General Transfer Formula for Stochastic Integral with respect to Multifractional Brownian Motion

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Abstract

In this work we show how results from stochastic integration with respect to multifractional Brownian motion (mBm) can be simply deduced from results of stochastic integration with respect to fractional Brownian motion (fBm), by using a"Transfer Principle". To illustrate this fact, we prove an Itô formula for integral *wrt* mBm by deriving it from Itô formula for integral *wrt* fBm, of any Hurst index H in $(0, 1)$.

Keywords: Fractional and multifractional Brownian motions, White Noise theory, Wick-Itô integral

1 Introduction

1.1 Fractional and multifractional Brownian motions

Fractional Brownian motion (fBm) is a centered Gaussian process with features that make it a useful model in various applications such as financial and Internet traffic modeling, image analysis and synthesis, physics, geophysics and more. These features include self-similarity, long range dependence and the ability to match any prescribed constant local regularity. For any H in $(0, 1)$, the covariance function of a fBm of Hurst index H, denoted R_H, reads:

$$R_H(t, s) := \frac{\gamma_H}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}),$$

where γ_H is a positive constant[1]. The fact that most of the properties of fBm are governed by the single real H restricts its application in some situations. In particular, its Hölder exponent remains the same all along its trajectory. This does not seem to be adapted to describe adequately phenomena that may have different regularity along the time. For instance, it seems

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[1]The fBm is said to be normalized when $\gamma_H = 1$.

better to model natural terrains with a 3-dimensional Gaussian process, the regularity of which can vary along the time, instead of using a 3-dimensional fBm with a single constant parameter H. In addition, long range dependence requires $H > 1/2$, and thus imposes paths smoother than the ones of Brownian motion. Multifractional Brownian motion was introduced to overcome these limitations. The basic idea is to replace the real H of $(0,1)$ by a deterministic function $t \mapsto h(t)$ ranging in $(0,1)$. Several definitions of mBm exist (*e.g.* [16, 2, 17]) and the reader interested in the evolution of these definitions may refer to [15]. Let first recall that a fractional Brownian field is a Gaussian process $\mathbf{B} := (\mathbf{B}(t,H))_{(t,H)\in\mathbf{R}\times(0,1)}$ such that, for every H in $(0,1)$, the process $\mathbf{B}^H := (\mathbf{B}(t,H))_{t\in\mathbf{R}}$ is a fractional Brownian motion. A multifractional Brownian motion is simply a "path traced" on a fractional Brownian field. More precisely, it is defined as follows. Let $h : \mathbf{R} \to (0,1)$ be a deterministic and continuous function. The mBm of functional parameter h, noted $B^h := (B^h_t)_{t\in\mathbf{R}}$, is defined by setting: $B^h_t := \mathbf{B}(t, h(t))$, for every real t. A word on notation: $B^H_{\cdot}$ or $B^{h(t)}_{\cdot}$ will always denote an fBm with Hurst index H or $h(t)$, while $B^h_{\cdot}$ will stand for an mBm. Hence it is clear that $B^h_t = B^{h(t)}_t$, for every real t. The function h gives the regularity of the mBm at any point t. We will say that a mBm is normalized when its covariance function, denoted R_h, is given by:

$$R_h(t,s) = \frac{c^2_{h_{t,s}}}{c_{h(t)}c_{h(s)}} \left[\tfrac{1}{2}\left(|t|^{2h_{t,s}} + |s|^{2h_{t,s}} - |t-s|^{2h_{t,s}}\right)\right] \tag{1.1}$$

where $h_{t,s} := \frac{h(t)+h(s)}{2}$ and where, for every x in $(0,1)$, $c_x := \left(\frac{2\cos(\pi x)\Gamma(2-2x)}{x(1-2x)}\right)^{\frac{1}{2}}$. As one can see, mBm does not point out one single Gaussian process but a class of Gaussian processes. Through this paper, and in order to simplify computations, we will chose a particular mBm (see (3.4)).

1.2 Approximation of mBm by piecewise fBms

Since an mBm is just a continuous path "traced" on a fractional Brownian field, it is a natural question to enquire whether it may be approximated by patching adequately chosen fBms, and in which sense. The answer to this question has been given in [15, Theorem 2.1]. Before giving the precise result, we need to introduce some additional notations. Let $T > 0$. Let $(q_n)_{n\in\mathbb{N}}$ be an increasing sequence of integers, which tends to $+\infty$, such that $q_0 := 1$. For n in $\mathbb{N}$, define $x^{(n)} := \{x^{(n)}_k; k \in [\![0, q_n]\!]\}$ where $x^{(n)}_k := \frac{kT}{q_n}$ (for integers p and q with $p < q$, $[\![p,q]\!]$ denotes the set $\{p, p+1, \cdots, q\}$). Define, for n in $\mathbb{N}$, the partition $\mathcal{A}_n := \{[x^{(n)}_k, x^{(n)}_{k+1}); k \in [\![0, q_n - 1]\!]\} \cup \{x^{(n)}_{q_n}\}$. Thus $\mathcal{A} := (\mathcal{A}_n)_{n\in\mathbb{N}}$ is a sequence of partitions of $[0,T]$ with mesh size that tends to 0 as n tends to $+\infty$. For t in $[0,T]$ and n in $\mathbb{N}$ there exists a unique integer p in $[\![0, q_n - 1]\!]$ such that $x^{(n)}_p \le t < x^{(n)}_{p+1}$. We will note $x^{(n)}_t$ the real $x^{(n)}_p$ in the sequel. The sequence $(x^{(n)}_t)_{n\in\mathbb{N}}$ converges to t as n tends to $+\infty$. Besides, define for n in $\mathbb{N}$, the function $h_n : [0,T] \to (0,1)$ by setting $h_n(T) = h(T)$ and, for any t in $[0,T)$, $h_n(t) := h(x^{(n)}_t)$. The sequence of step functions $(h_n)_{n\in\mathbb{N}}$ converges pointwise to h on $[0,T]$. Define, for t in $[0,T]$ and n in $\mathbb{N}$, the process

$$B^{h_n}_t := \mathbf{B}(t, h_n(t)) = \sum_{k=0}^{q_n-1} \mathbf{1}_{[x^{(n)}_k, x^{(n)}_{k+1})}(t)\, \mathbf{B}(t, h(x^{(n)}_k)) + \mathbf{1}_{\{T\}}(t)\, \mathbf{B}(T, h(T)). \tag{1.2}$$

Heuristically, we divide $[0,T)$ into "small" intervals $[t_i, t_{i+1})$, and replace on each of these B^h by the fBm B^{H_i} where $H_i := h(t_i)$. Note that, despite the notation, the process B^{h_n} is not an mBm, as h_n is not continuous. We believe however there is no risk of confusion in using this notation. B^{h_n} is almost surely càdlàg and discontinuous at times $x^{(n)}_k$, k in $[\![0, q_n]\!]$. Define the hypothesis $\mathcal{H}$ as follow.

$$\forall [a,b]\times[c,d]\subset \mathbf{R}\times(0,1), \exists(\Lambda,\delta)\in(\mathbf{R}_+^*)^2, s.t.$$
$$\mathbf{E}[(\mathbf{B}(t,H)-\mathbf{B}(s,H'))^2]\leq \Lambda\left(|t-s|^{2c}+|H-H'|^{\delta}\right), \tag{$\mathcal{H}$}$$
for all $(t,s,H,H')\in[a,b]^2\times[c,d]^2$.

The following approximation theorem, which is a corollary of [15, Theorem 2.1], shows that a mBm naturally appears as a limit of sums of fBms.

Theorem 1.1 (Approximation theorem)**.** *Let $\mathbf{B}$ be a fractional Brownian field, $h:\mathbf{R}\to(0,1)$ be a continuous deterministic function and B^h be the associated mBm. Let $[a,b]$ be a compact interval of $\mathbf{R}$, $\mathcal{A}$ be a sequence of partitions as defined above, and consider the sequence of processes defined in* (1.2)*. Then:*

1. *If $\mathbf{B}$ is such that the map $R:(t,H,s,H')\mapsto \mathbf{E}[\mathbf{B}(t,H)\,\mathbf{B}(s,H')]$ is continuous on $([a,b]\times h([a,b]))^2$ then the sequence of processes $(B^{h_n})_{n\in\mathbb{N}}$ converges in $L^2(\Omega)$ to B^h,i.e.*
$$\forall t\in[a,b],\ \lim_{n\to+\infty}\mathbf{E}\big[\big(B_t^{h_n}-B_t^h\big)^2\big]=0.$$
2. *If $\mathbf{B}$ satisfies assumption $(\mathcal{H})$ and if h is β-Hölder continuous for some positive real β, then the sequence of processes $(B^{h_n})_{n\in\mathbb{N}}$ converges*

   (**i**) *in law to B^h, i.e.* $\{B_t^{h_n};\ t\in[a,b]\}\xrightarrow[n\to+\infty]{law}\{B_t^h;\ t\in[a,b]\}$.

   (**ii**) *almost surely to B^h, i.e.* $P\big(\{\forall t\in[a,b],\ \lim_{n\to+\infty} B_t^{h_n}=B_t^h\}\big)=1$.

1.3 Integral with respect to mBm as a sum of integrals with respect to fBm

The results presented in Theorem 1.1 above suggest that one may define stochastic integrals with respect to mBm as limits of integrals with respect to approximating fBms. This is actually true and has been established in [15, Theorem 3.3], not only in the framework of Wick-Itô integral but also in the Skorohod and pathwise ones.
From now one, and for the remaining part of this paper we assume that $h:[0,T]\to(0,1)$ is a deterministic function of class C^1. The integral of a process Y with respect to a fBm of Hurst index H, using White Noise Theory is named fractional Wick-Itô integral and denoted $\int_{[0,T]} Y_t\, d^{\diamond}B_t^H$ or $\int_0^T Y_t\, d^{\diamond}B_t^H$ in the sequel. The integral of a process Y with respect to a mBm of functional parameter h, using White Noise Theory is named multifractional Wick-Itô integral and denoted $\int_{[0,T]} Y_t\, d^{\diamond}B_t^h$ or $\int_0^T Y_t\, d^{\diamond}B_t^h$ in the sequel. The fractional Wick-Itô integral was developed in [3, (26)] in the framework of Pettis integral and has been particularized in the framework of Bochner integral in [11, Def.3.2.]. The multifractional Wick-Itô integral has been developed in the framework of Pettis integral in [14] and has been particularized in the framework of Bochner integral in [11] and in [15]. The exact definition and meaning of these integrals will be given in Section 3.3. The way used in [15] to construct an integral *wrt* mBm using approximating integrals *wrt* fBms, that will be denoted $I_{p_0}^h(T)$ and named limiting fractional Wick-Itô integral in the sequel, is the following. Keeping in mind the notations given at the begining of Section 1.2, we denote $\int_0^T\ Y_s\ d^{\diamond}B_s^{h_n}$ the integral with respect to lumped fBms *i.e.*
$$\int_0^T\ Y_s\ d^{\diamond}B_s^{h_n} := \sum_{k=0}^{q_n-1}\int_0^T\ \mathbf{1}_{[x_k^{(n)},x_{k+1}^{(n)})}(s)\ \ Y_s\ dB_s^{\diamond h(x_k^{(n)})},\ n\in\mathbf{N}. \tag{1.3}$$

In other words, $\int_0^T Y_s \, d^\diamond B_s^{h_n}$ is just a sum of integrals with respect to fBm of different Hurst indices. Then, assuming the following quantities exist, in a sense to be precised in Section 3.3, we set

$$I_{p_0}^h(T) := \lim_{n\to\infty} \int_0^T Y_s \, d^\diamond B_s^{h_n} + \int_0^T h'(s) \, Y_s \diamond \tfrac{\partial \mathbf{B}}{\partial H}(s, h(s)) \, ds,$$

where $\diamond$ denotes the Wick product and $\frac{\partial \mathbf{B}}{\partial H}$ is the derivative of the field $\mathbf{B}$, with respect to its second variable. Besides, [15, Theorem 4.8] shows that the multifractional Wick-Itô integral of any process Y, can be obtained as limits of fractional Wick-Itô integral. This means that one has the following equality.

$$\int_0^T Y_t \, d^\diamond B_t^h = I_{p_0}^h(T).$$

Since the limiting fractional Wick-Itô integral does coincide with the multifractional Wick-Itô integral built in [14], the motivation of the present work thus appears clearly and can be summarized in the two following questions.

