A Limit Theorem Clarifying the Physical Origin of Fractional Brownian Motion and Related Gaussian Models of Anomalous Diffusion

Christian Bender*, Yana A. Butko†, Mirko D'Ovidio‡, Gianni Pagnini§

December 3, 2024

Abstract

We consider a dynamical system describing the motion of a test-particle surrounded by N Brownian particles with different masses. Physical principles of conservation of momentum and energy are met. We prove that, in the limit $N \to \infty$, the test-particle diffuses in time according to a quite general (non-Markovian) Gaussian process whose covariance function is determined by the distribution of the masses of the surround-particles. In particular, with proper choices of the distribution of the masses of the surround-particles, we obtain fractional Brownian motion, a mixture of independent fractional Brownian motions with different Hurst parameters and the classical Wiener process. Moreover, we present some distributions of masses of the surround-particles leading to limiting processes which perform transition from ballistic to superdiffusion or from ballistic to classical diffusion.

Keywords: fractional Brownian motion, limit theorems for stochastic processes, anomalous diffusion, heterogeneous environment, crowded environment

1 Introduction

Experimentally well-established [14, 1, 10, 6], anomalous diffusion (AD) is a phenomenon observed in many different natural systems belonging to different research fields [22, 15, 13]. In particular, AD has become foundational in living systems after a large use of single-particle tracking techniques in the recent years [20, 27, 28]. Generally speaking, AD labels all those diffusive processes that are governed by laws that differ from that of classical diffusion, namely, all those cases

*Saarland University, Department of Mathematics, Campus E2 4, 66123 Saarbrücken, Germany, bender@math.uni-saarland.de

†Kassel University, Institute of Mathematics, Heinrich-Plett-Str. 40, 34132 Kassel, Germany, kinderknecht@mathematik.uni-kassel.de

‡Department of Basic and Applied Sciences for Engineering, Sapienza University of Rome, Rome, Italy, mirko.dovidio@uniroma1.it

§BCAM–Basque Center for Applied Mathematics, Alameda de Mazarredo 14, 48009 Bilbao, Basque Country, Spain & Ikerbasque–Basque Foundation for Science, Plaza Euskadi 5, 48009 Bilbao, Basque Country, Spain, gpagnini@bcamath.org

when particles' displacements do not accommodate to the Gaussian density function and/or the variance of such displacements does not grow linearly in time.

In the present paper, we propose an attempt for establishing the physical origin of AD within the picture of a test-particle kicked by infinitely many heterogeneous surrounding particles. We consider a stochastic dynamical system where the microscopic thermal bath is the forcing for the mesoscopic Brownian motion of a bunch of N particles that embody the environment of a single test-particle. Physical conservation principles, namely the conservation of momentum and the conservation of energy, are met in the considered particle-system in the form of a coupling between the test-particle and the surround and the fluctuation-dissipation theorem for the motion of the surround-particles [7], respectively. The key feature of the considered particle-system is the distribution of the masses of the particles that compose the surround of the test-particle. When the number of mesoscopic Brownian particles N is large enough for providing a crowded environment, then the test-particle displays a particular AD motion specifically characterised by the distribution of the masses of the surround-particles. More precisely, we prove that, in the limit $N \to \infty$, the test-particle diffuses in time $t \geq 0$ according to a quite general (non-Markovian) Gaussian process $(Z_t)_{t\geq 0}$ with stationary increments characterised by a covariance function

$$\mathrm{Cov}(Z_t, Z_s) = \mathcal{D}\left(v(t) + v(s) - v(|t-s|)\right), \tag{1}$$

where $v(\cdot)$ is determined by the distribution of the masses of the surround-particles and the constant $\mathcal{D}$ depends on the strength of the coupling between the test particle and the surround-particles. In particular, for a proper choice of the distribution of the masses of the surround-particles, we obtain, as a special case, fractional Brownian motion (fBm) with Hurst parameter $H \in (1/2, 1)$. In this respect, we remind that the fBm has experimentally turned out to be the underlying stochastic motion in many living systems, see, inter alia [19, 30, 33]. In this paper, we present also some distributions of masses of the surround-particles which lead to a mixture of independent fBms with different Hurst parameters or to the classical Wiener process as the limiting process $(Z_t)_{t\geq 0}$. Moreover, we present some distributions of masses of the surround-particles leading to limiting processes which perform a transition from ballistic diffusion to superdiffusion, or from ballistic diffusion to classical diffusion. We expect that our approach allows to cover also the case of subdiffusive limiting processes after introducing a suitable potential into the considered particle-system, this is however left for a future paper.

The Brownian motion of the mesoscopic surround-particles is actually described through the under-damped Langevin equation. Namely, the velocity is provided by an Ornstein–Uhlenbeck process and the position by the integration in time of the velocity according to kinematics. Within this setting, the proof of our limit theorem exploits that, conditionally on the masses of surround-particles, the dynamics of the test particle is Gaussian, and we can make use of the theory of mixing convergence (see, e.g., [9]) to pass to a suitable scaling limit. The scaling is, however, worse than in the classical Central-Limit-Theorem (CLT), which is compensated by good properties of the Ornstein–Uhlenbeck process. The present result pushes forward in a rigorous way a preliminary analysis [5] aimed to derive models for AD on the basis of an unspecified (conditionally) Gaussian process generated by the superposition of Ornstein–Uhlenbeck processes.

Furthermore, the constant $\mathcal{D}$ in formula (1) depends on the coupling parameter between the test-particle and the surround (via the constant C_α appearing in Assumption 2.2 below). Therefore, if we consider several independent and identical copies of the same mesoscopic Brownian-surround and we immerse into any copy of this surround a single replication of a number of replicae of a test-particle that are all built-up of the same art but each with its own individual characteristics, we may obtain different coupling parameters and hence different coefficients $\mathcal{D}$ for the covariance of the limiting Gaussian process $(Z_t)_{t\geq 0}$ from different copies of the experiment. This is the case, for example, when our test-particle is a complex macromolecule which may differ from its replicae because of its individual structure features, shape, hydrodynamic radius, etc. These last considerations may serve as a physical basis for the formulation of AD also within the framework of the superstatistical fBm [24, 23, 18, 11], where further randomness is provided by a distribution of the diffusion coefficients associated to each diffusing test-particle, and also within the framework of its generalisation called diffusing-diffusivity approach [3, 2, 32, 31, 29, 4], where the diffusion coefficient of each test-particle is no longer a random variable but a process. In this respect, we remind that the experimental evidences of a population of diffusion coefficients have been reported, for example, in the motion of mRNA molecules in live E. coli cells [18], β-adrenergic receptors [8] and dendritic cell-specific intercellular adhesion molecule 3-grabbing nonintegrin (DC-SIGN) [21]. Thus, the superstatistical fBm [23, 18, 11, 25, 16, 17] together with the diffusing-diffusivity approach [3, 2] stand as promising methods. Rigorous mathematical exploration of the connection between our present results (with additional randomness affecting the constant $\mathcal{D}$) and models of AD within the framework of superstatistical fBm, or diffusing-diffusivity, are left for our further research.

The paper is organised a follows. In Section 2, we state the problem and present the main result together with some special cases. Its detailed proof is reported in the following Section 3.

2 Statement of the problem and main result

In the sequel, we consider the motion of a test-particle with mass M that is immersed into a surround composed by a heterogeneous ensemble of N Brownian particles with positive masses $m_{k,N}$, $k = 1, \ldots, N$, positions $Y_t^{k,N}$ and velocities $U_t^{k,N}$, $N \in \mathbb{N}$. Let X_t^N be the position and V_t^N be the velocity of the test-particle. This system of particles is described by the following system of Langevin equations:

$$\begin{cases} dX_t^N = V_t^N dt\,, & X_0^N = 0\,, \\ dV_t^N = \frac{1}{M}\sum_{k=1}^N \left(\beta_{k,N} U_t^{k,N} dt - \alpha_{k,N} V_t^N dt\right)\,, & V_0^N = 0\,, \\ dY_t^{k,N} = U_t^{k,N} dt\,, & Y_0^{k,N} = y_0^{k,N}\,, \\ dU_t^{k,N} = F(U_t^{k,N}, V_t^N, m_{k,N}, \alpha_{k,N}, \beta_0^{k,N}, \beta_{k,N})dt + \frac{\sqrt{2\sigma_0^{k,N}}}{m_{k,N}} dW_t^k\,, & U_0^{k,N} = u_0^{k,N}\,, \\ & k = 1, \ldots, N\,, \end{cases} \tag{2}$$

where $(W_t^1)_{t\geq 0}, \ldots, (W_t^n)_{t\geq 0}, \ldots$ is a sequence of independent Wiener processes on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\alpha_{k,N}$, $\beta_0^{k,N}$, $\beta_{k,N}$, $k = 1, \ldots, N$, $N \in \mathbb{N}$, are positive coupling constants and $\sigma_0^{k,N} > 0$ is the noise amplitude.

In view of establishing the physical origin of AD, we set system (2) such that conservation principles are met, that is $i)$ the conservation of momentum and $ii)$ the conservation of energy, this last in the form of the fluctuation-dissipation theorem for the diffusion of the surround particles. Therefore, we have for all $k = 1, \dots, N$

$$F(U_t^{k,N}, V_t^N, m_{k,N}, \alpha_{k,N}, \beta_0^{k,N}, \beta_{k,N}) = -\frac{\beta_0^{k,N} + \beta_{k,N}}{m_{k,N}} U_t^{k,N} + \frac{\alpha_{k,N}}{m_{k,N}} V_t^N \,, \tag{3}$$

and the fluctuation-dissipation theorem implies

$$\left(\beta_0^{k,N} + \beta_{k,N}\right) \kappa_B T = \sigma_0^{k,N}, \qquad k = 1, \dots, N \,, \quad N \in \mathbb{N} \,, \tag{4}$$

where κ_B and T are the Boltzman constant and the temperature, respectively. Furthermore, we impose the following assumption:

