r}}\right)^2} = \dot{\tilde{y}}_{1,r}\sqrt{1 + \left(\frac{\mathrm{d}f}{\mathrm{d}\tilde{y}_{1,r}}\right)^2}$$

   and from (3c)–(3d)

$$\tilde{\theta}_r = \arctan\frac{\dot{\tilde{y}}_{2,r}}{\dot{\tilde{y}}_{1,r}} = \arctan\frac{\mathrm{d}f}{\mathrm{d}\tilde{y}_{1,r}}$$

   The reference steering angle φ_r follows directly from replacing y_1 with $\tilde{y}_{1,r}$ in (3.39). These expressions along with those for $\frac{\mathrm{d}f}{\mathrm{d}\tilde{y}_{1,r}}$ and $\frac{\mathrm{d}^2 f}{\mathrm{d}\tilde{y}_{1,r}^2}$ and the reference system variables $(y_{1,r}, y_{2,r}, \theta_r)$ using the inverse transformation from Task 3 are calculated in the included file `KinematicCarReference.m`.

5. Simulate your results using `KinematicCarSim.m` from Tutorial 2 as a guide to show that you may navigate the car from rest for arbitrary $(y_{1,A}, y_{2,A}, \theta_A)$ to a chosen $(y_{1,B}, y_{2,B}, \theta_B)$.

   Solution: See provided file `KinematicCarSim.m`

6. What restrictions are there on the final configuration $(y_{1,B}, y_{2,B}, \theta_B)$ given the parametrization $y_2 = f(y_1)$?

   Solution: For $f(\tilde{y}_1)$ to be a smooth single-valued function, the derivative $\frac{\mathrm{d}f}{\mathrm{d}\tilde{y}_1}$ must be finite. As a result, $|\tilde{\theta}| < \frac{\pi}{2}$ and thus $|\theta_B - \theta_A| < \frac{\pi}{2}$. Moreover, then $\tilde{y}_{1,B} \neq \tilde{y}_{1,A} = 0$ such that $y_B \neq R^\top(\theta_A)(0, \tilde{y}_{2,B})^\top + y_A$.