- If we can derive the existence of a multifractional Wick-Itô integral from limiting fractional Wick-Itô integrals, can we derive some formulas involving stochastic integrals *wrt* mBm from the same formulas, but involving stochastic integrals *wrt* fBm, by simply taking the limit?
- If the answer is positive, do we just have to use the limiting argument used in [15] to do so?

Outline of the paper

This paper is structured as follow. We will briefly present, in the next section, the Transfer Principle, as well as, the Itô formula for integral with respect to mBm, which derives from it. The proof of the transfer principle will be presented and proved in Section 4.The required technical background on White Noise Theory, and on integral with respect to fBm and mBm is presented Section in 3. The auxiliary results required on fractional fields are given in Section 5. Finally, an Itô formula for integral with respect to mBm, holding in $L^2(\Omega)$ is established, in Section 6, as a consequence of the transfer principle. A short appendix on Bochner integral completes this work.

2 Itô's formula for mBm via the transfer principle

We are now ready to present the main application of the transfer principle. Denote, for any q in $\mathbf{N}$, $C^{1,q}([0,T] \times \mathbf{R}, \mathbf{R})$ the set of functions of two variables which belongs to, $C^1([0,T], \mathbf{R})$ as function of their first variable, and to $C^q(\mathbf{R}, \mathbf{R})$, as function of their second variable (such that $\frac{\partial^{i+j} f}{\partial t^i \partial x^j}$ is continuous on $[0,T] \times \mathbf{R}$). In all this section, let $T > 0$ be fixed and let f be in $C^{1,2}([0,T] \times \mathbf{R}, \mathbf{R})$. If there exists (C, λ) in $(\mathbf{R}_+^*)^2$, such that:

$$\forall (t,x) \in [0,T] \times \mathbf{R}, \qquad \max\Big\{ \big|f(t,x)\big|, \big|\tfrac{\partial f}{\partial t}(t,x)\big|, \big|\tfrac{\partial f}{\partial x}(t,x)\big|, \big|\tfrac{\partial^2 f}{\partial x^2}(t,x)\big| \Big\} \le C e^{\lambda x^2}, \qquad (\mathscr{E}_{C,\lambda})$$

we will write that f fulfills $(\mathscr{E}_{C,\lambda})$ or shortly that $(\mathscr{E}_{C,\lambda})$ is fulfilled. Starting from Itô formula with respect to fBm, established in [4, Theorem 5.3], and which reads:

$$f(T, B_T^H) = f(0,0) + \int_0^T \frac{\partial f}{\partial t}(t, B_t^H) \, dt + \int_0^T \frac{\partial f}{\partial x}(t, B_t^H) \, d^\diamond B_t^H + H \int_0^T \frac{\partial^2 f}{\partial x^2}(t, B_t^H) \, t^{2H-1} \, dt, \tag{2.1}$$

our goal is to prove the following Itô formula for mBm.

Theorem 2.1. *Let $h:[0,T]\to(0,1)$ be a function of class C^1. Assume moreover that there exist $C>0$ and λ in $(0,(4\max_{(t,H)\in\mathscr{D}_T^h} t^{2H})^{-1})$ such that $(\mathscr{E}_{C,\lambda})$ holds. Then the following equality holds in (L^2):*

$$f(T,B_T^h)=f(0,0)+\int_0^T \frac{\partial f}{\partial t}(t,B_t^h)\,dt+\int_0^T \frac{\partial f}{\partial x}(t,B_t^h)\,d^{\diamond}B_t^h\;+\frac{1}{2}\int_0^T \frac{\partial^2 f}{\partial x^2}(t,B_t^h)\,R_h'(t)\,dt, \tag{2.2}$$

where R_h denotes the variance function of the mBm B^h, R_h' its derivative and $\mathscr{D}_T^h:=[0,T]\times h([0,T])$.

Let's now briefly state the transfer principle which allows to prove the Itô formula for Multifractional Brownian Motion introduced above, from (2.1). $\mathbf{B}:=(\mathbf{B}(t,H))_{(t,H)\in\mathscr{D}_T^h}$ still denote the fractional Brownian field introduced at Section 1. Denote $(S)^*$ the set of Hida Distributions. Let $X,Y,Z:[0,T]\times(0,1)\to(S)^*$ be three $(S)^*$-valued fields. Given a C^1-function, denoted $h:[0,T]\to(0,1)$, we denote X_t^h for $X(t,h(t))$ and we identify the constant function $t\mapsto H$ with the real number H. We moreover assume that the following equation holds in $(S)^*$, for every $t\in[0,T]$ and H in $Im(h)$,

$$dX_t^H=Y_t^H d^{\diamond}B_t^H+Z_t^H dt.$$

In other words, one has:

$$X_t^H=X_0^H+\int_0^t Y_s^H\diamond W_s^H ds+\int_0^t Z_s^H ds. \tag{2.3}$$

Under suitable assumptions, one can go from the previous equality (where $h\equiv H$ is a constant function) to the following equality, which holds in $(S)^*$ and involved a non necessarily constant function h.

Theorem 2.2. *For every $t\in[0,T]$ and under two assumptions,*

$$X_t^h=X_0^h+\int_0^t Y_s^h\diamond W_s^h ds+\int_0^t Z_s^h ds+\underbrace{\int_0^t h'(s)\left(\frac{\partial X}{\partial H}(s,h(s))-Y_s^h\diamond\frac{\partial \mathbf{B}}{\partial H}(s,h(s))\right)ds}_{=:\text{“Multifractional Correction Term”}}. \tag{2.4}$$

Thus, the only thing we have to do, to go from the fractional Stochastic differential Equation (2.3) to the multifractional differential Equation (2.4) is to:

- replace H by h in (2.3)
- Add a Multifractional Correction Term

The importance of both Theorem 2.1 and 2.2 lies in the fact that: starting with any method that provides a stochastic integral *wrt* fBm (such as White Noise Theory, Skorohod integral, pathwise integral ...), we can derive an entire stochastic calculus *wrt* mBm, only using the existing stochastic calculus *wrt* fBm. This in particular includes Itô and Tanaka formulas both in (L^2) and in the sense of stochastic distributions or Hida's sense (see [14, 15] as well as [14, Theorem 6.1] and in [13, Theorems 3.4 & 4.1]). Beyond the result of both Theorem 2.1 and 2.2, the proof of these latters is important since it shows how one can effectively derive, starting from a known formula that involves stochastic integral *wrt* fBm, the analogue formula but where stochastic integral *wrt* fBm have been replaced by stochastic integral *wrt* mBm. This way of proceeding from stochastic integral *wrt* fBm to stochastic integral *wrt* mBm is independent of the integration method chosen (White Noise Theory, Skorohod integral, pathwise integral...) was foreshadowed from the results presented in [15].

3 Technical background

3.1 Background on White Noise Theory

The following subsection being on purpose extremely short. The reader who is no familiar with white noise theory should refer to [10] and references therein.
Define the measurable space $(\Omega, \mathcal{F})$ by setting $\Omega := \mathscr{S}'(\mathbf{R})$ and $\mathcal{F} := \mathcal{B}(\mathscr{S}'(\mathbf{R}))$, where $\mathcal{B}$ denotes the σ-algebra of Borel sets. Denotes μ the unique probability measure on $(\Omega, \mathcal{F})$ such that, for every f in $L^2(\mathbf{R})$, the map $< ., f >: \Omega \to \mathbf{R}$ defined by $< ., f > (\omega) =< \omega, f >$ (where $<,>$ continuously in $L^2(\mathbf{R})$ extends the action of tempered distributions on Schwartz functions) is a centered Gaussian random variable with variance equal to $\|f\|^2_{L^2(\mathbf{R})}$ under μ. We also denote (L^2) the space $L^2(\Omega, \mathcal{G}, \mu)$ where $\mathcal{G}$ is the σ-field generated by $(< ., f >)_{f \in L^2(\mathbf{R})}$, and for every n in $\mathbf{N}$, define the $n-$th Hermite function by $e_n(x) := (-1)^n \, \pi^{-1/4} (2^n n!)^{-1/2} e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2})$. Denote A the operator defined on $\mathscr{S}(\mathbf{R})$ by $A := -\frac{d^2}{dx^2} + x^2 + 1$ and $\Gamma(A)$ the second quantization operator of A (see [10, Section 4.2]). Denote, for φ in (L^2), $\|\varphi\|^2_0 := \|\varphi\|^2_{(L^2)}$ and, for n in $\mathbf{N}$, let $\mathbb{D}\text{om}(\Gamma(A)^n)$ be the domain of the $n-$th iteration of $\Gamma(A)$. Define the family of norms $(\| \, \|_p)_{p \in \mathbf{Z}}$ by:

$$\|\Phi\|_p := \|\Gamma(A)^p \Phi\|_0 = \|\Gamma(A)^p \Phi\|_{(L^2)}, \ \forall p \in \mathbf{Z}, \quad \forall \Phi \in (L^2) \cap \mathbb{D}\text{om}(\Gamma(A)^p).$$

For p in $\mathbf{N}$, define $(\mathcal{S}_p) := \{\Phi \in (L^2) : \ \Gamma(A)^p \Phi$ exists and belongs to $(L^2)\}$ and define $(\mathcal{S}_{-p})$ as the completion of the space (L^2) with respect to the norm $\| \, \|_{-p}$. As in [10], we let $(\mathcal{S})$ denote the projective limit of the sequence $((\mathcal{S}_p))_{p \in \mathbf{N}}$ and $(\mathcal{S})^*$ the inductive limit of the sequence $((\mathcal{S}_{-p}))_{p \in \mathbf{N}}$. This means that we have the equalities $(\mathcal{S}) = \underset{p \in \mathbf{N}}{\cap} (\mathcal{S}_p)$ (resp. $(\mathcal{S})^* = \underset{p \in \mathbf{N}}{\cup} (\mathcal{S}_{-p})$) and that convergence in $(\mathcal{S})$ (resp. in $(\mathcal{S})^*$) means convergence in $(\mathcal{S}_p)$ for every p in $\mathbf{N}$ (resp. convergence in $(\mathcal{S}_{-p})$ for some p in $\mathbf{N}$).
The space $(\mathcal{S})$ is called the space of stochastic test functions and $(\mathcal{S})^*$ the space of Hida distributions. Since $(\mathcal{S})^*$ is the dual space of $(\mathcal{S})$. We will note $\ll, \gg$ the duality bracket between $(\mathcal{S})^*$ and $(\mathcal{S})$. If ϕ and Φ both belong to (L^2) then we have the equality $\ll \Phi, \varphi \gg = < \Phi, \varphi >_{(L^2)} = \mathbf{E}[\Phi \, \varphi]$. A function $\Phi : \mathbf{R} \to (\mathcal{S})^*$ is called a stochastic distribution process, or an $(\mathcal{S})^*-$process, or a Hida process. A Hida process Φ is said to be differentiable at $t_0 \in \mathbf{R}$ if $\lim\limits_{r \to 0} r^{-1}(\Phi(t_0 + r) - \Phi(t_0))$ exists in $(\mathcal{S})^*$.
The S-transform of an element Φ of $(\mathcal{S}^*)$, noted $S(\Phi)$, is defined as the function from $\mathscr{S}(\mathbf{R})$ to $\mathbf{R}$ given, for every η in $\mathscr{S}(\mathbf{R})$, by $S(\Phi)(\eta) := \ll \Phi, : e^{<.,\eta>} : \gg$, where $: e^{<.,\eta>} :$ is, by definition, $e^{<.,f> - \frac{1}{2}|f|^2_0}$ and where $(| \ \, |_p)_{p \in \mathbf{Z}}$ is the family norms defined by

$$|f|^2_p := \sum_{k=0}^{+\infty} (2k+2)^{2p} < f, e_k >^2_{L^2(\mathbf{R})}, \quad \forall (p, f) \in \mathbf{Z} \times L^2(\mathbf{R}).$$

Finally for every $(\Phi, \Psi) \in (\mathcal{S})^* \times (\mathcal{S})^*$, there exists a unique element of $(\mathcal{S})^*$, called the Wick product of Φ and Ψ and noted $\Phi \diamond \Psi$, such that $S(\Phi \diamond \Psi)(\eta) = S(\Phi)(\eta) \, S(\Psi)(\eta)$ for every η in $\mathscr{S}(\mathbf{R})$. We will use a lot the following results in the sequel.