Assumption 2.1. We assume that there exists a constant $\sigma > 0$ such that for any $N \in \mathbb{N}$ and any $k = 1, \dots, N$,

$$\frac{\sqrt{\sigma_0^{k,N}}}{m_{k,N}} = \sqrt{\sigma} \,. \tag{5}$$

Therefore, combining (3), (4) and (5) and using $\gamma := \frac{\sigma}{\kappa_B T}$, we obtain the followng system of equations:

$$\begin{cases} dX_t^N = V_t^N dt \,, & X_0^N = 0 \,, \\ dV_t^N = \frac{1}{M} \sum_{k=1}^N \left(\beta_{k,N} U_t^{k,N} dt - \alpha_{k,N} V_t^N dt\right) \,, & V_0^N = 0 \,, \\ dY_t^{k,N} = U_t^{k,N} dt \,, & Y_0^{k,N} = y_0^{k,N} \,, \\ dU_t^{k,N} = -\gamma m_{k,N} U_t^{k,N} dt + \frac{\alpha_{k,N}}{m_{k,N}} V_t^N dt + \sqrt{2\sigma} dW_t^k \,, & U_0^{k,N} = u_0^{k,N} \,, \\ & k = 1, \dots, N \,. \end{cases} \tag{6}$$

For each fixed $t_0 > 0$, we are interested in the behaviour of the system (6) during the time interval $[0, t_0]$ in the limit $N \to \infty$. In order to streamline the notation, we write $g_1 \simeq g_2$ if and only if $g_1(N)/g_2(N) \to 1$ as $N \to \infty$, $g_1 \propto g_2$ if and only if $\exists\, C > 0$ such that $g_1 \simeq C\, g_2$, and $g_1 \lesssim g_2$ if and only if $\exists\, C > 0$ such that $g_1 \leq C\, g_2$ for all $N \in \mathbb{N}$. In the sequel, we impose the following conditions on the parameters of system (6):

Assumption 2.2. For given parameters $a > 0$, $b > 0$, $C_\alpha > 0$, $C_\beta > 0$, we assume that

$$\begin{aligned} \alpha_{k,N} &\simeq C_\alpha N^{-a}, \qquad k = 1, \dots, N, \\ \beta_{k,N} &\simeq C_\beta N^{-b}, \qquad k = 1, \dots, N. \end{aligned}$$

Assumption 2.3. (i) For each fixed $N \in \mathbb{N}$, we assume that $m_{1,N}, \dots, m_{N,N}$ are i.i.d. random variables on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with distribution μ_N supported in $[m_{min}^N, \infty) \subset (0, \infty)$. We suppose that

$$m_{min}^N \simeq N^{-d} \quad \text{for some } \; d \in \mathbb{R}$$

and there exist constants $\delta \geq 0$ and $C_\delta > 0$, a sequence $(m_N^*)_{N\in\mathbb{N}}$ and a nondecreasing function $\dot{v} : [0, t_0] \to [0, \infty)$ such that

$$m_N^* \simeq C_\delta N^{-\delta} \qquad \text{and} \qquad \frac{\mathfrak{e}_N(t)}{m_N^*} \uparrow \dot{v}(t), \quad t \in [0, t_0].$$

Here,

$$\mathfrak{e}_N(t) := \int_{(0,\infty)} \frac{1 - e^{-\gamma y t}}{y^2} \mu_N(dy),$$

and we will use the notation $v(t) := \int_0^t \dot{v}(\tau) d\tau$. Moreover, we assume that there exists $d' \in \mathbb{R}$ such that

$$\int_{(0,\infty)} y^{-4} \mu_N(dy) \lesssim N^{d'}.$$

(ii) Furthermore, let $\mathcal{M} := (m_{1,N}, \dots, m_{N,N})_{N\in\mathbb{N}}$ denote the collection of all random masses. We assume that $\mathcal{M}$ is a family of independent random variables and that all random variables $m_{k,N}$, $N \in \mathbb{N}$, $k = 1, \dots, N$, are independent also from the Wiener processes $(W_t^1)_{t\geq 0}, \dots, (W_t^n)_{t\geq 0}, \dots$.

Note that, since $m_{k,N}$ are random, so are $\sigma_0^{k,N}$ and $\beta_0^{k,N}$ due to Assumption 2.1 and equality (4).

Assumption 2.4. For a part of our results, we will use also the following additional assumptions on the distribution μ_N. They are either

(i) There exist $\varepsilon > 1$ and $C = C(\varepsilon, t_0) > 0$ such that the variance function $v(t)$ satisfies

$$v(t) \leq C t^{1+\varepsilon}, \qquad t \in [0, t_0]. \tag{7}$$

Moreover, there exist $\epsilon \in (0, 1)$, $C = C(\epsilon) > 0$ and $N_0 \in \mathbb{N}$ such that for every $N > N_0$

$$\frac{1}{m_N^*} \left(\int_{(0,\infty)} \left[\frac{\gamma}{y} \mathbb{1}_{\{0<y\leq 1\}} + \frac{\gamma^\epsilon}{y^{2-\epsilon}} \mathbb{1}_{\{y>1\}} \right] \mu_N(dy) \right) < C. \tag{8}$$

or

(ii) The masses $m_{k,N}$, $k = 1, \dots, N$, $N \in \mathbb{N}$, are deterministic, i.e. the distribution μ_N is the Dirac measure concentrated at some point on $(0, \infty)$; this point may depend on N. Moreover, there exists $C = C(t_0) > 0$ such that

$$v(t) \leq C t, \qquad t \in [0, t_0]. \tag{9}$$

Assumption 2.5. We assume that $\left(u_0^{k,N},\, N \in \mathbb{N},\, k = 1, \dots, N\right)$ is a family of independent random variables which are independent also from all Wiener processes $(W_t^1)_{t\geq 0}$,..., $(W_t^n)_{t\geq 0}, \dots$. And, given $\mathcal{M}$, $u_0^{k,N}$ has Gaussian distribution $\mathcal{N}(0, \sigma/(\gamma m_{k,N}))$, $k = 1, \dots, N$.

Assumption 2.6. We pose the following conditions on the parameters:

$$\begin{cases} 0 < a < 1,\, b > 0,\, d, d' \in \mathbb{R},\, \delta \geq 0, \\ 2(a-b) - \delta = 1, \\ d' < 5 + 8(b-a) \\ b > d. \end{cases}$$

The main result of this paper is the following theorem:

Theorem 2.1. *(i) Fix any $t_0 > 0$. Under Assumptions 2.1, 2.2, 2.3, 2.5, 2.6, consider a centered Gaussian process $(Z_t)_{t\in[0,t_0]}$ with covariance function*

$$\mathrm{Cov}(Z_t, Z_s) = \left(\frac{2\sigma C_\beta^2 C_\delta}{\gamma^2 C_\alpha^2}\right) \frac{1}{2} \left(v(t) + v(s) - v(|t-s|)\right).$$

Then the processes $\left(X_t^N\right)_{t\in[0,t_0]}$ in (6), $N \in \mathbb{N}$, converge as $N \to \infty$ to $(Z_t)_{t\in[0,t_0]}$ in finite dimensional distributions.
(ii) If, additionally, Assumption 2.4 (i) or Assumption 2.4 (ii) is true, the processes $\left(X_t^N\right)_{t\in[0,t_0]}$ in (6), $N \in \mathbb{N}$, converge as $N \to \infty$ to $(Z_t)_{t\in[0,t_0]}$ in distribution on the space $C[0, t_0]$ of continuous functions from $[0, t_0]$ to $\mathbb{R}$.

Remark 2.1. Note that the process $(Z_t)_{t\in[0,t_0]}$ has stationary increments. Indeed, $\mathbb{E}[Z_{t+h} - Z_t] = 0$, $\mathrm{Var}(Z_{t+h} - Z_t) = 2\mathcal{D}v(|h|)$ with $\mathcal{D} := \frac{\sigma C_\beta^2 C_\delta}{\gamma^2 C_\alpha^2}$ and, since the process $(Z_t)_{t\in[0,t_0]}$ is Gaussian, the distribution of $Z_{t+h} - Z_t$ does not depend on t.

We present now some important special cases when the Assumptions 2.2, 2.3 (i), 2.4, 2.6 are satisfied and then outline the ideas of the proof. The rigorous proof of Theorem 2.1 with all technical details is given in the next Section.

Example 2.1. Suppose ν is the Lévy measure of a pure jump subordinator with Laplace exponent $\Phi(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda y})\nu(dy)$. We assume that ν is continuous and $\int_{(1,\infty)} y^2\nu(dy) = \infty$. Now, fix $d, \delta > 0$ and define sequences $(m_{max}^N)_{N\in\mathbb{N}}$ and $(m_{min}^N)_{N\in\mathbb{N}}$ via $m_{min}^N := N^{-d}$ and $\int_{(0,m_{max}^N]} y^2\nu(dy) = N^\delta$. We consider the distribution μ_N given by

$$\frac{d\mu_N}{d\nu}(y) = m_N^* y^2 \mathbb{1}_{(m_{min}^N, m_{max}^N]}(y), \tag{10}$$

where the normalizing constant m_N^* turns μ_N into a probability measure. Then, $m_N^* \simeq N^{-\delta}$ and for any $t_0 > 0$ and any $t \in [0, t_0]$

$$\frac{\mathfrak{e}_N(t)}{m_N^*} = \int_{(m_{min}^N, m_{max}^N]} (1 - e^{-\gamma y t})\, \nu(dy) \uparrow \Phi(\gamma t) =: \dot{v}(t).$$

Note that $\Phi(\cdot)$ is nonnegative and nondecreasing as a Bernstein function. The limiting variance function is then

$$v(t) = \int_0^t \Phi(\gamma\tau) d\tau.$$

Moreover, since the Lévy measure of any subordinator satisfies the condition $\int_{(0,\infty)} 1 \wedge y\,\nu(dy) < \infty$, we have

$$\int_{(0,\infty)} y^{-4}\mu_N(dy) = m_N^* \int_{(m_{min}^N, m_{max}^N]} y^{-2}\nu(dy)$$
$$\leq m_N^* \left((m_{min}^N)^{-3} \int_{(m_{min}^N, 1]} y\nu(dy) + \int_{(1,\infty)} \nu(dy) \right) \lesssim N^{3d-\delta}.$$

Hence, we may take $d' = 3d - \delta$. And Assumption 2.3 (i) is satisfied. Then, Assumption 2.6 is satisfied in the following situation:

$$b \in (0, 1/2), \quad a \in (b + 1/2, (b + 2/3) \wedge 1), \quad \delta = 2(a - b) - 1, \quad d < (4/3 - 2(a - b)) \wedge b.$$

Furthermore, condition (8) of Assumption 2.4 has then the following view:

$$\exists\, \epsilon \in (0, 1) \text{ and } C = C(\epsilon) > 0 \; : \qquad \int_{(1,\infty)} y^{\epsilon}\nu(dy) \leq C, \tag{11}$$

since the condition $\int_{(0,1]} y\nu(dy) < \infty$ is satisfied by the Lévy measure of any subordinator.