Lemma 3.1. *For any (p, q) in $\mathbf{N}^2$ and (X, Y) in $(\mathcal{S}_{-p}) \times (\mathcal{S}_{-q})$,*

$$|S(X \diamond Y)(\eta)| \leq \|X\|_{-p} \, \|Y\|_{-q} \, e^{|\eta|^2_{\max\{p;q\}}}.$$

Lemma 3.2. *[10, Remark 2 p.92] For every p in $\mathbf{N}^*$, c in $[1, \infty)$ and every integer $q \geq p + 1$, the following inequality holds for all X and Y in $(\mathcal{S}_{-p})$*

$$\|X \diamond Y\|_{-q} \leq c \, \|X\|_{-p} \, \|Y\|_{-p}. \tag{3.1}$$

3.2 Background on operators M_H and on the derivative of a Gaussian process

We now introduce two operators, denoted M_H and $\frac{\partial M_H}{\partial H}$, that will prove useful for the definition of the integral with respect to fBm and mBm.

Operators M_H and $\frac{\partial M_H}{\partial H}$

Let H be a fixed real in $(0,1)$ and recall that c_H has been defined right after (1.1). Following [5] and references therein, define the operator M_H, specified in the Fourier domain, by $\widehat{M_H(u)}(y) := \frac{\sqrt{2\pi}}{c_H}\, |y|^{1/2-H} \widehat{u}(y)$ for every y in $\mathbf{R}^*$. This operator is well defined on the homogeneous Sobolev space of order $1/2-H$, denoted $L^2_H(\mathbf{R})$ and defined by $L^2_H(\mathbf{R}) := \{u \in \mathscr{S}'(\mathbf{R}) \,:\, \widehat{u} \in L^1_{loc}(\mathbf{R}) \text{ and } \|u\|_H < +\infty\}$, where the norm $\|\ \|_H$ derives from the inner product $<,>_H$ defined on $L^2_H(\mathbf{R})$ by $< u, v >_H := \frac{1}{c_H^2} \int_{\mathbf{R}} |\xi|^{1-2H} \widehat{u}(\xi)\, \overline{\widehat{v}(\xi)}\, d\xi$. The definition of the operator $\frac{\partial M_H}{\partial H}$ is quite similar. More precisely, define for every H in $(0,1)$, the space $\Gamma_H(\mathbf{R}) := \{u \in \mathscr{S}'(\mathbf{R}) \,:\, \widehat{u} \in L^1_{loc}(\mathbf{R}) \text{ and } \|u\|_{\delta_H(\mathbf{R})} < +\infty\}$, where the norm $\|\ \|_{\delta_H(\mathbf{R})}$ derives from the inner product on $\Gamma_H(\mathbf{R})$ defined by setting:

$$< u, v >_{\delta_H} := \frac{1}{c_H^2} \int_{\mathbf{R}} (\beta_H + \ln|\xi|)^2\, |\xi|^{1-2H}\, \widehat{u}(\xi)\, \overline{\widehat{v}(\xi)}\, d\xi.$$

Following [14], define the operator $\frac{\partial M_H}{\partial H}$ from $(\Gamma_H(\mathbf{R}), <,>_{\delta_H(\mathbf{R})})$ to $(L^2(\mathbf{R}), <,>_{L^2(\mathbf{R})})$, in the Fourier domain, by: $\widehat{\frac{\partial M_H}{\partial H}(u)}(y) := -(\beta_H + \ln|y|)\, \frac{\sqrt{2\pi}}{c_H}\, |y|^{1/2-H}\, \widehat{u}(y)$, for every y in $\mathbf{R}^*$. It is easy to verify that the Gaussian field $\mathbf{B} := (\mathbf{B}(t,H))_{t\in[0,T]}$, defined by setting:

$$\mathbf{B}(t,H) :=< ., M_H(\mathbf{1}_{[0,t]}) >, \tag{3.2}$$

is a fractional Brownian field. Indeed, define $\hat{H} := \frac{H+H'}{2}$ and having in mind the definition of R given in Theorem 1.1, the proof of [14, Lemma 3.2 (ii)] allows us to write that:

$$\begin{aligned} R(t,H,s,H') := \mathbf{E}[\mathbf{B}(t,H)\mathbf{B}(s,H')] &= < M_H(\mathbf{1}_{[0,t]}), M_{H'}(\mathbf{1}_{[0,s]}) >_{L^2(\mathbf{R})} \\ &= \tfrac{c^2_{\widehat{H}}}{c_H c_{H'}}\, \big[\tfrac{1}{2}\big(|t|^{2\widehat{H}} + |s|^{2\widehat{H}} - |t-s|^{2\widehat{H}}\big)\big]. \end{aligned} \tag{3.3}$$

for every (s,t,H,H') in $\mathbf{R}^2 \times (0,1)^2$. Thus, the processes $B^H := (B^H_t)_{t\in[0,T]}$ and $B^h := (B^h_t)_{t\in[0,T]}$, defined by:

$$B^H_t := \mathbf{B}(t,H); \qquad \text{and} \qquad B^h_t := B^{h(t)}_t := \mathbf{B}(t,h(t)) \tag{3.4}$$

are, respectively, a fBm of Hurst index H and a mBm of functional parameter h. Thanks to [15, Propositions 3.1, 3.2 and Remark 3] we know that, for every t in $[a,b] \subset \mathbf{R}$, the map $H \mapsto \mathbf{B}(t,H)$ is C^1, in the $L^2(\Omega)$ sense, from $(0,1)$ to $L^2(\Omega)$. Moreover, for all $[a,b]\times[c,d] \subset \mathbf{R}\times(0,1)$, there exists $(\Lambda,\delta) \in (\mathbf{R}^*_+)^2$, such that, for all $(t,s,H,H') \in [a,b]^2 \times [c,d]^2$,

$$\mathbf{E}[(\mathbf{B}(t,H) - \mathbf{B}(t',H'))^2] \leq \Lambda\, \left(|t-t'|^{2c} + |H-H'|^{\delta}\right). \tag{3.5}$$

In particular the time derivative of B^H_t and B^h_t are the $(\mathcal{S})^*$-valued process defined by setting:

$$W^H_t := \sum_{k=0}^{+\infty} M_H(e_k)(t) < ., e_k >, \tag{3.6}$$

$$W^h_t := \sum_{k=0}^{+\infty} M_{h(t)}(e_k)(t) < ., e_k > + h'(t) \sum_{k=0}^{+\infty} \big(\int_0^t \tfrac{\partial M_H}{\partial H}(e_k)(s)\big|_{H=h(t)} ds\big) < ., e_k >, \tag{3.7}$$

where both equalities hold in $(\mathcal{S})^*$. The $(\mathcal{S})^*$-process $W^H := (W_t^H)_{t\in[0,T]}$ is called fractional white noise while $W^h := (W_t^h)_{t\in[0,T]}$ is called multifractional white noise. Note moreover that (3.7) may be written as:

$$W_t^h = W_t^{h(t)} + h'(t) \cdot \frac{\partial \mathbf{B}}{\partial H}(t, h(t)), \tag{3.8}$$

where $W_t^{h(t)}$ is nothing but $W_t^H|_{H=h(t)}$ and where the equality holds in $(\mathcal{S})^*$. For more details on the properties of M_H and $\frac{\partial M_H}{\partial H}$ one may refer to [14, Sections 2.2 and 4.2].

3.3 Background on integrals with respect to fBm and mBm

We give below the definition of both fractional Wick-Itô integral and multifractional Wick-Itô integral (or Wick-Itô integrals *wrt* fBm and mBm), in the framework of Bochner integral. Readers who are not familiar with Bochner integral may refer to Appendix 6.2.
Let $G := (G_t)_{t\in[0,T]}$ be either a fBm of Hurst index H or a mBm with functional parameter h.

Definition 1 (Wick-Itô integral *wrt* to G, in the Bochner sense)**.** *Let I be a Borel subset of $[0,T]$, and $Y := (Y_t)_{t\in I}$ be an $(\mathcal{S})^*$-valued process. Denote $W^{(G)} := (W_t^{(G)})_{t\in[0,T]}$ the derivative, in Hida's sense of the process G. Assume that:*

(i) there exists $p \in \mathbf{N}$ such that $Y_t \in (\mathcal{S}_{-p})$ for almost every $t \in I$, .

(ii) the process $t \mapsto Y_t \diamond W_t^{(G)}$ is Bochner integrable on I.

then, Y is said to be Bochner-integrable with respect to G on I and its integral, denoted $\int_I Y_s \, d^\diamond G_s$ is an element of $(\mathcal{S})^*$, which is defined by setting:

$$\int_I Y_s \, d^\diamond G_s := \int_I Y_s \diamond W_s^{(G)} ds. \tag{3.9}$$

As it has been shown in [11, Lemma 3.1] (*resp.* in [11, Remark 5.1]), the Bochner integrability of an $(\mathcal{S})^*$-valued process Y is a simple condition that ensures the Wick-Itô integrability of Y with respect to a fBm of any Hurst index H in $(0,1)$ (*resp. wrt* a mBm with any C^1 deterministic function h).

Remark 1. *In order to keep the name given in [5] and in [14], and since the Pettis and Bochner integrals both coincide, when they both exist, there is no risk of confusion by calling fractional Wick-Itô integral the Wick-Itô integral wrt fBm, in the Bochner sense. It is the same for multifractional Wick-Itô integral and the Wick-Itô integral wrt mBm, in the Bochner sense Moreover, and in order to simplify notations, we will denote W_s^H instead of $W_s^{(B^H)}$ and W_s^h instead of $W_s^{(B^h)}$, for every real s.*

Integral with respect to mBm as a sum of integrals with respect to fBm

The way used in [15] to construct an integral *wrt* mBm using approximating integrals *wrt* fBms, that will be denoted $\int_0^T Y_t \; d^\diamond B_t^h$ and that will be named limiting fractional Wick-Itô integral, is the following. For any integer p_0, define the set Λ_{p_0} by setting:

$$\Lambda_{p_0} := \{Y := (Y_t)_{t\in[0,T]} \in (\mathcal{S}_{-p_0})^{\mathbf{R}} : Y \text{ is Bochner integrable of index } p_0 \text{ on } [0,T]\}.$$

Having in mind the expression of $\int_0^T Y_s \, d^\diamond B_s^{h_n}$, given in (1.3), we can now give a rigorous definition of limiting fractional Wick-Itô integral.