Let us consider now some special cases of the above setting:

(i) **Fractional Brownian Motion:** Let $H \in (1/2, 1)$ and let ν be the Lévy measure of the $(2H - 1)$-stable subordinator, i.e.,

$$\nu(dy) := \mathbb{1}_{(0,\infty)}(y) y^{-2H} dy.$$

We define μ_N in accordance with (10). Note that $m_{max}^N = (3 - 2H)^{\frac{1}{3-2H}} N^{\frac{\delta}{3-2H}}$. Since

$$\Phi(\lambda) = \frac{\Gamma(2 - 2H)}{2H - 1} \lambda^{2H-1},$$

we find the variance function

$$v(t) = \frac{\Gamma(2 - 2H)\gamma^{2H-1}}{2H(2H - 1)} t^{2H}.$$

Hence, up to a multiplicative factor, the limiting process is a fractional Brownian motion with Hurst parameter H. Condition (7) is satisfied, e.g., with $\varepsilon := 2H - 1$ and $C := \frac{\Gamma(2-2H)\gamma^{2H-1}}{2H(2H-1)}$; condition (11) is satisfied with any $\epsilon \in (0, 2H - 1)$. Hence, Assumption 2.4 (i) is satisfied.

(ii) **A Mixture of Fractional Brownian Motions with Different Hurst Parameters:** Let $K \in \mathbb{N}$ and $H_1 < \ldots < H_K$ with $H_1, H_K \in (1/2, 1)$. Let

$$\nu(dy) := \mathbb{1}_{(0,\infty)}(y) \sum_{k=1}^{K} y^{-2H_k}\, dy.$$

We define μ_N in accordance with (10). Then

$$\Phi(\lambda) = \sum_{k=1}^{K} \frac{\Gamma(2-2H_k)}{2H_k-1} \lambda^{2H_k-1},$$

and we find the variance function

$$v(t) = \sum_{k=1}^{K} \frac{\Gamma(2-2H_k)\gamma^{2H_k-1}}{2H_k(2H_k-1)} t^{2H_k}.$$

Hence, the limiting process is a sum of independent fractional Brownian motions with Hurst parameters $H_1, \dots, H_K$ (up to multiplicative factors). In this case, the character of anomalous diffusion changes with time: $v(t) \approx \frac{\Gamma(2-2H_1)\gamma^{2H_1-1}}{2H_1(2H_1-1)} t^{2H_1}$ for small t and $v(t) \approx \frac{\Gamma(2-2H_K)\gamma^{2H_K-1}}{2H_K(2H_K-1)} t^{2H_K}$ for large t. Assumption 2.4 (i) is satisfied, e.g., with $\varepsilon := 2H_1 - 1$ and $\epsilon \in (0, 2H_1 - 1)$.

(iii) Further couples (ν, Φ) satisfying the setting (cp. [26]):

$$\nu(dy) := \frac{(2-y)e^{-1/y} + y}{2\sqrt{\pi} y^{5/2}} dy \qquad \text{and} \qquad \Phi(\lambda) := \sqrt{\lambda}\left(1 - e^{-2\sqrt{\lambda}}\right);$$

$$\nu(dy) := \frac{1 - e^{-y}(1+y)}{y^2} dy \qquad \text{and} \qquad \Phi(\lambda) := \lambda \log(1 + 1/\lambda).$$

Example 2.2. Suppose ν is the Lévy measure of a pure jump subordinator with Laplace exponent $\Phi(\lambda) = \int_{(0,\infty)} (1 - e^{-\lambda y}) \nu(dy)$. We now assume that $\int_{(1,\infty)} y^2 \nu(dy) < \infty$. Let $m_{min}^N = N^{-d}$ for some $d > 0$ and consider the distribution μ_N given by

$$\frac{d\mu_N}{d\nu}(y) = m_N^* y^2 \mathbb{1}_{(m_{min}^N, \infty)}(y), \tag{12}$$

where the normalizing constant m_N^* turns μ_N into a probability measure. Therefore, $m_N^* \simeq (\int_{(0,\infty)} y^2 \nu(dy))^{-1}$, i.e., $\delta = 0$. Moreover

$$\frac{\mathfrak{e}_N(t)}{m_N^*} := \int_{(m_{min}^N, \infty)} \left(1 - e^{-\gamma y t}\right) \nu(dy) \uparrow \Phi(\gamma t) =: \dot{v}(t),$$

leading, as before, to the limiting variance function

$$v(t) = \int_0^t \Phi(\gamma \tau) d\tau.$$

A similar reasoning as before shows

$$\int_{(0,\infty)} y^{-4} \mu_N(dy) \leq m_N^* (m_{min}^N)^{-3} \int_{(m_{min}^N, \infty)} y \nu(dy) \lesssim N^{3d}.$$

Hence, we may take $d' = 3d$. And Assumption 2.3 (i) is satisfied. Assumption 2.6 is, then, satisfied in the following situation:

$$b \in (0, 1/2), \quad a = b + 1/2, \quad d < (1/3) \wedge b.$$

Condition (8) of Assumption 2.4 has again the form (11) and is satisfied for any $N \in \mathbb{N}$ and any $\epsilon \in (0,1]$.

Let us consider now some special cases of the above setting:

(i) **Transition from Ballistic Diffusion to Superdiffusion:** Let $H \in (1/2, 1)$ and let ν be the Lévy measure of a tempered $(2H-1)$-stable subordinator, i.e.,

$$\nu(dy) := \mathbb{1}_{(0,\infty)}(y) e^{-y} y^{-2H} dy.$$

Hence, $\int_{(0,\infty)} y^2 \, \nu(dy) < \infty$. We define μ_N in accordance with (12). Then (cp. [26]),

$$\Phi(\lambda) = (\lambda + 1)^{2H-1} - 1.$$

Hence, we obtain the variance function $v(t) = \frac{(\gamma t+1)^{2H}-1}{2H\gamma} - t$. In this case, the character of anomalous diffusion changes with time: $v(t) \approx Ct^2$ for small t and $v(t) \approx Ct^{2H}$ for large t. Condition (7) is then satisfied, e.g., with $\varepsilon := 2H - 1$. Hence, Assumption 2.4 (i) is true.

(ii) **Transition from Ballistic Diffusion to Classical Diffusion:** Let ν be the following Lévy measure:

$$\nu(dy) := \mathbb{1}_{(0,\infty)}(y) \gamma e^{-\gamma y} dy.$$

Hence, $\int_{(0,\infty)} y^2 \, \nu(dy) < \infty$. We define μ_N in accordance with (12). Then (cp. [26]),

$$\Phi(\lambda) = \frac{\lambda}{\lambda + \gamma}.$$

Hence, we obtain the variance function $v(t) = t - \log(t+1)$. In this case, the character of anomalous diffusion changes with time: $v(t) \approx Ct^2$ for small t and $v(t) \approx t$ for large t. Therefore, condition (7) is satisfied for any $\varepsilon \in (0,1]$ with some suitable constant $C = C(\varepsilon, t_0) > 0$.

(iii) Further couples (ν, Φ) satisfying the setting (cp. [26]):

$$\nu(dy) = \mathbb{1}_{(0,\infty)}(y) e^{-y} y^{-1} dy, \qquad \Phi(\lambda) = \log(1-\lambda);$$
$$\nu(dy) = \mathbb{1}_{(0,1)}(y) \frac{1}{\Gamma(1-\alpha)} e^{-y} y^{-\alpha} dy, \qquad \Phi(\lambda) = 1 - (1+\lambda)^{\alpha-1}, \quad \alpha \in (0,1).$$

Example 2.3. In this example, the masses $m_{k,N}$, $k = 1, \ldots, N$, $N \in \mathbb{N}$, are deterministic.

(i) **Classical Diffusion:** Fix some $\delta > 0$ and define μ_N to be the Dirac measure concentrated at $N^{\delta/2}$. Then,

$$N^\delta \mathfrak{e}_N(t) = \int_{(0,\infty)} \frac{1 - e^{-\gamma y t}}{y^2} \mu_N(dy) = 1 - e^{-\gamma N^{\delta/2} t} \uparrow 1_{(0,t_0]}(t)$$

leading to $v(t) = t$, and, hence, to a Wiener process as limiting process. Hence, Assumption 2.4 (ii) is satisfied. Further, we may take $d = -\delta/2 < 0$ and $m_N^* = N^{-\delta}$. Moreover,

$$\int_{(0,\infty)} y^{-4} \mu_N(dy) = N^{-2\delta},$$

i.e., $d' = -2\delta < 0$. With such choice of parameters, Assumption 2.3 (i) is satisfied. Assumption 2.6 is, then, satisfied in the following situation:

$$b \in (0, 1/2), \quad a \in (b + 1/2, (b + 3/4) \wedge 1), \quad \delta = 2(a - b) - 1.$$

(ii) **Transition from Ballistic Diffusion to Classical Diffusion:** Let now the masses $m_{k,N}$, $k = 1, \dots, N$, $N \in \mathbb{N}$ be deterministic and do not change with N, say, μ_N is the Dirac measure concentrated at 1. Then $m^N_{min} = 1$, $d = 0$,

$$\mathfrak{e}_N(t) = \int_{(0,\infty)} \frac{1 - e^{-\gamma y t}}{y^2} \mu_N(dy) = 1 - e^{-\gamma t} = \dot{v}(t),$$

and hence $\delta = 0$, $m^*_N = 1$, $d' = 0$. With such choice of parameters, Assumption 2.3 (i) is satisfied. Assumption 2.6 is, then, satisfied in the following situation:

$$b \in (0, 1/2), \quad a = b + 1/2.$$

The variance function is $v(t) = t + \frac{1}{\gamma}\left(e^{-\gamma t} - 1\right)$. In this case, the character of anomalous diffusion changes with time: $v(t) \approx Ct^2$ for small t and $v(t) \approx t$ for large t. Hence, Assumption 2.4 (ii) is satisfied.

Below we outline the ideas and the structure of the proof of Theorem 2.1. A complete rigorous proof can be found in the next Section.