Theorem-Definition 3.3. *(Limiting fractional Wick-Itô integral [15], Corollary 4.7) For any fixed integer p_0 and any element $Y := (Y_t)_{t\in[0,T]}$ in Λ_{p_0}, the quantity:*

$$I^h_{p_0}(T) := \lim_{n\to\infty} \int_0^T Y_t \, d^\diamond B^{h_n}_t + \int_0^T h'(t)\, Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t))\, dt,$$

where the limit and the equality both hold in $(\mathcal{S})^*$, is well-defined and belongs to $(\mathcal{S})^*$.

The multifractional Wick-Itô integral of Y built in Definition 1, can be obtained as limit of fractional Wick-Itô integrals. More precisely we have the following result, stated in [15, Theorem 4.8].

Theorem 3.4. *Let $Y := (Y_t)_{t\in[0,T]}$ be a Bochner integrable process of index p_0 in $\mathbf{N}$. Then Y is integrable wrt mBm in both senses of Definition 1 and Theorem-Definition 3.3. Moreover the following equality holds in $(\mathcal{S})^*$.*

$$\int_0^T Y_t \, d^\diamond B^h_t = I^h_{p_0}(T),$$

where, once again, the limit and the equality both hold in $(\mathcal{S})^*$.

Since both quantities $\int_0^T Y_t \, d^\diamond B^h_t$ and $I^h_{p_0}(T)$ are equal, we will use only the notation $\int_0^T Y_t \, d^\diamond B^h_t$ for both the multifractional Wick-Itô integral and the limiting fractional Wick-Itô integral $I^h_{p_0}(T)$ in the sequel.

Remark 2. *The reader interested in the main properties of the limiting fractional Wick-Itô integral integral may refer to [11, Section 4] as well as to [15, Sections 3 & 4]. Moreover a complete comparison between the limiting fractional Wick- Itô integral and the multifractional integral wrt mBm defined in [14] can be found in [11, Section 5].*

4 The Transfer principle

Let's now describe the transfer principle, a consequence of which will be an Itô formula for Multifractional Brownian Motion. This latter result will be proved in Section 6.
$\mathbf{B} := (\mathbf{B}(t,H))_{(t,H)\in\mathscr{D}^h_T}$ still denote the fractional Brownian field introduced at Section 1. Let p_0 be a positive integer and let X, Y, Z be three fields

$$X, Y, Z : [0,T] \times (0,1) \to (S_{-p_0})$$

such that:

$$Y \,\&\, Z \text{ are both continuous in } H \text{ and } \int_0^T (\sup_{H\in\mathcal{K}} \|Y(t,H)\|_{-p_0} + \sup_{H\in\mathcal{K}} \|Z(t,H)\|_{-p_0})dt < \infty, \qquad (\mathscr{F}_{Y,Z})$$

for every compact interval $\mathcal{K} \subset (0,1)$.

Denote $\frac{\partial X}{\partial H} : [0,T] \times (0,1) \to (S_{-p_0})$ the partial derivative of X with respect to H, when it exists. We will make another assumption on X which is:

$$X \text{ is partially differentiable in } H \,\&\, \frac{\partial X}{\partial H} : [0,T] \times (0,1) \to (S_{-p_0}) \text{ is continuous in both variables.} \qquad (\mathscr{F}_X)$$

All Hida processes below are supposed to be weakly measurable. Given a C^1-function, denoted $h:[0,T]\to(0,1)$, we denote X_t^h for $X(t,h(t))$ and we identify the constant function $t\mapsto H$ with the real number H.
We moreover assume that the following equation holds in $(S)^*$, for every $t\in[0,T]$ and H in $Im(h)$,

$$X_t^H = X_0^H + \int_0^t Y_s^H \diamond W_s^H ds + \int_0^t Z_s^H ds. \tag{4.1}$$

One can think about (4.1) as:

$$dX_t^H = Y_t^H d^{\diamond}B_t^H + Z_t^H dt.$$

Note however that only Equation (4.1) has a clear and rigorous meaning. The following result constitutes the main result of this section.

Theorem 4.1. *Under both Assumptions ($\mathscr{F}_{Y,Z}$) and ($\mathscr{F}_X$), the following equality holds in $(S)^*$. For every $t\in[0,T]$,*

$$X_t^h = X_0^h + \int_0^t Y_s^h \diamond W_s^h ds + \int_0^t Z_s^h ds + \underbrace{\int_0^t h'(s)\left(\frac{\partial X}{\partial H}(s,h(s)) - Y_s^h \diamond \frac{\partial \mathbf{B}}{\partial H}(s,h(s))\right) ds}_{=:\text{“Multifractional Correction Term”}}. \tag{4.2}$$

Proof. We here mainly follow [11, Section 4]. A standard argument, as in [11, Lemma 3.1], shows that all integrals in (4.1)–(4.2) exist in $(S)^*$. Let h_n still denote the function defined at (1.2), for the piecewise constant approximation of h on $[0,t]$ instead of $[0,T]$. Then, thanks to [11, Theorem 4.1], one can write:

$$\int_0^t Y_s^h \diamond W_s^h ds - \int_0^t h'(s) Y_s^h \diamond \frac{\partial B}{\partial H}(s,h(s)) ds = \lim_{n\to\infty} \int_0^t Y_s^h \diamond W_s^{h_n} ds =: \mathcal{I}, \tag{4.3}$$

where the last limit stands in $(S)^*$. Now, for sufficiently large $q\in\mathbf{N}$, one can write:

$$\begin{aligned}
J_n^{(q)}(t) &:= \left\| \int_0^t Y_s^h \diamond W_s^{h_n} ds - \int_0^t Y_s^{h_n} \diamond W_s^{h_n} ds \right\|_{-q} \le \left\| \sum_{k=0}^{q_n-1} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} (Y_s^{h(s)} - Y_s^{h_n(s)}) \diamond W_s^{h_n} ds \right\|_{-q} \\
&\le \sum_{k=0}^{q_n-1} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} \left\| (Y_s^{h(s)} - Y_s^{h_n(s)}) \diamond W_s^{h_n(s)} \right\|_{-q} ds \\
&\le \sum_{k=0}^{q_n-1} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} \left\| (Y_s^{h(s)} - Y_s^{h_n(s)}) \right\|_{-p_0} \left\| W_s^{h_n(s)} \right\|_{-2} ds = \int_0^t \left\| Y_s^{h(s)} - Y_s^{h_n} \right\|_{-p_0} \left\| W_s^{h_n} \right\|_{-2} ds \\
&\le \sup_{(u,H)\in\mathcal{D}_t^h} \|W_u^H\|_{-2} \int_0^t \|Y_s^h - Y_s^{h_n}\|_{-p_0} ds \underset{n\to\infty}{\longrightarrow} 0,
\end{aligned} \tag{4.4}$$

by dominated convergence and the assumptions made on Y.
Besides, writing (4.1) with $(t,H) = (x_{k+1}^{(n)}, h(x_k^{(n)}))$, then with $(t,H) = (x_k^{(n)}, h(x_k^{(n)}))$ and finally subtracting the latter one to the first one, we get:

$$\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} Y_s^{h(x_k^{(n)})} \diamond W_s^{h(x_k^{(n)})} ds = X_{x_{k+1}^{(n)}}^{h(x_k^{(n)})} - X_{x_k^{(n)}}^{h(x_k^{(n)})} - \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} Z_s^{h(x_k^{(n)})} ds.$$

We then add up this latter equality from $k=0$ up to $k=q_{n-1}$ and hence get:

$$\int_0^t Y_s^{h_n} \diamond W_s^{h_n} ds = \sum_{k=0}^{q_n-1} \left(X_{x_{k+1}^{(n)}}^{h(x_k^{(n)})} - X_{x_k^{(n)}}^{h(x_k^{(n)})} \right) - \int_0^t Z_s^{h_n} ds. \tag{4.5}$$

On the other hand, we can write:

$$\begin{aligned}X_t^h - X_0^h &= X(t, h(t)) - X(0, h(0)) \\ &= \sum_{k=0}^{q_n-1} \left(X\left(x_{k+1}^{(n)}, h(x_{k+1}^{(n)})\right) - X\left(x_{k+1}^{(n)}, h(x_k^{(n)})\right)\right) \\ &\quad + \sum_{k=0}^{q_n-1} \left(X\left(x_{k+1}^{(n)}, h(x_k^{(n)})\right) - X\left(x_k^{(n)}, h(x_k^{(n)})\right)\right) =: \sum_{k=0}^{q_n-1} A_k^{(n)} + \sum_{k=0}^{q_n-1} B_k^{(n)}. \end{aligned} \tag{4.6}$$

Gathering (4.5) and (4.6), one can write:

$$\int_0^t Y_s^{h_n} \diamond W_s^{h_n} ds = X_t^h - X_0^h - \sum_{k=0}^{q_n-1} A_k^{(n)} - \int_0^t Z_s^{h_n} ds.$$

and thus, using (4.3) and (4.4),

$$\mathcal{I} = \lim_{n\to\infty} \int_0^t Y_s^{h_n} \diamond W_s^{h_n} ds = X_t^h - X_0^h + \lim_{n\to\infty} \left(\sum_{k=0}^{q_n-1} \left(X_{x_{k+1}^{(n)}}^{h(x_k^n)} - X_{x_{k+1}^{(n)}}^{h(x_{k+1}^{(n)})} \right) - \int_0^t Z_s^{h_n} ds \right). \tag{4.7}$$

The convergence of $(\int_0^t Z_s^{h_n} ds)_{n\in\mathbf{N}}$ to $\int_0^t Z_s^h ds$, in $(S)^*$, is just a consequence of the assumptions made on the process Z and an even simpler reasoning than the one provided in (4.4). Hence, the sequence involving only the sums converges as well. Moreover, (4.7) now reads:

$$X_t^h = X_0^h + \mathcal{I} + \int_0^t Z_s^h ds + \lim_{n\to\infty} \sum_{k=0}^{q_n-1} \left(X_{x_{k+1}^{(n)}}^{h(x_{k+1}^{(n)})} - X_{x_{k+1}^{(n)}}^{h(x_k^{(n)})} \right). \tag{4.8}$$

In view of (4.3), one then has:

$$X_t^h = X_0^h + \int_0^t Y_s^h \diamond W_s^h ds + \int_0^t Z_s^h ds - \int_0^t h'(s) Y_s^h \diamond \frac{\partial B}{\partial H}(s, h(s)) ds + \lim_{n\to\infty} \sum_{k=0}^{q_n-1} \left(X_{x_{k+1}^{(n)}}^{h(x_{k+1}^{(n)})} - X_{x_{k+1}^{(n)}}^{h(x_k^{(n)})} \right). \tag{4.9}$$

As convergence of the sequence has already been argued we only need to identify the limit by means of the S-transform.

$$f_\eta(s, h(s)) := S(X(s, h(s)))(\eta).$$

A straight consequence of $(\mathscr{F}_X)$ is that, for every η in $\mathscr{S}(\mathbf{R})$, the map $s \mapsto S(X(s, h(s)))(\eta)$ is C^1-differentiable on $[0, t]$. Using the properties of S-transform[2], as well as fundamental Theorem of calculus to the S-transform, we can write, for every (a, b) in $[0, t]^2$,

$$\begin{aligned} S(X(b, h(b)))(\eta) - S(X(b, h(a)))(\eta) &= f_\eta(b, h(b)) - f_\eta(b, h(a)) = \int_a^b \frac{\partial f_\eta}{\partial r_2}(b, h(u)) \cdot h'(u)\, du \\ &= \int_a^b S\left(\tfrac{\partial X}{\partial H}(b, h(u))\right)(\eta) \cdot h'(u)\, du = S\left(\int_a^b \frac{\partial X}{\partial H}(b, h(u)) \cdot h'(u)\, du\right)(\eta). \end{aligned} \tag{4.10}$$

The integrability of $S\left(\frac{\partial X}{\partial H}(t, h(u))\right)(\eta) \cdot h'(u)$ results from the C^1-differentiability of the map $s \mapsto S(X(s, h(s)))(\eta)$. Applying (4.10) for $(a, b) = (x_k^{(n)}, x_{k+1}^{(n)})$ and taking the sum from 0 to q_{n-1}, we get

$$S\left(\sum_{k=0}^{q_n-1} \left(X_{x_{k+1}^{(n)}}^{h(x_{k+1}^{(n)})} - X_{x_{k+1}^{(n)}}^{h(x_k^{(n)})} \right) \right)(\eta) = S\left(\int_0^t \frac{\partial X}{\partial H}(\bar{u}^{(n)}, h(u)) \cdot h'(u)\, du \right)(\eta).$$

[2] see e.g. [14, Lemma 2.6]

where $\bar{u}^{(n)}$ denotes the rounding to the closest point larger or equal to u in the q_n-th partition. The convergence, in $(S)^*$, of $\sum_{k=0}^{q_n-1}\left(X_{x_{k+1}^{(n)}}^{h(x_{k+1}^{(n)})} - X_{x_{k+1}^{(n)}}^{h(x_k^{(n)})}\right)$ is granted since (4.8). This ensures us that $\int_0^t \frac{\partial X}{\partial H}(\bar{u}^{(n)}, h(u)) \cdot h'(u)\, du$ is also convergent in $(S)^*$. The equality between $\lim_{n\to\infty}\int_0^t \frac{\partial X}{\partial H}(\bar{u}^{(n)}, h(u)) \cdot h'(u)\, du$ and $\int_0^t \frac{\partial X}{\partial H}(u, h(u)) \cdot h'(u)\, du$ is just a consequence of the assumptions made on $\frac{\partial X}{\partial H}$ and an even simpler reasoning than the one provided in (4.4). The uniform continuity of the field X on $\mathscr{D}_t^h$ therefore allows us to write:

$$\lim_{n\to\infty}\sum_{k=0}^{q_n-1}\left(X_{x_{k+1}^{(n)}}^{h(x_{k+1}^{(n)})} - X_{x_{k+1}^{(n)}}^{h(x_k^{(n)})}\right) = \lim_{n\to\infty}\int_0^t \frac{\partial X}{\partial H}(\bar{u}^{(n)}, h(u)) \cdot h'(u)\, du = \int_0^t \frac{\partial X}{\partial H}(u, h(u)) \cdot h'(u)\, du,$$

where the convergence holds in $(S)^*$. Thus (4.9) now reads:

$$X_t^h = X_0^h + \int_0^t Y_s^h \diamond W_s^h ds + \int_0^t Z_s^h ds - \int_0^t h'(s) Y_s^h \diamond \frac{\partial B}{\partial H}(s, h(s)) ds + \int_0^t \frac{\partial X}{\partial H}(u, h(u)) \cdot h'(u)\, du,$$

which is nothing but (4.2) and thus ends the proof. □

5 Proof of auxiliary results

Until the end of this paper, M will denote an universal positive constant, that may differ from a line to another, but which is independent of all parameters $n, k, \eta, ...$

5.1 Auxiliary results on White Noise Theory

The following results will be used extensively in the next section to establish the Itô formula stated in Equality (2.2).
Let us start with a result given in [9, p.217 − 218] and which is a consequence of the Cameron-Martin shift.

Fact 1. *Let* $X := \sum_{k=0}^{+\infty} a_k < ., e_k >$ *and* $Y := \sum_{k=0}^{+\infty} b_k < ., e_k >$ *be two Gaussian random variables. Since* $\mathbf{E}[: e^{<.,\eta>} :] = 1$*, for every* η *in* $\mathscr{S}(\mathbf{R})$*, we can define a probability measure, denoted* $\mathbf{Q}_\eta$*, by setting:*

$$\frac{d\mathbf{Q}_\eta}{d\mu} \stackrel{def}{=} : e^{<.,\eta>} : .$$

Moreover, it is easy to check that:

$$\mathcal{L}_{(X,Y)}^{\mathbf{Q}_\eta} = \mathcal{L}_{(X+S(X)(\eta), Y+S(Y)(\eta))}^{\mu}, \tag{5.1}$$

i.e. that the law of (X, Y)*, under the probability measure* $\mathbf{Q}_\eta$*, is the same as the law of* $(X + S(X)(\eta), Y + S(Y)(\eta))$*, under the probability measure* μ*.*

The result presented in the following lemma, that will be used in the next section, is very well-known although it is usually formulated in terms of Malliavin calculus (see e.g. [8, Proposition 4.7]), we provide below an elementary proof which does not make use of Malliavin derivatives. One can also refer to [7] to see how one applies this result in an SPDE's framework.

Lemma 5.1. *Let* $X := \sum_{k=0}^{+\infty} a_k < ., e_k >$ *and* $Y := \sum_{k=0}^{+\infty} b_k < ., e_k >$ *be two Gaussian random variables and let* $j : \mathbf{R} \to \mathbf{R}$ *be a a map of class* C^1*, the derivative of which is denoted* j'*. If there exists* (C, λ) *in* $(\mathbf{R}_+^*)^2$*, such that:*

$$\forall x \in \mathbf{R}, \qquad |k(x)| \leq Ce^{\lambda x^2}, \tag{5.2}$$

for every k *in* $\{j, j'\}$ *and where* $\lambda < (4\,\mathbf{E}[X^2])^{-1}$*, then the following equality holds in* (L^2)*:*

$$j(X) \cdot Y = j(X) \diamond Y + \mathbf{E}[X \cdot Y]\, j'(X). \tag{5.3}$$

Proof. Denote (σ_1, σ_2) the couple of positive reals defined by $\sigma_1^2 := \mathbf{E}[X^2]$, $\sigma_2^2 := \mathbf{E}[Y^2]$. Denote $\sigma_{1,2} := \mathbf{E}[XY]$ and $d^2 := \sigma_1^2\, \sigma_2^2 - \sigma_{1,2}^2$. We recall that, for every positive real number σ and every γ in $[0, \frac{1}{2\sigma^2})$, we have the following equality:

$$I_{\gamma,\sigma^2} := \mathbf{E}[e^{\gamma Z^2}] = \frac{1}{\sqrt{1 - 2\gamma\sigma^2}}, \tag{5.4}$$

where $Z \rightsquigarrow \mathcal{N}(0, \sigma^2)$. Let k be fixed in $\{j, j'\}$. For any ε in $(0, 2)$, one can write:

$$\mathbf{E}[(k(X)\; Y)^2] \leq \Big(\mathbf{E}\big[|k(X)|^{2-\varepsilon}\big]\Big)^{\frac{2}{1-\varepsilon/2}} \cdot \big(\mathbf{E}\big[|Y|^{2+\varepsilon}\big]\big)^{\frac{2}{1+\varepsilon/2}} \tag{5.5}$$

Since Y is a centered Gaussian random variable, it is clear that one can write:

$$\mathbf{E}\big[|Y|^{2+\varepsilon}\big] \leq 1 + \mathbf{E}\big[|Y|^4 \mathbf{1}_{|Y|\geq 1}\big] \leq 1 + \sqrt{105}\, \sigma_2^2. \tag{5.6}$$

Besides, using (5.4), one gets:

$$\mathbf{E}\big[|k(X)|^{2-\varepsilon}\big] \leq C^{2-\varepsilon} \cdot \mathbf{E}\big[e^{(2-\varepsilon)\lambda X^2}\big] = C^{2-\varepsilon} \cdot (1 - 2\lambda(2-\varepsilon) \cdot \mathbf{E}[X^2])^{-1/2} \tag{5.7}$$

The use of (5.4) here is authorized since the inequality $\lambda < (4\,\mathbf{E}[X^2])^{-1}$, which is provided by the assumption, allows us to prove that the right hand side of the following equivalence holds:

$$\lambda(2-\varepsilon) \in (0, (2\sigma_1^2)^{-1}) \Longleftrightarrow 0 < 4\lambda\sigma_1^2 - 2\varepsilon\lambda\sigma_1^2 < 1.$$

The right hand side quantity of (5.7) is therefore finite for the same reasons. This proves that both $j(X) \cdot Y$ and $j'(X) \cdot Y$ belongs to (L^2). Moreover, replacing Y by 1 in (5.5) allows us to state that both $j(X)$ and $j'(X)$ belongs to (L^2). Since $j(X) \diamond Y$ belongs to $(\mathcal{S})^*$, it is sufficient, in order to establish (5.3), to show that both right and left members have the same S-transform. Let η be in $\mathscr{S}(\mathbf{R})$, define $s_1 := S(X)(\eta)$, $s_2 := S(Y)(\eta)$, $I := S(j(X) \cdot Y)(\eta)$ and $J := S(j(X) \diamond Y + \mathbf{E}[X \cdot Y]\; \cdot j'(X))(\eta)$, . Thanks to (5.1) we get:

$$\begin{aligned}
I &= \mathbf{E}_{\mathbf{Q}_\eta}[j(X)\, Y] = \mathbf{E}[j(X + S(X)(\eta))\, (Y + S(Y)(\eta))] \\
&= \int_{\mathbf{R}^2} j(x+s_1)\, (y+s_2)\, \frac{1}{2\pi\sqrt{d^2}}\, e^{\frac{-1}{2d^2}\, (\sigma_2^2 x^2 + \sigma_1^2 y^2 + \sigma_{1,2} xy)}\, dx\, dy \\
&= \frac{s_2}{\sigma_1\,\sqrt{2\pi}} \int_{\mathbf{R}} j(x+s_1)\, e^{\frac{-x^2}{2\sigma_1^2}}\, dx + \frac{\sigma_{1,2}}{\sigma_1^3\,\sqrt{2\pi}} \int_{\mathbf{R}} j(x+s_1)\, x\, e^{\frac{-x^2}{2\sigma_1^2}}\, dx \\
&= \frac{s_2}{\sqrt{2\pi}} \int_{\mathbf{R}^2} j(\sigma_1\, u + s_1)\, e^{-u^2/2}\, du + \frac{\sigma_{1,2}}{\sigma_1^3\,\sqrt{2\pi}} \int_{\mathbf{R}} j(x+s_1)\, x\, e^{\frac{-x^2}{2\sigma_1^2}}\, dx =: I_1 + I_2.
\end{aligned} \tag{5.8}$$

An integration by parts allows us to write that:

$$I_2 = \frac{\sigma_{1,2}}{\sqrt{2\pi}} \int_{\mathbf{R}} j'(u\,\sigma_1 + s_1) \cdot e^{-u^2/2}\, du. \tag{5.9}$$

Besides, we have the equality:

$$S(Y)(\eta) S(j(X))(\eta) = s_2\, \mathbf{E}_{\mathbf{Q}_\eta}[j(X)] = s_2\, \mathbf{E}[j(X{+}S(X)(\eta))] = \frac{s_2}{\sqrt{2\pi}} \int_{\mathbf{R}^2} j(\sigma_1\, u{+}s_1){\cdot}e^{-u^2/2}\, du = I_1. \tag{5.10}$$

Using j' instead of j in (5.10) , as well as (5.9), allows us to claim that:

$$\mathbf{E}[XY]\; S(j'(X))(\eta) = \frac{\sigma_{1,2}}{\sqrt{2\pi}} \int_{\mathbf{R}^2} j'(\sigma_1\, u + s_1) \cdot e^{-u^2/2}\, du = I_2. \tag{5.11}$$

In view of (5.10) and (5.11), (5.8) can now be written under the following form:

$$I = S(Y)(\eta)\; S(j(X))(\eta)\; + \mathbf{E}[XY]\; S(j'(X))(\eta) = S(j(X) \diamond Y + \mathbf{E}[XY]\; j'(X))(\eta) = J.$$

□

5.2 Proof of a technical lemma

M still denote an universal positive constant, that may differ from a line to another, but which is independent of all parameters $n, k, \eta, \ldots$. Let K be any element in $\left\{f, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}\right\}$.