Step 1: We show that we may neglect the drift terms $\frac{\alpha_{k,N}}{m_{k,N}} V^N_t dt$ in the system (6) when $N \to \infty$. Hence, we may replace the processes $(U^{k,N}_t)_{t\in[0,t_0]}$ in the system (6) by the corresponding Ornstein-Uhlenbeck processes $(\widetilde{U}^{k,N}_t)_{t\in[0,t_0]}$,

$$\widetilde{U}^{k,N}_t := u^{k,N}_0 e^{-\gamma m_{k,N} t} + \sqrt{2\sigma} \int_0^t e^{-\gamma m_{k,N}(t-s)} dW^k_s, \qquad k = 1, \dots, N. \tag{13}$$

Therefore, solving the ODEs in the first two lines of the system (6), we may approximate the position of the test-particle as $N \to \infty$ by the process $\left(\widetilde{X}^N_t\right)_{t\in[0,t_0]}$, given by

$$\widetilde{X}^N_t := \int_0^t \widetilde{V}^N_\tau d\tau := \frac{1}{M} \int_0^t e^{-\frac{\mathcal{A}_N}{M} s} \sum_{k=1}^N \beta_{k,N} \int_s^t \widetilde{U}^{k,N}_{\tau-s} d\tau ds, \qquad t \in [0, t_0], \tag{14}$$

where $\mathcal{A}_N := \sum_{k=1}^N \alpha_{k,N}$.

Step 2: We show that, as $N \to \infty$, the process $\left(\widetilde{X}^N_t\right)_{t\in[0,t_0]}$ can be approximated by the process $\left(\widetilde{Z}^N_t\right)_{t\in[0,t_0]}$, where

$$\widetilde{Z}^N_t := \frac{1}{\mathcal{A}_N} \sum_{k=1}^N \beta_{k,N} \int_0^t \widetilde{U}^{k,N}_\tau d\tau. \tag{15}$$

Step 3: We show that the processes $\left(\widetilde{Z}_t^N\right)_{t\in[0,t_0]}$, $N \in \mathbb{N}$, converge in a suitable sense to the process $(Z_t)_{t\in[0,t_0]}$ as in the statement of Theorem 2.1. Finally, combining the results of all three steps, we obtain the statement of the Theorem.

Remark 2.2. Pay attention, that the scaling coefficients $\frac{\beta_{k,N}}{\mathcal{A}_N} \simeq CN^{a-b-1}$. In the case $\delta = 0$, this scaling correspond to the scaling in the classical Central Limit Theorem. In the case $\delta > 0$, this scaling is worse than that one in the Central Limit Theorem since $a - b - 1 > -1/2$ due to Assumption 2.6. This is however compensated by the good properties of Ornstein-Uhlenbeck processes $(\widetilde{U}_t^N)_{t\in[0,t_0]}$.

3 Proofs

Let us start with some preparatory results. In the sequel, we use the constant $C > 0$ which may change from line to line but is always independent of N, k, M, t and any other time indices. We denote by $\mathbb{E}[\cdot \,|\, \mathcal{M}]$ the expectation given $\mathcal{M}$.

Lemma 3.1. *Under Assumptions of Theorem 2.1 (i) consider the Ornstein-Uhlenbeck processes given by* (13)*. Then we have*

$$\mathbb{E}\left[\widetilde{U}_t^{k,N}\widetilde{U}_s^{k,N} \,|\, \mathcal{M}\right] = \frac{\sigma}{\gamma m_{k,N}} e^{-\gamma m_{k,N}|t-s|}. \tag{16}$$

Proof. Due to independence of $u_0^{k,N}$ and the Wiener process W^k, we have

$$\begin{aligned}
&\mathbb{E}\left[\widetilde{U}_t^{k,N}\widetilde{U}_s^{k,N} \,|\, \mathcal{M}\right] \\
&= \mathbb{E}[(u_0^{k,N})^2 \,|\, \mathcal{M}]e^{-\gamma m_{k,N}(t+s)} + 2\sigma e^{-\gamma m_{k,N}(t+s)}\mathbb{E}\left[\left(\int_0^{t\wedge s} e^{\gamma m_{k,N}\tau} dW_\tau^k\right)^2 |\mathcal{M}\right] \\
&= \frac{\sigma}{\gamma m_{k,N}} e^{-\gamma m_{k,N}(t+s)} + 2\sigma e^{-\gamma m_{k,N}(t+s)} \int_0^{t\wedge s} e^{2\gamma m_{k,N}\tau} d\tau \\
&= \frac{\sigma}{\gamma m_{k,N}} e^{-\gamma m_{k,N}|t-s|}.
\end{aligned}$$

□

3.1 Step 1: Elimination of the cross-interaction terms in equations for speeds of surrounding particles

Lemma 3.2. *Fix any $t_0 > 0$. Let $(X_t^N)_{t\in[0,t_0]}$ be the solution of system* (6) *and $(\widetilde{X}_t^N)_{t\in[0,t_0]}$ be the process given in formula* (14)*. Under Assumptions 2.2, 2.3, 2.5, 2.6, we have with a suitable constant $C > 0$*

$$\mathbb{E}\left[\sup_{t\in[0,t_0]}\left|X_t^N - \widetilde{X}_t^N\right|^2\right] \leq Cv(t_0)t_0^2 N^{-2(b-d)} \exp\left(CN^{-2(b-d)}t_0^2\right) \longrightarrow 0, \tag{17}$$

as $N \to \infty$.

Proof. We first estimate $|V_t^N - \widetilde{V}_t^N|$ for $(V_t^N)_{t\in[0,t_0]}$ and $(\widetilde{V}_t^N)_{t\in[0,t_0]}$ as in system (6) and formula (14) respectively. For this aim, we consider the following weights

$$w_\tau := \frac{1}{M}\sum_{k=1}^{N}\frac{\beta_{k,N}\alpha_{k,N}}{m_{k,N}}\int_\tau^t e^{-\frac{\mathcal{A}_N}{M}(t-s)}e^{-\gamma m_{k,N}(s-\tau)}ds$$

$$\leq \frac{1}{\mathcal{A}_N}\sum_{k=1}^{N}\frac{\beta_{k,N}\alpha_{k,N}}{m_{k,N}} \leq CN^{-(b-d)}; \tag{18}$$

$$\overline{w}_\rho := \int_\rho^t e^{-\frac{\mathcal{A}_N}{M}(\tau-\rho)}w_\tau d\tau \leq C\frac{M}{\mathcal{A}_N}N^{-(b-d)}. \tag{19}$$

Further, for the Ornstein-Uhlenbeck process $(\widetilde{U}_t^{k,N})_{t\in[0,t_0]}$ and the process $(U_t^{k,N})_{t\in[0,t_0]}$ as in system (6), we have

$$d\left(U_t^{k,N} - \widetilde{U}_t^{k,N}\right) = -\gamma m_{k,N}\left(U_t^{k,N} - \widetilde{U}_t^{k,N}\right)dt + \frac{\alpha_{k,N}}{m_{k,N}}V_t^N dt.$$

And hence

$$U_t^{k,N} - \widetilde{U}_t^{k,N} = \int_0^t \frac{\alpha_{k,N}}{m_{k,N}}e^{-\gamma m_{k,N}(t-\tau)}V_\tau^N d\tau.$$

Therefore, using the Fubini Theorem,

$$V_t^N - \widetilde{V}_t^N = \frac{1}{M}\sum_{k=1}^{N}\beta_{k,N}\int_0^t e^{-\frac{\mathcal{A}_N}{M}(t-s)}\left(U_s^{k,N} - \widetilde{U}_s^{k,N}\right)ds$$

$$= \frac{1}{M}\sum_{k=1}^{N}\beta_{k,N}\int_0^t e^{-\frac{\mathcal{A}_N}{M}(t-s)}\left(\int_0^s \frac{\alpha_{k,N}}{m_{k,N}}e^{-\gamma m_{k,N}(s-\tau)}V_\tau^N d\tau\right)ds$$

$$= \int_0^t V_\tau^N\left(\frac{1}{M}\sum_{k=1}^{N}\frac{\beta_{k,N}\alpha_{k,N}}{m_{k,N}}\int_\tau^t e^{-\frac{\mathcal{A}_N}{M}(t-s)}e^{-\gamma m_{k,N}(s-\tau)}ds\right)d\tau$$

$$= \int_0^t \left(V_\tau^N - \widetilde{V}_\tau^N\right)w_\tau d\tau + \int_0^t \widetilde{V}_\tau^N w_\tau d\tau. \tag{20}$$

Moreover, again using the Fubini Theorem,

$$\int_0^t \widetilde{V}_\tau^N w_\tau d\tau = \int_0^t\left(\frac{1}{M}\sum_{k=1}^{N}\beta_{k,N}\int_0^\tau e^{-\frac{\mathcal{A}_N}{M}(\tau-\rho)}\widetilde{U}_\rho^{k,N}d\rho\right)w_\tau d\tau$$

$$= \frac{1}{M}\sum_{k=1}^{N}\beta_{k,N}\int_0^t \widetilde{U}_s^{k,N}\left(\int_\rho^t e^{-\frac{\mathcal{A}_N}{M}(\tau-\rho)}w_\tau d\tau\right)d\rho$$

$$= \frac{1}{M}\sum_{k=1}^{N}\beta_{k,N}\int_0^t \widetilde{U}_\rho^{k,N}\overline{w}_\rho d\rho. \tag{21}$$

Therefore,

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t \widetilde{V}_\tau^N w_\tau d\tau\right)^2 \,|\, \mathcal{M}\right] \\
&= \frac{1}{M^2}\sum_{k=1}^N \beta_{k,N}^2 \int_0^t\int_0^t \mathbb{E}\left[\widetilde{U}_\rho^{k,N}\widetilde{U}_\tau^{k,N} \,|\, \mathcal{M}\right] \overline{w}_\rho \overline{w}_\tau d\rho d\tau \\
&= \frac{C}{M^2}\sum_{k=1}^N \beta_{k,N}^2 \int_0^t\int_0^t \frac{\overline{w}_\rho\overline{w}_\tau}{m_{k,N}} e^{-\gamma m_{k,N}|\rho-\tau|} d\rho d\tau \\
&\le CN^{-2(b-d)}\frac{1}{\mathcal{A}_N^2}\sum_{k=1}^N \beta_{k,N}^2 \int_0^t\int_0^t \frac{1}{m_{k,N}} e^{-\gamma m_{k,N}|\rho-\tau|} d\rho d\tau.
\end{aligned} \tag{22}$$