Lemma 5.2. *Define the maps*

$$\begin{array}{cccc} \frac{\partial \mathbf{B}}{\partial H} : & \mathscr{D}_T^h & \to & (L^2) \\ & (t,H) & \mapsto & \frac{\partial \mathbf{B}}{\partial H}(t,H) \end{array} \qquad \begin{array}{cccc} \rho : & \mathscr{D}_T^h \times \mathscr{D}_T^h & \to & (L^2) \\ & (t_1,t_2,H_1,H_2) & \mapsto & B_{t_1}^{H_1} \cdot \frac{\partial \mathbf{B}}{\partial H}(t_2,H_2) \end{array} \tag{5.12}$$

$$\begin{array}{cccc} \mathscr{K} : & [0,T] \times \mathscr{D}_T^h & \to & (L^2) \\ & (s,t,H) & \mapsto & K(s, B_t^H). \end{array}$$

All the maps defined above are continuous.

Proof. The continuity of the map $\frac{\partial \mathbf{B}}{\partial H}$ is a straightforward consequence of the fact that Hypothesis $(\mathcal{H}_2)$ in [15, p.684] is verified for the field $\mathbf{B}$ defined in (3.2), according to [15, Proposition 3.1]. This entails, in particular, that there exists $(\Delta, \alpha, \lambda) \in (\mathbf{R}_+^*)^3$ such that, for all (t,s,H,H') in $[a,b]^2 \times [c,d]^2$,

$$\mathbf{E}\left[\left(\tfrac{\partial \mathbf{B}}{\partial H}(t,H) - \tfrac{\partial \mathbf{B}}{\partial H}(s,H')\right)^2\right] \le \Delta\; \left(|t-s|^\alpha + |H-H'|^\lambda\right). \tag{5.13}$$

By Gaussianity, both functions $\frac{\partial \mathbf{B}}{\partial H}(t_2,H_2)$ and $B_{t_1}^{H_1}$ are even continuous in (L^4) and their product is continuous in (L^2) by Hölder. Moreover, the (L^2)-continuity of ρ is obvious since it is a product of two continuous functions, in (L^2). Denote

$$m_{T,h} := \sup_{(t,H)\, \in\, \mathscr{D}_T^h} t^{2H} \qquad \& \qquad \varepsilon_{T,h} := 2\big(\frac{1}{4\lambda m_{T,h}} - 1\big) \tag{5.14}$$

For every (t,H) in $\mathscr{D}_T^h$ and ε in $[0, \varepsilon_{T,h})$, we easily get, using (5.4),

$$\mathbf{E}[e^{\lambda(2+\varepsilon)(B_t^H)^2}] = (1 - 2\lambda(2+\varepsilon)t^{2H})^{-1/2}$$

and thus

$$\sup_{(t,H)\,\in\,\mathscr{D}_T^h} \mathbf{E}[e^{\lambda(2+\varepsilon)(B_t^H)^2}] \leq \big(1-2\lambda(2+\varepsilon)m_{T,h}\big)^{-1/2}. \tag{5.15}$$

The right-hand side quantity of (5.15) is finite since ε is in $[0,\varepsilon_{T,h})$. On can therefore deduces that

$$\forall \varepsilon \in [0,\varepsilon_{T,h}), \quad \sup_{(s,t,H)\,\in\,[0,T]\times\mathscr{D}_T^h} \mathbf{E}\left[\left|K(s,B_t^H)\right|^{2+\varepsilon}\right] < \infty. \tag{5.16}$$

According to La Vallée Poussin's criterion, we deduce from (5.16) the uniform integrability of $\{\big(K(s,B_t^H)\big)^2, (s,t,H) \in [0,T]\times\mathscr{D}_T^h\}$. Furthermore, we know that map K is continuous on $[0,T]\times\mathbf{R}$. Besides, gathering [1, Proposition 2.1 & Proposition 2.2 (a)], we know that, almost surely, the map $(t,H)\mapsto B_t^H(\omega)$ is continuous on $\mathscr{D}_T^h$. From these facts we immediately deduce that the map $\mathscr{K} : [0,T]\times\mathscr{D}_T^h \to (L^2)$ defined by $\mathscr{K}(s,t,H) := K(s,B_t^H)$ is continuous. This ends the proof. □

Remark 3. *Define*

$$\delta_{\mathscr{D}_T^h}(\varepsilon) := \max\left\{ \sup_{(s,t,H)\,\in\,[0,T]\times\mathscr{D}_T^h} \mathbf{E}\left[\left|K(s,B_t^H)\right|^{2+\varepsilon}\right],\ \ K \in \left\{f, \tfrac{\partial f}{\partial t}, \tfrac{\partial f}{\partial x}, \tfrac{\partial^2 f}{\partial x^2}\right\} \right\}. \tag{5.17}$$

In view of $(\mathscr{E}_{C,\lambda})$ *this quantity does not depend on* K*. Besides, it is finite for every* ε *in* $(0,\varepsilon_{T,h})$*, according to* (5.16)*.*

The proof of the following result, can be found in [12, (4.7)]

Lemma 5.3. *For every* η *in* $\mathscr{S}(\mathbf{R})$*, we have the following equality:*

$$S(K(t,B_t^H))(\eta) = \int_{\mathbf{R}} K\big(t, u\, t^H + < M_H(\mathbf{1}_{[0,T]}), \eta >\big)\ \tfrac{1}{\sqrt{2\pi}}\ e^{-u^2/2}\ du. \tag{5.18}$$

6 Proof of Itô formula

Let us prove theorem 2.1, given page 5, in two steps. First we prove that the Itô formula holds in $(S)^*$ and then that it also holds in (L^2).

6.1 Proof of the Itô formula in $(S)^*$

We start by proving the Itô Formula in $(S)^*$. To do so we first have to use Theorem 4.1 with:

- $X_t^H := X(t,H) := f(t,B_t^H) = f(t,\mathbf{B}(t,H))$,
- $Y_t^H := Y(t,H) := \dfrac{\partial f}{\partial x}(t,B_t^H) = \dfrac{\partial f}{\partial x}(t,\mathbf{B}(t,H))$ and
- $Z_t^H := Z(t,H) := \dfrac{\partial f}{\partial t}(t,B_t^H) + Ht^{2H-1}\dfrac{\partial^2 f}{\partial x^2}(t,B_t^H) = \dfrac{\partial f}{\partial t}(t,\mathbf{B}(t,H)) + Ht^{2H-1}\dfrac{\partial^2 f}{\partial x^2}(t,\mathbf{B}(t,H))$.

6.1.1 Assumption $(\mathscr{F}_{Y,Z})$ holds

Note first that (5.16) ensures us that all quantities X_t^H, Y_t^H and Z_t^H belong to $L^2(\Omega)$. Moreover, using (5.17), we easily get the following inequality:

$$\int_0^T (\sup_{H\in\mathcal{K}} \|Y(t,H)\|_0 + \sup_{H\in\mathcal{K}} \|Z(t,H)\|_0)\, dt \leq 2\delta_{\mathscr{D}_T^h}(0) \cdot \int_0^T t^{2H-1}\, dt < \infty, \tag{6.1}$$

for every compact interval $\mathcal{K} \subset (0,1)$. The continuity in H of both Y and Z is obvious [3], in view of Lemma 5.2.

6.1.2 Assumption $(\mathscr{F}_X)$ holds

We are going to prove more than that. Precisely, let's prove that:

A: $\dfrac{\partial f}{\partial x}(t, B_t^H) \cdot \dfrac{\partial \mathbf{B}}{\partial H}(t,H)$ belongs to (L^2),

B: $\displaystyle\lim_{\varepsilon\to 0} \frac{X(t,H+\varepsilon) - X(t,H)}{\varepsilon} \overset{(L^2)}{=} \frac{\partial f}{\partial x}(t, B_t^H) \cdot \frac{\partial \mathbf{B}}{\partial H}(t,H).$

The lemma below will help us in the sequel.

Lemma 6.1. *Let $Q := \{Q(t,H), (t,H) \in [0,T]\times(0,1)\}$ be a centered Gaussian field, continuous from $[0,T]\times(0,1)$ to $L^2(\Omega)$, denote $\Gamma_Q(t,H) := \dfrac{\partial f}{\partial x}(t, B_t^H) \cdot Q(t,H)$.*
For every (t,H) in $[0,T]\times(0,1)$, the random variable $\Gamma_Q(t,H)$ belongs to (L^2). Moreover, for every ε in $(0, \varepsilon_{T,h})$, one has the following inequalities:

$$\mathbf{E}\big[\big|\Gamma_Q(t,H)\big|^{2+\varepsilon/2}\big] \leq M_{T,\varepsilon}\left(\mathbf{E}\left[\left|\frac{\partial f}{\partial x}(t,B_t^H)\right|^{2+\varepsilon}\right]\right)^{1/p} \cdot \left(\mathbf{E}\left[|Q_{t,H}|^2\right]\right)^{1+\varepsilon/4} \tag{6.2}$$

$$\leq M_{T,\varepsilon} \cdot \left(\delta_{\mathscr{D}_T^h}(\varepsilon)\right)^{1/p} \cdot \sup_{(t,H)\in\mathscr{D}_T^h} \left(\mathbf{E}\left[|Q(t,H)|^2\right]\right)^{1+\varepsilon/4}., \tag{6.3}$$

where we have set $M_{T,\varepsilon} := \kappa((2+\varepsilon/2)q)^{\varepsilon/(4+2\varepsilon)}$.