Since $2(a-b)-1-\delta = 0$ by Assumption 2.6,

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t \widetilde{V}_\tau^N w_\tau d\tau\right)^2\right] \\
&\le CN^{-2(b-d)}\frac{2}{\mathcal{A}_N^2}\sum_{k=1}^N \beta_{k,N}^2 \int_0^t\int_0^\tau \mathbb{E}\left[\frac{1}{m_{k,N}} e^{-\gamma m_{k,N}(\tau-\rho)}\right] d\rho d\tau \\
&\le CN^{-2(b-d)}\frac{2}{\gamma\mathcal{A}_N^2}\sum_{k=1}^N \beta_{k,N}^2 \int_0^t \mathbb{E}\left[\frac{1}{m_{k,N}^2}\left(1-e^{-\gamma m_{k,N}\tau}\right)\right] d\tau \\
&= CN^{-2(b-d)}\frac{2}{\gamma\mathcal{A}_N^2}\sum_{k=1}^N \beta_{k,N}^2 \int_0^t \mathfrak{e}_N(\tau) d\tau \\
&\le CN^{-2(b-d)}N^{-2+2a-2b+1-\delta}v(t) = CN^{-2(b-d)}v(t).
\end{aligned} \tag{23}$$

Combining (18), (20) and (23) yields by the Fubini theorem and by the inequality $ab \le (a^2+b^2)/2$ for any positive a, b

$$\begin{aligned}
\mathbb{E}\left[|V_t^N - \widetilde{V}_t^N|^2\right] &\le 2\mathbb{E}\left[\left|\int_0^t (V_\tau^N - \widetilde{V}_\tau^N) w_\tau d\tau\right|^2\right] + 2\mathbb{E}\left[\left|\int_0^t \widetilde{V}_\tau^N w_\tau d\tau\right|^2\right] \\
&\le CN^{-2(b-d)} t \int_0^t \mathbb{E}\left[\left|V_\tau^N - \widetilde{V}_\tau^N\right|^2\right] d\tau + Cv(t)N^{-2(b-d)}.
\end{aligned}$$

Since v is nondecreasing, we have by Grönwall's inequality, for every fixed $t_0 > 0$ and every $t \in [0, t_0]$

$$\mathbb{E}\left[|V_t^N - \widetilde{V}_t^N|^2\right] \le Cv(t_0)N^{-2(b-d)} \exp\left(CN^{-2(b-d)}t_0^2\right).$$

Therefore, for every $t_0 > 0$

$$\sup_{t\in[0,t_0]} \mathbb{E}\left[|V_t^N - \widetilde{V}_t^N|^2\right] \le Cv(t_0)N^{-2(b-d)} \exp\left(CN^{-2(b-d)}t_0^2\right). \tag{24}$$

Finally,

$$\mathbb{E}\left[\sup_{t\in[0,t_0]}|X_t^N-\widetilde{X}_t^N|^2\right]=\mathbb{E}\left[\sup_{t\in[0,t_0]}\left|\int_0^t(V_s^N-\widetilde{V}_s^N)ds\right|^2\right]$$
$$\le t_0\int_0^{t_0}\mathbb{E}\left[|V_s^N-\widetilde{V}_s^N|^2\right]ds\le t_0^2\sup_{s\in[0,t_0]}\mathbb{E}\left[|V_s^N-\widetilde{V}_s^N|^2\right].$$

Thus, by (24) and Assumption 2.6,

$$\mathbb{E}\left[\sup_{t\in[0,t_0]}|X_t^N-\widetilde{X}_t^N|^2\right]\le Cv(t_0)t_0^2N^{-2(b-d)}\exp\left(CN^{-2(b-d)}t_0^2\right)\longrightarrow 0,$$

as $N\to\infty$. □

3.2 Step 2: Comparison of $\left(\widetilde{X}_t^N\right)_{t\in[0,t_0]}$ with $\left(\widetilde{Z}_t^N\right)_{t\in[0,t_0]}$.

Lemma 3.3. *For each fixed $t_0>0$ consider $(\widetilde{X}_t^N)_{t\in[0,t_0]}$ and $(\widetilde{Z}_t^N)_{t\in[0,t_0]}$ as in formulas (14) and (15) respectively. Under assumptions of Theorem 2.1 (i), we have*

$$\sup_{t\in[0,t_0]}\mathbb{E}\left[|\widetilde{X}_t^N-\widetilde{Z}_t^N|^2\right]\le CM\dot{v}(t_0)N^{a-1}\longrightarrow 0,\qquad N\to\infty.$$

Proof. We have by formula (14) and by the Fubini theorem

$$\begin{aligned}\widetilde{X}_t^N&=\frac{1}{M}\sum_{k=1}^N\beta_{k,N}\int_0^t e^{-\frac{\mathcal{A}_N}{M}s}\int_s^t\widetilde{U}_{\tau-s}^{k,N}d\tau ds\\&=\frac{1}{M}\sum_{k=1}^N\beta_{k,N}\int_0^t e^{-\frac{\mathcal{A}_N}{M}s}\int_0^{t-s}\widetilde{U}_{\tau}^{k,N}d\tau ds\\&=\frac{1}{M}\sum_{k=1}^N\beta_{k,N}\int_0^t \widetilde{U}_{\tau}^{k,N}\int_0^{t-\tau}e^{-\frac{\mathcal{A}_N}{M}s}ds d\tau\\&=\frac{1}{\mathcal{A}_N}\sum_{k=1}^N\beta_{k,N}\int_0^t \widetilde{U}_{\tau}^{k,N}\left(1-e^{-\frac{\mathcal{A}_N}{M}(t-\tau)}\right)d\tau\\&=\widetilde{Z}_t^N-\frac{1}{\mathcal{A}_N}\sum_{k=1}^N\beta_{k,N}\int_0^t \widetilde{U}_{\tau}^{k,N}e^{-\frac{\mathcal{A}_N}{M}(t-\tau)}d\tau.\end{aligned}$$

Consider $\mathcal{R}_t^N := \frac{1}{\mathcal{A}_N}\sum_{k=1}^N \beta_{k,N}\int_0^t \widetilde{U}_\tau^{k,N} e^{-\frac{\mathcal{A}_N}{M}(t-\tau)}d\tau$. Hence, by (16) and Assumptions 2.3, 2.6,

$$
\begin{aligned}
&\mathbb{E}\left[\left|\widetilde{X}_t^N - \widetilde{Z}_t^N\right|^2\right] = \mathbb{E}\left[\left|\mathcal{R}_t^N\right|^2\right] \\
&= \frac{1}{\mathcal{A}_N^2}\sum_{k=1}^N \beta_{k,N}^2 \int_0^t\int_0^t e^{-\frac{\mathcal{A}_N}{M}(t-\tau)}e^{-\frac{\mathcal{A}_N}{M}(t-\rho)}\mathbb{E}\left[\widetilde{U}_\tau^{k,N}\widetilde{U}_\rho^{k,N}\right]d\rho d\tau \\
&= \frac{2}{\mathcal{A}_N^2}\sum_{k=1}^N \beta_{k,N}^2 \int_0^t\int_0^\tau e^{-\frac{\mathcal{A}_N}{M}(t-\tau)}e^{-\frac{\mathcal{A}_N}{M}(t-\rho)}\mathbb{E}\left[\frac{\sigma}{\gamma m_{k,N}}e^{-\gamma m_{k,N}(\rho-\tau)}\right]d\rho d\tau \\
&\le \frac{2\sigma}{\gamma^2\mathcal{A}_N^2}\sum_{k=1}^N \beta_{k,N}^2\int_0^t e^{-\frac{\mathcal{A}_N}{M}(t-\tau)}\mathfrak{e}_N(\tau)d\tau \\
&\le CN^{-2+2a+1-2b-\delta}\dot{v}(t_0)\int_0^t e^{-\frac{\mathcal{A}_N}{M}(t-\tau)}d\tau \\
&\le C\dot{v}(t_0)\frac{M}{\mathcal{A}_N} \le CM\dot{v}(t_0)N^{a-1} \longrightarrow 0, \quad N\to\infty.
\end{aligned}
$$

□

3.3 Step 3: Convergence to the Gaussian Process $(Z_t)_{t\in[0,t_0]}$

Let us introduce the following notations:

$$
\begin{aligned}
\varphi^{t,s}(m) &:= \int_0^t\int_0^s \frac{1}{m}e^{-\gamma m|\tau-\rho|}d\tau d\rho; \\
\xi_k^{N,t,s} &:= \frac{\sigma}{\gamma\mathcal{A}_N^2}\beta_{k,N}^2\varphi^{t,s}(m_{k,N}); \\
\eta_k^{N,t,s} &:= \xi_k^{N,t,s} - \mathbb{E}\left[\xi_k^{N,t,s}\right]; \\
\eta^{N,t,s} &:= \sum_{k=1}^N \eta_k^{N,t,s}; \\
\xi^{N,t,s} &:= \sum_{k=1}^N \xi_k^{N,t,s} = \mathrm{Cov}(\widetilde{Z}_t^N, \widetilde{Z}_s^N \mid \mathcal{M}),
\end{aligned}
$$

where the last identity is due to Lemma 3.1 and (15).

Lemma 3.4. *Under assumptions of Theorem 2.1 (i), we have*

$$
\mathbb{E}\left[\xi^{N,t,s}\right] \longrightarrow \left(\frac{2\sigma C_\beta^2 C_\delta}{\gamma^2 C_\alpha^2}\right)\frac{1}{2}\left(v(t)+v(s)-v(|t-s|)\right), \qquad N\to\infty.
$$

Proof. Let

$$\bar{Z}_t^{k,N} := \int_0^t \widetilde{U}_\tau^{k,N} d\tau.$$

Since $\widetilde{U}^{k,N}$ is (conditionally on $\mathcal{M}$) a stationary Ornstein-Uhlenbeck process, $\bar{Z}^{k,N}$ has stationary increments conditionally on $\mathcal{M}$. Hence, using the equality $ab = \frac{1}{2}\left(a^2 + b^2 - (a-b)^2\right)$ for $a, b > 0$, we obtain

$$\begin{aligned}\mathrm{Cov}(\widetilde{Z}_t^N, \widetilde{Z}_s^N \mid \mathcal{M}) &= \frac{1}{\mathcal{A}_N^2} \sum_{k=1}^N \beta_{k,N}^2 \mathbb{E}[\bar{Z}_t^{k,N} \bar{Z}_s^{k,N} | \mathcal{M}] \\ &= \frac{1}{2\mathcal{A}_N^2} \sum_{k=1}^N \left(\beta_{k,N}^2 \left(v_{k,N}(t) + v_{k,N}(s) - v_{k,N}(|t-s|)\right)\right), \end{aligned} \tag{25}$$

where $v_{k,N}(t) := \mathbb{E}[(\bar{Z}_t^{k,N})^2 | \mathcal{M}]$. Now, by Lemma 3.1,

$$\begin{aligned} v_{k,N}(t) &= 2\int_0^t \int_0^\tau \mathbb{E}[\widetilde{U}_\tau^{k,N} \widetilde{U}_\rho^{k,N} | \mathcal{M}] d\rho d\tau \\ &= \int_0^t \int_0^\tau \frac{2\sigma}{\gamma m_{k,N}} e^{-\gamma m_{k,N}(\tau-\rho)} d\rho d\tau \\ &= \frac{2\sigma}{\gamma^2} \int_0^t \frac{1 - e^{-\gamma m_{k,N}\tau}}{m_{k,N}^2} d\tau. \end{aligned}$$

Taking expectation and applying Fubini's theorem and the monotone convergence theorem, we obtain

$$N^\delta \mathbb{E}[v_{k,N}(t)] = \frac{2\sigma}{\gamma^2} N^\delta m_N^* \int_0^t \frac{\mathfrak{e}_N(\tau)}{m_N^*} d\tau \to \frac{2\sigma C_\delta}{\gamma^2} v(t). \tag{26}$$

In view of (25),

$$\mathrm{Cov}(\widetilde{Z}_t^N, \widetilde{Z}_s^N) \to \frac{2\sigma C_\beta^2 C_\delta}{\gamma^2 C_\alpha^2} \, \frac{1}{2} \left(v(t) + v(s) - v(|t-s|)\right).$$

□

Lemma 3.5. *Under assumptions of Theorem 2.1 (i) we have*

$$\sum_{N=1}^\infty \mathbb{E}\left[|\eta^{N,t,s}|^4\right] < \infty.$$

Proof. Since $\mathbb{E}\left[\eta_k^{N,t,s}\right] = 0$, we obtain by the i.i.d. property of masses $m_{1,N}, \ldots, m_{N,N}$

$$\mathbb{E}\left[|\eta^{N,t,s}|^4\right] = N\mathbb{E}\left[|\eta_1^{N,t,s}|^4\right] + 3N(N-1)\mathbb{E}\left[|\eta_1^{N,t,s}|^2\right]^2.$$

Hence

$$\mathbb{E}\left[|\eta^{N,t,s}|^4\right] \leq 3N^2\mathbb{E}\left[\left(\xi_1^{N,t,s} - \mathbb{E}\left[\xi_1^{N,t,s}\right]\right)^4\right] \leq CN^2\mathbb{E}\left[\left|\xi_1^{N,t,s}\right|^4\right]. \tag{27}$$

Now,

$$\left|\xi_1^{N,t,s}\right|^4 \leq CN^{-8(1+b-a)}\left|\varphi^{t,s}(m_{1,N})\right|^4.$$

Noting that

$$\varphi^{t,s}(m) = \int_0^t \int_0^s \frac{1}{m} e^{-\gamma m|\tau-\rho|} d\tau d\rho \leq \frac{ts}{m},$$

we observe,

$$\mathbb{E}\left[\left|\xi_1^{N,t,s}\right|^4\right] \leq Ct_0^8 N^{-8(1+b-a)} \int_{(0,\infty)} y^{-4}\mu_N(dy) \leq Ct_0^8 N^{-8(1+b-a)+d'}$$

Therefore,

$$\sum_{N=1}^{\infty} \mathbb{E}\left[|\eta^{N,t,s}|^4\right] \leq Ct_0^8 \sum_{N=1}^{\infty} N^{-1+(8a-8b-5+d')} < \infty,$$

since $8a - 8b - 5 + d' < 0$ due to Assumption 2.6. □

Lemma 3.6. *Under assumptions of Theorem 2.1 (i), we have $\mathbb{P}$-almost surely*

$$\operatorname{Cov}(\widetilde{Z}_t^N, \widetilde{Z}_s^N \,|\, \mathcal{M}) \longrightarrow \left(\frac{2\sigma C_\beta^2 C_\delta}{\gamma^2 C_\alpha^2}\right) \frac{1}{2}\left(v(t) + v(s) - v(|t-s|)\right), \qquad N \to \infty.$$

Proof. For every $\varepsilon > 0$, we have by Lemma 3.5 and Markov's inequality

$$\sum_{N=1}^{\infty} \mathbb{P}\left(\left\{\left|\eta^{N,t,s}\right| > \varepsilon\right\}\right) \leq \frac{1}{\varepsilon^4} \sum_{N=1}^{\infty} \mathbb{E}\left[|\eta^{N,t,s}|^4\right] < \infty.$$

Hence, $\eta^{N,t,s} \to 0$, $N \to \infty$, $\mathbb{P}$-almost surely by the Borel–Cantelli Lemma. Therefore, the statement follows from Lemma 3.4. □

Let us now recall the notion of mixing convergence of random elements.

Definition 3.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{X}$ be a separable metrizable topological space endowed with its Borel σ-field $\mathcal{B}(\mathcal{X})$. Let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-field. A sequence $(\xi_N)_{N\in\mathbb{N}}$ of $(\mathcal{X}, \mathcal{B}(\mathcal{X}))$-valued random elements is said to *converge $\mathcal{G}$-mixing* to a (random element with) probability distribution ν on $\mathcal{X}$, if the conditional distributions $\mathcal{P}_{\xi_N|\mathcal{G}}$ converge weakly to ν as $N \to \infty$, i.e., if for every $f \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ and every bounded continuous function h on $\mathcal{X}$

$$\lim_{N\to\infty} \mathbb{E}\left[f\mathbb{E}\left[h(\xi_N)|\mathcal{G}\right]\right] = \mathbb{E}[f] \int_{\mathcal{X}} h d\nu.$$

Remark 3.1. Note that the $\mathcal{G}$-mixing convergence is a special case of the so called $\mathcal{G}$-stable convergence; and $\mathcal{G}$-stable convergence implies also convergence in distribution (see [9] for further details).

Lemma 3.7. *Let $\sigma(\mathcal{M})$ be the σ-field generated by the family $\mathcal{M}$ given in Assumption 2.3. In the setting of Theorem 2.1 (i), the sequence of stochastic processes $\left((\widetilde{Z}_t^N)_{t\in[0,t_0]}\right)_{N\in\mathbb{N}}$ converges $\sigma(\mathcal{M})$-mixing in finite dimensional distributions to the process $(Z_t)_{t\geq 0}$, i.e., for any $n\in\mathbb{N}$ and any $t_1,\dots,t_n\in[0,t_0]$, the $(\mathbb{R}^n,\mathcal{B}(\mathbb{R}^n))$-valued random elements $(\widetilde{Z}_{t_1}^N,\dots,\widetilde{Z}_{t_n}^N)$ converge $\sigma(\mathcal{M})$-mixing to $(Z_{t_1},\dots,Z_{t_n})$.*

Proof. By Lemma 3.6, conditionally on $\mathcal{M}$, we have a sequence of Gaussian stochastic processes $\left((\widetilde{Z}_t^N)_{t\in[0,t_0]}\right)_{N\in\mathbb{N}}$ with continuous paths whose mean functions and covariance functions converge to those of the process $(Z_t)_{t\in[0,t_0]}$. Therefore, conditionally on $\mathcal{M}$, all finite dimensional marginal distributions of processes $(\widetilde{Z}_t^N)_{t\in[0,t_0]}$ converge to those of the process $(Z_t)_{t\geq 0}$. □

Lemma 3.8. *In the setting of Theorem 2.1 (i), the sequence of processes $\left((X_t^N)_{t\in[0,t_0]}\right)_{N\in\mathbb{N}}$ converges $\sigma(\mathcal{M})$-mixing in finite dimensional distributions to the process $(Z_t)_{t\in[0,t_0]}$.*

Proof. Since, by Lemma 3.7, the sequence of processes $\left((\widetilde{Z}_t^N)_{t\in[0,t_0]}\right)_{N\in\mathbb{N}}$ converges $\sigma(\mathcal{M})$-mixing in finite dimensional distributions to the process $(Z_t)_{t\in[0,t_0]}$, it is enough to show that for any $\varepsilon>0$, any $n\in\mathbb{N}$ and any $t_1,\dots,t_n\in[0,t_0]$

$$\mathbb{P}\left(\left\|(X_{t_1}^N,\dots,X_{t_n}^N)-(\widetilde{Z}_{t_1}^N,\dots,\widetilde{Z}_{t_n}^N)\right\|_{\mathbb{R}^n}>\varepsilon\right)\longrightarrow 0,\qquad N\to\infty,$$

and to apply Theorem 3.7 (a) of [9]. By Markov's inequality, Lemma 3.2 and Lemma 3.3, we have

$$\begin{aligned}
&\mathbb{P}\left(\left\|(X_{t_1}^N,\dots,X_{t_n}^N)-(\widetilde{Z}_{t_1}^N,\dots,\widetilde{Z}_{t_n}^N)\right\|_{\mathbb{R}^n}>\varepsilon\right)\\
&\leq\frac{1}{\varepsilon^2}\mathbb{E}\left[\left\|(X_{t_1}^N,\dots,X_{t_n}^N)-(\widetilde{Z}_{t_1}^N,\dots,\widetilde{Z}_{t_n}^N)\right\|_{\mathbb{R}^n}^2\right]\\
&\leq\frac{2}{\varepsilon^2}\sum_{j=1}^n\left(\mathbb{E}\left[|X_{t_j}^N-\widetilde{X}_{t_j}^N|^2\right]+\mathbb{E}\left[|\widetilde{X}_{t_j}^N-\widetilde{Z}_{t_j}^N|^2\right]\right)\\
&\leq\frac{2n}{\varepsilon^2}\left(\sup_{t\in[0,t_0]}\mathbb{E}\left[|X_t^N-\widetilde{X}_t^N|^2\right]+\sup_{t\in[0,t_0]}\mathbb{E}\left[|\widetilde{X}_t^N-\widetilde{Z}_t^N|^2\right]\right)\longrightarrow 0,
\end{aligned}$$

as $N\to\infty$. □

Let us now refine the previous result to the convergence of processes $\left((X_t^N)_{t\in[0,t_0]}\right)_{N\in\mathbb{N}}$ to the process $(Z_t)_{t\in[0,t_0]}$. To this aim, we need the following preparatory result:

Lemma 3.9. *Fix N_0 sufficiently large such that $\mathcal{A}_N \geq M$ for every $N \geq N_0$. Then, there is a constant $C > 0$ such that for every $0 \leq \epsilon \leq 1$, $N \geq N_0$ and $0 \leq s \leq t \leq t_0$*

$$\mathbb{E}\left[|\widetilde{X}_t^N - \widetilde{X}_s^N|^2\right]$$
$$\leq C\Bigg(v(|t-s|) + \Big(\frac{1}{m_N^*}\int\limits_{(0,\infty)} \left[\frac{\gamma}{y}\mathbb{1}_{\{0<y\leq 1\}} + \frac{\gamma^\epsilon}{y^{2-\epsilon}}\mathbb{1}_{\{y>1\}}\right]\mu_N(dy)\Big)\,|t-s|^{1+\epsilon}\Bigg)$$