Proof. In view of (5.17); it is clear that one can write, for every ε in $(0, \varepsilon_{T,h})$,

$$\mathbf{E}\left[|\Gamma_Q(t,H)|^{2+\varepsilon/2}\right] \leq \mathbf{E}\left[\Big|\underbrace{\frac{\partial f}{\partial x}(t,B_t^H)}_{:=A_{t,H}} \cdot \underbrace{Q(t,H))}_{:=Q_{t,H}}\Big|^{2+\varepsilon/2}\right]. \tag{6.4}$$

Define $p = (4+2\varepsilon)/(4+\varepsilon)$ and $q = 2 + 4/\varepsilon$, by Hölder's inequality, we get:

$$\mathbf{E}\big[\big|\Gamma_Q(t,H)\big|^{2+\varepsilon/2}\big] \leq \left(\mathbf{E}\left[|A_{t,H}|^{(2+\varepsilon/2)p}\right]\right)^{1/p} \cdot \left(\mathbf{E}\left[|Q_{t,H}|^{(2+\varepsilon/2)q}\right]\right)^{1/q}. \tag{6.5}$$

We remind the following equality:

$$\mathbf{E}\left[|X|^\alpha\right] = \kappa(\alpha) \cdot \left(\mathbf{E}\left[|X|^2\right]\right)^{\alpha/2}, \tag{6.6}$$

[3] One can replace $Z(0,H)$ by 0 in order to eliminate the singularity at $t=0$, when $2H-1 \leq 0$.

where X is a centered Gaussian random variable and where $\kappa(\alpha) := 2^{\alpha/2}\frac{\Gamma(\frac{\alpha+1}{2})}{\sqrt{\pi}}$, for every $\alpha > 0$. In view of (6.6), one can therefore write, since $Q_{t,H}$ is a centered Gaussian random variable,

$$\left(\mathbf{E}\left[|Q_{t,H}|^{(2+\varepsilon/2)q}\right]\right)^{1/q} = (\kappa((2+\varepsilon/2)q))^{1/q} \cdot \left(\mathbf{E}\left[|Q_{t,H}|^2\right]\right)^{1+\varepsilon/4}.$$

Using this latter result, as well as Remark 3, Inequality (6.5) therefore reads:

$$\begin{aligned}\mathbf{E}\big[\big|\Gamma_Q(t,H)\big|^{2+\varepsilon/2}\big] &\leq M_{T,\varepsilon}\left(\mathbf{E}\left[\left|\frac{\partial f}{\partial x}(t,B_t^H)\right|^{2+\varepsilon}\right]\right)^{1/p} \cdot \left(\mathbf{E}\left[|Q_{t,H}|^2\right]\right)^{1+\varepsilon/4} &(6.7)\\ &\leq M_{T,\varepsilon} \sup_{(t,H)\,\in\,\mathscr{D}_T^h}\left(\mathbf{E}\left[\left|\frac{\partial f}{\partial x}(t,B_t^H)\right|^{2+\varepsilon}\right]\right)^{1/p} \cdot \sup_{(t,H)\in\mathscr{D}_T^h}\left(\mathbf{E}\left[|Q_{t,H}|^2\right]\right)^{1+\varepsilon/4}\\ &\leq M_{T,\varepsilon}\cdot\left(\delta_{\mathscr{D}_T^h}(\varepsilon)\right)^{1/p} \cdot \sup_{(t,H)\in\mathscr{D}_T^h}\left(\mathbf{E}\left[|Q(t,H)|^2\right]\right)^{1+\varepsilon/4}, &(6.8)\end{aligned}$$

$M_{T,\varepsilon} := \kappa((2+\varepsilon/2)q)^{\varepsilon/(4+2\varepsilon)}$. On the right-hand side of (6.8), the last factor is finite by the continuity of $Q : [0,T]\times(0,1) \to L^2(\Omega)$ and $\delta_{\mathscr{D}_T^h}(\varepsilon)$ is also finite since ε belongs to $(0,\varepsilon_{T,h})$, according to Remark 3. This proves that $\Gamma_Q(t,H) := \frac{\partial f}{\partial x}(t,B_t^H)\cdot Q(t,H)$ belongs to $L^2(\Omega)$. $\square$

6.1.2-A The random variable $\frac{\partial f}{\partial x}(t,B_t^H)\cdot\frac{\partial \mathbf{B}}{\partial H}(t,H)$ belongs to (L^2)

This result is obvious in view of Lemma 6.1 with $Q(t,H) := \frac{\partial \mathbf{B}}{\partial H}(t,H)$.

6.1.2-B $\lim_{\varepsilon\to 0}\frac{X(t,H+\varepsilon)-X(t,H)}{\varepsilon} \overset{(L^2)}{=} \frac{\partial f}{\partial x}(t,B_t^H)\cdot\frac{\partial \mathbf{B}}{\partial H}(t,H)$.

Let (t,H_0,H) be fixed in $[0,1]\times(h([0,T]))^2$ such that $H \neq H_0$. Applying a first order Taylor with integral remainder formula to the map $u \mapsto f(t,u)$, between points $\mathbf{B}(t,H_0)$ and $\mathbf{B}(t,H)$, provides us with the equality:

$$\begin{aligned}f(t,B_t^H) - f(t,B_t^{H_0}) - (\mathbf{B}(t,H)-\mathbf{B}(t,H_0))\frac{\partial f}{\partial x}(t,\mathbf{B}(t,H_0)) &= \int_{\mathbf{B}(t,H_0)}^{\mathbf{B}(t,H)}\frac{\partial^2 f}{\partial x^2}(t,u)\,(\mathbf{B}(t,H)-u)\,du\\ &= (\mathbf{B}(t,H)-\mathbf{B}(t,H_0))^2\cdot\int_0^1\frac{\partial^2 f}{\partial x^2}\left(t,(\mathbf{B}(t,H_0)+u(\mathbf{B}(t,H)-\mathbf{B}(t,H_0))\right)\cdot(1-u)\,du.\end{aligned}$$

One can therefore write:

$$\underbrace{\frac{X(t,H)-X(t,H_0)}{H-H_0} - \frac{\partial f}{\partial x}(t,\mathbf{B}(t,H_0))\cdot\frac{(\mathbf{B}(t,H)-\mathbf{B}(t,H_0))}{H-H_0}}_{=:D_{H,H_0}}$$

$$= \underbrace{\frac{(\mathbf{B}(t,H)-\mathbf{B}(t,H_0))}{H-H_0}}_{:=E_{H,H_0}}\cdot\underbrace{(\mathbf{B}(t,H)-\mathbf{B}(t,H_0)}_{:=F_{H,H_0}}\cdot\underbrace{\int_0^1\frac{\partial^2 f}{\partial x^2}\left(t,(\mathbf{B}(t,H_0)+u(\mathbf{B}(t,H)-\mathbf{B}(t,H_0))\right)(1-u)du}_{:=G_{H,H_0}}, \tag{6.9}$$

where the equality holds in (L^2). Note first that D_{H,H_0} belongs to (L^2) as a difference of elements of (L^2). Besides, using (5.14), one can write, for every u in $[0,1]$:

$$\begin{aligned}\mathrm{Var}(\mathbf{B}(t,H_0)+u(\mathbf{B}(t,H)-\mathbf{B}(t,H_0))) &= \mathrm{Var}((1-u)\mathbf{B}(t,H_0)+u\mathbf{B}(t,H))\\ &\leq \left((1-u)\|\mathbf{B}(t,H_0)\|_{L^2(\Omega)} + u\|\mathbf{B}(t,H)\|_{L^2(\Omega)}\right)^2 \leq m_{T,h}. &(6.10)\end{aligned}$$

Moreover, by definition of $\varepsilon_{T,h}$, given at (5.14), one can therefore write, for every ε in $(0, \varepsilon_{T,h})$:

$$\|G_{H,H_0}\|_{L^{2(1+\varepsilon)}(\Omega)} \leq \int_0^1 \left\| \frac{\partial^2 f}{\partial x^2} \left(t, (\mathbf{B}(t,H_0) + u(\mathbf{B}(t,H) - \mathbf{B}(t,H_0))\right) \right\|_{L^{2(1+\varepsilon)}(\Omega)} du$$

$$\leq \int_0^1 \left(\mathbf{E}\left[\left(e^{\lambda((1-u)\mathbf{B}(t,H_0)+u\mathbf{B}(t,H))^2} \right)^{2(1+\varepsilon)} \right] \right)^{\frac{1}{2(1+\varepsilon)}} du = \int_0^1 \left(\mathbf{E}\left[e^{2\lambda(1+\varepsilon)Z_u^2} \right] \right)^{\frac{1}{2(1+\varepsilon)}} du, \tag{6.11}$$

where we have set $Z_u := (1-u)\mathbf{B}(t,H_0) + u\mathbf{B}(t,H)$. Using (5.4), and (6.10) one gets:

$$\mathbf{E}[e^{2\lambda(1+\varepsilon)Z_u^2}] = \frac{1}{\sqrt{1 - 4\lambda(1+\varepsilon)\operatorname{Var}(Z_u)}} \leq \frac{1}{\sqrt{1 - 4\lambda(1+\varepsilon_{T,h})\, m_{T,h}}} \tag{6.12}$$

Both (6.11) and (6.12) allow one to state that $\|G_{H,H_0}\|_{L^{2(1+\varepsilon)}(\Omega)} < \infty$, for every ε in $(0, \varepsilon_{T,h})$. Besides, for every ε in $(0, \varepsilon_{T,h})$, one can apply Hölder inequality, with $p := 1 + \varepsilon$ and then $q := \frac{\varepsilon}{1+\varepsilon}$, and then Cauchy-Schwarz inequality for $\mathbf{E}\left[|E_{H,H_0} \cdot F_{H,H_0}|^{\frac{2(1+\varepsilon)}{\varepsilon}} \right]$; to get:

$$\mathbf{E}[|E_{H,H_0} \cdot F_{H,H_0} \cdot G_{H,H_0}|^2] \leq \mathbf{E}\left[|G_{H,H_0}|^{2(1+\varepsilon)} \right]^{\frac{1}{1+\varepsilon}} \cdot \mathbf{E}\left[|E_{H,H_0} \cdot F_{H,H_0}|^{\frac{2(1+\varepsilon)}{\varepsilon}} \right]^{\frac{\varepsilon}{1+\varepsilon}}$$

$$\leq \mathbf{E}\left[|G_{H,H_0}|^{2(1+\varepsilon)} \right]^{\frac{1}{1+\varepsilon}} \cdot \mathbf{E}\left[|E_{H,H_0}|^{\frac{4(1+\varepsilon)}{\varepsilon}} \right]^{\frac{\varepsilon}{2(1+\varepsilon)}} \cdot \mathbf{E}\left[|F_{H,H_0}|^{\frac{4(1+\varepsilon)}{\varepsilon}} \right]^{\frac{\varepsilon}{2(1+\varepsilon)}} \tag{6.13}$$

Using again Equality (6.6), one easily gets:

$$\mathbf{E}\left[|E_{H,H_0}|^{\frac{4(1+\varepsilon)}{\varepsilon}} \right]^{\frac{\varepsilon}{2(1+\varepsilon)}} = \left(\kappa\left(\tfrac{4(1+\varepsilon)}{\varepsilon} \right) \right)^{\frac{\varepsilon}{2(1+\varepsilon)}} \cdot \mathbf{E}\left[|E_{H,H_0}|^2 \right]$$

&

$$\mathbf{E}\left[|F_{H,H_0}|^{\frac{4(1+\varepsilon)}{\varepsilon}} \right]^{\frac{\varepsilon}{2(1+\varepsilon)}} = \left(\kappa\left(\tfrac{4(1+\varepsilon)}{\varepsilon} \right) \right)^{\frac{\varepsilon}{2(1+\varepsilon)}} \cdot \mathbf{E}\left[|F_{H,H_0}|^2 \right].$$