Proof. Let $0 \leq s \leq t \leq T$. We write (cp. the proof of Lemma 3.3)

$$\begin{aligned}
&\widetilde{X}_t^N - \widetilde{X}_s^N \\
=\;& \frac{1}{\mathcal{A}_N}\sum_{k=1}^N \beta_{k,N}\int_s^t \widetilde{U}_\tau^{k,N}\left(1 - e^{-\frac{\mathcal{A}_N}{M}(t-\tau)}\right)d\tau \\
&+\frac{1}{\mathcal{A}_N}\sum_{k=1}^N \beta_{k,N}\int_0^s \widetilde{U}_\tau^{k,N}\left(e^{-\frac{\mathcal{A}_N}{M}(s-\tau)} - e^{-\frac{\mathcal{A}_N}{M}(t-\tau)}\right)d\tau \\
=\;& (I) + (II)
\end{aligned}$$

By standard calculations, making use of (16),

$$\begin{aligned}
&\mathbb{E}[|(I)|^2] \\
=\;& \left(\frac{2\sigma}{\gamma\mathcal{A}_N^2}\sum_{k=1}^N \beta_{k,N}^2\right)\int_s^t\int_s^\tau\int_{(0,\infty)} \frac{e^{-\gamma y(\tau-\rho)}}{y}(1 - e^{-\frac{\mathcal{A}_N}{M}(t-\rho)}) \\
&\times(1 - e^{-\frac{\mathcal{A}_N}{M}(t-\tau)})\mu_N(dy)d\rho d\tau \\
\leq\;& \left(\frac{2\sigma}{\gamma\mathcal{A}_N^2}\sum_{k=1}^N \beta_{k,N}^2\right)\int_s^t\int_{(0,\infty)}\int_s^\tau \frac{e^{-\gamma y(\tau-\rho)}}{y}d\rho\,\mu_N(dy)\,d\tau \\
=\;& \left(\frac{2\sigma}{\gamma^2\mathcal{A}_N^2}\sum_{k=1}^N \beta_{k,N}^2\right)\int_s^t\int_{(0,\infty)}(1 - e^{-\gamma y(\tau-s)})y^{-2}\,\mu_N(dy)\,d\tau \\
=\;& \left(\frac{2\sigma}{\gamma^2\mathcal{A}_N^2}\sum_{k=1}^N \beta_{k,N}^2\right)\int_0^{t-s}\int_{(0,\infty)}(1 - e^{-\gamma y\tau})y^{-2}\,\mu_N(dy)\,d\tau \\
=\;& \left(\frac{2\sigma m_N^*}{\gamma^2\mathcal{A}_N^2}\sum_{k=1}^N \beta_{k,N}^2\right)\int_0^{t-s}\frac{\mathfrak{e}_N(\tau)}{m_N^*}\,d\tau \leq \left(\frac{2\sigma m_N^*}{\gamma^2\mathcal{A}_N^2}\sum_{k=1}^N \beta_{k,N}^2\right)v(|t-s|).
\end{aligned}\tag{28}$$

For the estimate of (II), we fix some $0 \leq \epsilon \leq 1$ and assume that $\mathcal{A}_N \geq M$. Moreover, we abbreviate

$$g(t,s;\tau) = e^{-\frac{\mathcal{A}_N}{M}(s-\tau)} - e^{-\frac{\mathcal{A}_N}{M}(t-\tau)}$$

Then,

$$\begin{aligned}
&\mathbb{E}[|(II)|^2] \\
=\;& \left(\frac{2\sigma}{\gamma\mathcal{A}_N^2}\sum_{k=1}^N \beta_{k,N}^2\right)\int_0^s\int_{(0,\infty)}\int_0^\tau \frac{e^{-\gamma y(\tau-\rho)}}{y} g(t,s;\rho)d\rho\,\mu_N(dy)\,g(t,s;\tau)\,d\tau
\end{aligned}$$

Now,

$$\begin{aligned}
&\int_0^\tau \frac{e^{-\gamma y(\tau-\rho)}}{y} g(t,s;\rho)d\rho \\
=\;& \frac{1}{y(\gamma y+\frac{\mathcal{A}_N}{M})}\left(e^{-\frac{\mathcal{A}_N}{M}(s-\tau)} - e^{-\frac{\mathcal{A}_N}{M}(t-\tau)} + (e^{-\frac{\mathcal{A}_N}{M}t} - e^{-\frac{\mathcal{A}_N}{M}s})e^{-\gamma y\tau}\right) \\
\leq\;& \frac{1}{y(\gamma y+\frac{\mathcal{A}_N}{M})}\left(e^{-\frac{\mathcal{A}_N}{M}(s-\tau)} - e^{-\frac{\mathcal{A}_N}{M}(t-\tau)}\right)^{1-\epsilon}\left(e^{-\frac{\mathcal{A}_N}{M}(s-\tau)} - e^{-\frac{\mathcal{A}_N}{M}(t-\tau)}\right)^{\epsilon} \\
\leq\;& \frac{1}{y}\frac{1}{(\gamma y+\frac{\mathcal{A}_N}{M})^{\epsilon}}\frac{1}{(\gamma y+\frac{\mathcal{A}_N}{M})^{1-\epsilon}}\left(e^{-\frac{\mathcal{A}_N}{M}(s-\tau)} - e^{-\frac{\mathcal{A}_N}{M}(t-\tau)}\right)^{\epsilon} \\
\leq\;& \left(\frac{M}{y\mathcal{A}_N}\mathbb{1}_{\{0<y\leq 1\}} + \frac{M^\epsilon}{y^{2-\epsilon}\gamma^{1-\epsilon}\mathcal{A}_N^\epsilon}\mathbb{1}_{\{y>1\}}\right)\left(e^{-\frac{\mathcal{A}_N}{M}(s-\tau)} - e^{-\frac{\mathcal{A}_N}{M}(t-\tau)}\right)^{\epsilon} \\
\leq\;& \left(\frac{1}{y}\mathbb{1}_{\{0<y\leq 1\}} + \frac{1}{y^{2-\epsilon}\gamma^{1-\epsilon}}\mathbb{1}_{\{y>1\}}\right)\left(\frac{M}{\mathcal{A}_N}\left(e^{-\frac{\mathcal{A}_N}{M}(s-\tau)} - e^{-\frac{\mathcal{A}_N}{M}(t-\tau)}\right)\right)^{\epsilon} \\
\leq\;& \left(\frac{1}{y}\mathbb{1}_{\{0<y\leq 1\}} + \frac{1}{y^{2-\epsilon}\gamma^{1-\epsilon}}\mathbb{1}_{\{y>1\}}\right)|t-s|^{\epsilon}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\mathbb{E}[|(II)|^2] \\
\leq\;& \left(\frac{2\sigma}{\gamma\mathcal{A}_N^2}\sum_{k=1}^N \beta_{k,N}^2\right)\left(\int\limits_{(0,\infty)}\left[\frac{1}{y}\mathbb{1}_{\{0<y\leq 1\}} + \frac{1}{y^{2-\epsilon}\gamma^{1-\epsilon}}\mathbb{1}_{\{y>1\}}\right]\mu_N(dy)\right) \\
&\times\left(\int_0^s g(t,s,\tau)d\tau\right)|t-s|^{\epsilon}
\end{aligned}$$

Noting that

$$\int_0^s g(t,s,\tau)d\tau \quad\leq\quad \frac{M}{\mathcal{A}_N}\left(1-e^{-\frac{\mathcal{A}_N}{M}(t-s)}\right)\leq |t-s|,$$

we finally obtain

$$\begin{aligned}
\mathbb{E}[|(II)|^2] \leq& \left(\frac{2\sigma m_N^*}{\gamma\mathcal{A}_N^2}\sum_{k=1}^N \beta_{k,N}^2\right) \\
&\times\left(\frac{1}{m_N^*}\int\limits_{(0,\infty)}\left[\frac{1}{y}\mathbb{1}_{\{0<y\leq 1\}} + \frac{1}{y^{2-\epsilon}\gamma^{1-\epsilon}}\mathbb{1}_{\{y>1\}}\right]\mu_N(dy)\right)|t-s|^{1+\epsilon}.
\end{aligned}$$

Noting that the factor $m_N^* \mathcal{A}_N^{-2} \sum_{k=1}^N \beta_{k,N}^2$ is bounded in N by Assumption 2.6, the proof is finished. $\square$

Lemma 3.10. *Let $\mathcal{X}$ be the space of continuous functions $C[0,t_0]$ endowed with its Borel σ-field $\mathcal{B}(\mathcal{X})$. In the setting of Theorem 2.1 (ii), the sequence of processes $\big((X_t^N)_{t\in[0,t_0]}\big)_{N\in\mathbb{N}}$, considered as $(\mathcal{X},\mathcal{B}(\mathcal{X}))$-valued random elements, converges $\sigma(\mathcal{M})$-mixing in distribution to the process $(Z_t)_{t\in[0,t_0]}$.*

Proof. Note that each process $(X_t^N)_{t\in[0,t_0]}$, $N\in\mathbb{N}$, as well as each process $(\widetilde{X}_t^N)_{t\in[0,t_0]}$, $N\in\mathbb{N}$, has continuous paths as (a part of) a solution of a multidimensional linear in a narrow sense stochastic differential equation driven by a (multidimensional) Wiener process. Further, let us first show that the sequence of processes $\left((\widetilde{X}_t^N)_{t\in[0,t_0]}\right)_{N\in\mathbb{N}}$ is tight in $\mathcal{X}$. By Corollary 14.9 of [12], it is enough to check that $(\widetilde{X}_0^N)_{N\in\mathbb{N}}$ is tight in $\mathbb{R}$ and $\mathbb{E}\left[|\widetilde{X}_t^N-\widetilde{X}_s^N|^q\right]\le C|t-s|^p$ for some $p>1$, $q,C>0$ not depending on N. Recall that $\widetilde{X}_0^N=0$ for all $N\in\mathbb{N}$, hence $(\widetilde{X}_0^N)_{N\in\mathbb{N}}$ is tight.

Consider the setting of Assumption 2.4 (i). We have by Lemma 3.9 (with ε, ϵ and N_0 as in Assumption 2.4 (i))

$$\mathbb{E}\left[|\widetilde{X}_t^N-\widetilde{X}_s^N|^2\right]\le C|t-s|^{1+\epsilon\wedge\varepsilon},\qquad t,s\in[0,t_0],\quad N>N_0.$$

Hence, the family $\left((\widetilde{X}_t^N)_{t\in[0,t_0]}\right)_{N\in\mathbb{N}}$ is tight in $\mathcal{X}$.

In the setting of Assumption 2.4 (ii), we have by Lemma 3.9 with $\epsilon:=0$

$$\mathbb{E}\left[|\widetilde{X}_t^N-\widetilde{X}_s^N|^2\right]\le C|t-s|,\qquad t,s\in[0,t_0],\quad N>N_0.$$

Since, in the setting of Assumption 2.4 (ii), $\widetilde{X}_t^N-\widetilde{X}_s^N$ is Gaussian (and not only conditionally Gaussian as in the case of random masses $m_{k,N}$), we get,

$$\mathbb{E}\left[|\widetilde{X}_t^N-\widetilde{X}_s^N|^4\right]\le 3C^2|t-s|^2.$$

Therefore, the family $\left((\widetilde{X}_t^N)_{t\in[0,t_0]}\right)_{N\in\mathbb{N}}$ is again tight in $\mathcal{X}$.

Further, it follows from the proof of Lemma 3.8 that the sequence of stochastic processes $\left((\widetilde{X}_t^N)_{t\in[0,t_0]}\right)_{N\in\mathbb{N}}$ converges $\sigma(\mathcal{M})$-mixing in finite dimensional distributions to the process $(Z_t)_{t\in[0,t_0]}$. Therefore, the processes $\left((\widetilde{X}_t^N)_{t\in[0,t_0]}\right)_{N\in\mathbb{N}}$ converge to the process $(Z_t)_{t\in[0,t_0]}$ $\sigma(\mathcal{M})$-mixing in distribution by Proposition 3.9 of [9].

Moreover, we have for any $\varepsilon>0$ by Markov's inequality

$$\mathbb{P}\left(\left\|X^N-\widetilde{X}^N\right\|_{C([0,t_0])}>\varepsilon\right)\le\frac{1}{\varepsilon^2}\mathbb{E}\left[\left\|X^N-\widetilde{X}^N\right\|_{C([0,t_0])}^2\right]$$
$$=\frac{1}{\varepsilon^2}\mathbb{E}\left[\sup_{t\in[0,t_0]}\left|X_t^N-\widetilde{X}_t^N\right|^2\right]\longrightarrow 0,\qquad N\to\infty,$$

by Lemma 3.2. Therefore, the sequence of processes $\big((X_t^N)_{t\in[0,t_0]}\big)_{N\in\mathbb{N}}$ converges $\sigma(\mathcal{M})$-mixing in distribution to the process $(Z_t)_{t\in[0,t_0]}$ by Theorem 3.7 (a) of [9]. In particular, the sequence of processes $\big((X_t^N)_{t\in[0,t_0]}\big)_{N\in\mathbb{N}}$ converges to the process $(Z_t)_{t\in[0,t_0]}$ in distribution (cp. Remark 3.1). □

4 Acknowledgments

Gianni Pagnini acknowledges the support by the Basque Government through the BERC 2022–2025 program and by the Ministry of Science and Innovation: BCAM Severo Ochoa accreditation CEX2021-001142-S / MICIN / AEI / 10.13039/501100011033.

Yana Kinderknecht acknowledges the Basque Center of Applied Mathematics (BCAM) and Kassel University for the financial support of her regular research visits to BCAM.

References

[1] E. Barkai, Y. Garini, and R. Metzler. Strange kinetics of single molecules in living cells. *Phys. Today*, 65(8):29, 2012.

[2] A. V. Chechkin, F. Seno, R. Metzler, and I. M. Sokolov. Brownian yet non-Gaussian diffusion: From superstatistics to subordination of diffusing diffusivities. *Phys. Rev. X*, 7:021002, 2017.

[3] M. Chubynsky and G. Slater. Diffusing diffusivity: A model for anomalous, yet Brownian, diffusion. *Phys. Rev. Lett.*, 113:098302, 2014.

[4] M. A. F. dos Santos and L. Menon Junior. Random diffusivity models for scaled Brownian motion. *Chaos Solitons Fract.*, 144:110634, 2021.

[5] M. D'Ovidio, S. Vitali, V. Sposini, O. Sliusarenko, P. Paradisi, G. Castellani, and G. Pagnini. Centre-of-mass like superposition of Ornstein–Uhlenbeck processes: A pathway to non-autonomous stochastic differential equations and to fractional diffusion. *Fract. Calc. Appl. Anal.*, 21:1420–1435, 2018.

[6] I. Golding and E. C. Cox. Physical nature of bacterial cytoplasm. *Phys. Rev. Lett.*, 96:098102, 2006.

[7] P. Grassia. Dissipation, fluctuations, and conservation laws. *Am. J. Phys.*, 69:113–119, 2001.

[8] J. Grimes, Z. Koszegi, Y. Lanoiselée, T. Miljus, S. O'Brien, T. Stepniewski, B. Medel–Lacruz, M. Baidya, M. Makarova, R. Mistry, J. Goulding, J. Drube, C. Hoffmann, D. Owen, A. Shukla, J. Selent, S. Hill, and D. Calebiro. Plasma membrane preassociation drives β-arrestin coupling to receptors and activation. *Cell*, 186:2238–2255, 2023.

[9] E. Häusler and H. Luschgy. *Stable convergence and stable limit theorems*, volume 74 of *Probab. Theory Stoch. Model.* Cham: Springer, 2015.

[10] F. Höfling and T. Franosch. Anomalous transport in the crowded world of biological cells. *Rep. Prog. Phys.*, 76:046602, 2013.

[11] Y. Itto and C. Beck. Superstatistical modelling of protein diffusion dynamics in bacteria. *J. R. Soc. Interface*, 18:20200927, 2021.

[12] O. Kallenberg. *Foundations of modern probability.* Probab. Appl. New York, NY: Springer, 1997.

[13] J. Klafter, S.-C. Lim, and R. Metzler, editors. *Fractional Dynamics: Recent Advances.* World Scientific, Singapore, 2012.

[14] J. Klafter and I. M. Sokolov. Anomalous diffusion spread its wings. *Physics World*, 18:29–32, 2005.

[15] R. Klages, G. Radons, and I. M. Sokolov, editors. *Anomalous Transport: Foundations and Applications.* Wiley–VCH Verlag GmbH & Co. KGaA, Weinheim, 2008.

[16] N. Korabel, D. Han, A. Taloni, G. Pagnini, S. Fedotov, V. Allan, and T. A. Waigh. Local analysis of heterogeneous intracellular transport: slow and fast moving endosomes. *Entropy*, 23:958, 2021.

[17] N. Korabel, A. Taloni, G. Pagnini, V. Allan, S. Fedotov, and T. A. Waigh. Ensemble heterogeneity mimics ageing for endosomal dynamics within eukaryotic cells. *Sci. Rep.*, 13:8789, 2023.

[18] A. Maćkała and M. Magdziarz. Statistical analysis of superstatistical fractional Brownian motion and applications. *Phys. Rev. E*, 99:012143, 2019.

[19] M. Magdziarz, A. Weron, K. Burnecki, and J. Klafter. Fractional Brownian motion versus the Continuous-Time Random Walk: A simple test for subdiffusive dynamics. *Phys. Rev. Lett.*, 103:180602, 2009.

[20] C. Manzo and M. F. Garcia-Parajo. A review of progress in single particle tracking: from methods to biophysical insights. *Rep. Progr. Phys.*, 78:124601, 2015.

[21] C. Manzo, J. A. Torreno–Pina, P. Massignan, G. J. Lapeyre, M. Lewenstein, and M. F. Garcia–Parajo. Weak ergodicity breaking of receptor motion in living cells stemming from random diffusivity. *Phys. Rev. X*, 5:011021, 2015.

[22] R. Metzler and J. Klafter. The restaurant at the end of the random walk: recent developments in fractional dynamics descriptions of anomalous dynamical processes. *J. Phys. A: Math. Theor.*, 37(31):R161–R208, 2004.

[23] D. Molina-García, T. Minh Pham, P. Paradisi, C. Manzo, and G. Pagnini. Fractional kinetics emerging from ergodicity breaking in random media. *Phys. Rev. E*, 94:052147, 2016.

[24] A. Mura and G. Pagnini. Characterizations and simulations of a class of stochastic processes to model anomalous diffusion. *J. Phys. A: Math. Theor.*, 41:285003, 2008.

[25] C. Runfola, S. Vitali, and G. Pagnini. The Fokker–Planck equation of the superstatistical fractional Brownian motion with application to passive tracers inside cytoplasm. *R. Soc. Open Sci.*, 9:221141, 2022.

[26] R. L. Schilling, R. Song, and Z. Vondraček. *Bernstein functions*, volume 37 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, second edition, 2012. Theory and applications.

[27] H. Shen, L. J. Tauzin, R. Baiyasi, W. Wang, N. Moringo, B. Shuang, and C. F. Landes. Single particle tracking: from theory to biophysical applications. *Chem. Rev.*, 117:7331–7376, 2017.

[28] F. Simon, L. E. Weiss, and S. van Teeffelen. A guide to single-particle tracking. *Nat. Rev. Methods Primers*, 4:66, 2024.

[29] V. Sposini, D. S. Grebenkov, R. Metzler, G. Oshanin, and F. Seno. Universal spectral features of different classes of random-diffusivity processes. *New J. Phys.*, 22:063056, 2020.

[30] J. Szymanski and M. Weiss. Elucidating the origin of anomalous diffusion in crowded fluids. *Phys. Rev. Lett.*, 103:038102, 2009.

[31] W. Wang, A. G. Cherstvy, A. V. Chechkin, S. Thapa, F. Seno, X. Liu, and R. Metzler. Fractional Brownian motion with random diffusivity: emerging residual nonergodicity below the correlation time. *J. Phys. A: Math. Theor.*, 53:474001, 2020.

[32] W. Wang, F. Seno, I. M. Sokolov, A. V. Chechkin, and R. Metzler. Unexpected crossovers in correlated random-diffusivity processes. *New J. Phys.*, 22:083041, 2020.

[33] M. Weiss. Single-particle tracking data reveal anticorrelated fractional Brownian motion in crowded fluids. *Phys. Rev. E*, 88:010101(R), 2013.