In view of the previous result, and starting from (6.9), one can write, for every ε in $(0, \varepsilon_{T,h})$,

$$\begin{aligned} \|D_{H,H_0}\|^2_{L^2(\Omega)} &= \mathbf{E}[|E_{H,H_0} \cdot F_{H,H_0} \cdot G_{H,H_0}|^2] \\ &\leq \left(\kappa\left(\tfrac{4(1+\varepsilon)}{\varepsilon} \right) \right)^{\frac{\varepsilon}{1+\varepsilon}} \cdot \|G_{H,H_0}\|^2_{L^{2(1+\varepsilon)}(\Omega)} \cdot \|E_{H,H_0}\|^2_{L^2(\Omega)} \cdot \|F_{H,H_0}\|^2_{L^2(\Omega)} \\ &\leq w_{T,h} \cdot \|E_{H,H_0}\|^2_{L^2(\Omega)} \cdot \|F_{H,H_0}\|^2_{L^2(\Omega)}. \end{aligned} \tag{6.14}$$

where we have set $w_{T,h} := (1 - 4\lambda(1+\varepsilon_{T,h})\, m_{T,h})^{-\frac{1}{4(1+\varepsilon)}} \cdot \left(\kappa\left(\frac{4(1+\varepsilon)}{\varepsilon} \right) \right)^{\frac{\varepsilon}{1+\varepsilon}}$. Since E_{H,H_0} converges in (L^2), as H tends to H_0, $\|E_{H,H_0}\|^2_{L^2(\Omega)}$ is bounded. Besides $\|F_{H,H_0}\|^2_{L^2(\Omega)}$ clearly converges to 0. Inequality (6.14) then allows us to claim that D_{H,H_0} converges to 0 in (L^2), as H tends to H_0. This proves that $X : [0,T] \times h([0,T]) \to (L^2)$ is partially differentiable in H. Finally the continuity of $\frac{\partial X}{\partial H} : [0,T] \times (0,1) \to (L^2)$ results from Lemma 5.2. This achieves to prove that Assumption $(\mathscr{F}_X)$ holds. □

6.1.3 Proof of Itô Formula in $(S)^*$

This will provide us with the following equality, in $(S)^*$:

$$\begin{aligned} f(t, B_t^h) = f(0, B_0^h) &+ \int_0^t \frac{\partial f}{\partial x}(s, B_s^h) \diamond W_s^h ds + \int_0^t \left(\frac{\partial f}{\partial t}(s, B_s^h) + h(s) s^{2h(s)-1} \frac{\partial^2 f}{\partial x^2}(s, B_s^h) \right) ds \\ &+ \int_0^t h'(s) \left(\frac{\partial X}{\partial H}(s, h(s)) - \frac{\partial f}{\partial x}(s, B_s^h) \diamond \frac{\partial \mathbf{B}}{\partial H}(s, h(s)) \right) ds. \end{aligned} \tag{6.15}$$

Applying Definition 1 (Equality (3.9)) with the mBm *i.e.* with $G = B^h$, and in view of the equality:

$$\frac{\partial X}{\partial H}(t, H) = \frac{\partial f}{\partial x}(t, B_t^H) \cdot \frac{\partial \mathbf{B}}{\partial H}(t, H),$$

one can write Equality (6.15) under the following form:

$$f(t, B_t^h) = f(0, B_0^h) + \int_0^t \frac{\partial f}{\partial x}(s, B_s^h)\, d^\diamond B_s^h + \int_0^t \left(\frac{\partial f}{\partial t}(s, B_s^h) + h(s)s^{2h(s)-1}\frac{\partial^2 f}{\partial x^2}(s, B_s^h)\right) ds$$
$$+ \int_0^t h'(s)\left(\frac{\partial f}{\partial x}(s, B_s^h) \cdot \frac{\partial \mathbf{B}}{\partial H}(s, h(s)) - \frac{\partial f}{\partial x}(s, B_s^h) \diamond \frac{\partial \mathbf{B}}{\partial H}(s, h(s))\right) ds. \qquad (6.16)$$

Using Lemma 5.1 with $X := B_s^{h(s)}$, $Y := \frac{\partial \mathbf{B}}{\partial H}(s, h(s))$ and $j := \frac{\partial f}{\partial x}(s, .)$, we get the following equality:

$$\frac{\partial f}{\partial x}(s, B_s^h) \cdot \frac{\partial \mathbf{B}}{\partial H}(s, h(s)) - \frac{\partial f}{\partial x}(s, B_s^h) \diamond \frac{\partial \mathbf{B}}{\partial H}(s, h(s)) = \frac{\partial^2 f}{\partial x^2}(s, B_s^h)\, \mathbf{E}\left[B_s^h \frac{\partial \mathbf{B}}{\partial H}(s, h(s))\right]. \quad (6.17)$$

The quantity $\mathbf{E}\left[B_s^h \frac{\partial \mathbf{B}}{\partial H}(s, h(s))\right]$, introduced in the previous equality, can be easily computed using the isometry $\mathbf{E}[< ., f > \cdot < ., g >] = < f, g >_{L^2(\mathbf{R})}$ (valid for all (f, g) in $L^2(\mathbf{R}) \times L^2(\mathbf{R})$). Indeed, one gets, for every (t, H) in $[0, T] \times (0, 1)$, the equality:

$$\mathbf{E}[B_s^H \cdot \tfrac{\partial \mathbf{B}}{\partial H}(s, H)] = \mathbf{E}[< ., M_H(\mathbf{1}_{[0,s]}) \cdot < ., \tfrac{\partial M_H}{\partial H}(\mathbf{1}_{[0,s]}) >] = < M_H(\mathbf{1}_{[0,s]}), \tfrac{\partial M_H}{\partial H}(\mathbf{1}_{[0,s]}) >_{L^2(\mathbf{R})}$$
$$= \tfrac{1}{2} \cdot \tfrac{d}{dH}[< M_H(\mathbf{1}_{[0,s]}), M_H(\mathbf{1}_{[0,s]}) >_{L^2(\mathbf{R})}] = \tfrac{1}{2} \cdot \tfrac{d}{dH}[s^{2H}] = s^{2H} \cdot \ln s. \quad (6.18)$$

Plug (6.18) in (6.17), Equality (6.16) now reads:

$$f(t, B_t^h) = f(0, 0) + \int_0^t \frac{\partial f}{\partial x}(s, B_s^h)\ d^\diamond B_s^h + \int_0^t \left(\frac{\partial f}{\partial t}(s, B_s^h) + h(s)s^{2h(s)-1}\frac{\partial^2 f}{\partial x^2}(s, B_s^h)\right) ds$$
$$+ \int_0^t h'(s)\frac{\partial^2 f}{\partial x^2}(s, B_s^h)\ s^{2h(s)} \ln s\ ds$$
$$= f(0, 0) + \int_0^t \frac{\partial f}{\partial t}(s, B_s^h)\ ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s^h)\ d^\diamond B_s^h$$
$$+ \int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s^h) \cdot \left(h(s)s^{2h(s)-1} + h'(s)\ s^{2h(s)} \ln s\right)\ ds,$$

and finally, we get the following equality, in $(S)^*$:

$$f(t, B_t^h) = f(0,0) + \int_0^t \frac{\partial f}{\partial t}(s, B_s^h)\ ds + \int_0^t \frac{\partial f}{\partial x}(s, B_s^h)\ d^\diamond B_s^h + \tfrac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s^h) \cdot\ R'_h(s)\ ds, \quad (6.19)$$

which establishes the Itô formula for mBm in $(S)^*$.

6.2 Proof of Itô formula in $L^2(\Omega)$

In order to prove that (6.19) also holds in $L^2(\Omega)$, one first notice that this latter equality can be rewritten as:

$$\int_0^t \frac{\partial f}{\partial x}(s, B_s^h)\ d^\diamond B_s^h = f(t, B_t^h) - f(0,0) - \int_0^t \frac{\partial f}{\partial t}(s, B_s^h)\ ds - \tfrac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s^h) \cdot\ R'_h(s)\ ds. \quad (6.20)$$

Hence, one just need to prove that all terms of the right hand side of Equality (6.20) belong to $L^2(\Omega)$. To do so, we use the arguments given in [12, p.23], which will be briefly given here, in the particular case of mBm, for reader's convenience. Thanks to $(\mathscr{E}_{C,\lambda})$ and (5.17), we may write, for every K in $\{f, \frac{\partial f}{\partial t}, \frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}\}$ and s in $[0,t]$, that $\mathbf{E}\big[K(t,B^h_s)^2\big] \leq M^2$, where we set $M^2 := C^2\,(1-4\lambda\overline{R_h})^{-1/2}$ and $\overline{R_h} := \sup\{t^{2h(t)};\ s \in [0,T]\}$. Moreover, $s \mapsto \|K(s,B^h_s)\|_0$ belongs to $L^1([0,T],dt)$ while $t \mapsto R'_h(s) \cdot \|\frac{\partial^2 f}{\partial x^2}(t,B^h_s)\|_0$ belongs to $L^1([0,T],ds)$. The measurability of the maps $s \mapsto S(K(s,B^h_s)(\eta)$ is clear in view of (5.18). A simple application of [12, Theorem 4.3] then yields that all members on the right hand side of (6.20) exist and are in (L^2). Moreover, Lemma 3.1, as well as [12, Point 2 of Example 3.3.1.], provides the upper-bound

$$|S(\tfrac{\partial f}{\partial x}(s,B^h_s) \diamond W^h_s)(\eta)| \leq M\ \|W^h_s\|_{-q}\ e^{|\eta|^2_q},$$

for all (η, s) in $\mathscr{S}(\mathbf{R}) \times [0,T]$ and all $q \geq 2$. A straightforward application of Theorem A.3, given in the Appendix of this work, then shows that $\int_0^t \frac{\partial f}{\partial x}(s,B^h_s)\ d^{\diamond}B^h_s$ belongs to $(\mathcal{S})^*$and thus, achieves the proof.

As we stated in the introduction, the proof of this result shows that, starting with any method that provides a stochastic integral *wrt* fBm (such as White Noise Theory, Skorohod integral, pathwise integral $\cdots$), we can derive an entire stochastic calculus *wrt* mBm, only using the existing the stochastic calculus *wrt* fBm. Beyond the result of Theorem 2.1, the proof of this latter is important since it shows how one can effectively derive, starting from a known formula that involves stochastic integral *wrt* fBm, the analogue formula but where stochastic integral *wrt* fBm have been replaced by stochastic integral *wrt* mBm. This way of proceeding from stochastic integral *wrt* fBm to stochastic integral *wrt* mBm is independent of the integration method chosen (White Noise Theory, Skorohod integral, pathwise integral $\cdots$).

Appendix

Background on the Bochner integral

In order not to weigh down this statement we will only give the necessary tools to proceed. On can refer to [10, p.247] as well as to [6] for more details about Bochner integral.

Definition 2 (Bochner integral [10], p.247). *Let I be a Borel subset of $[0,1]$ and $\Phi := (\Phi_t)_{t\in I}$ be an $(\mathcal{S})^*$-valued process verifying:*

(i) the process Φ is weakly measurable on I i.e. the map $t \mapsto \ll \Phi_t, \varphi \gg$ is measurable on I, for every φ in $(\mathcal{S})$.

(ii) there exists $p \in \mathbf{N}$ such that $\Phi_t \in (\mathcal{S}_{-p})$ for almost every $t \in I$ and $t \mapsto \|\Phi_t\|_{-p}$ belongs to $L^1(I)$.

Then there exists an unique element in $(\mathcal{S})^*$, noted $\int_I \Phi_u\,du$, called the Bochner integral of Φ on I such that, for all φ in $(\mathcal{S})$,

$$\ll \int_I \Phi_u\,du, \varphi \gg = \int_I \ll \Phi_u, \varphi \gg\ du. \tag{A.21}$$

In this latter case one says that Φ is Bochner-integrable on I with index p.

Proposition A.2. *If $\Phi\colon I \to (\mathcal{S})^*$ is Bochner-integrable on I with index p then $\|\int_I \Phi_t\,dt\|_{-p} \leq \int_I \|\Phi_t\|_{-p}\,dt$.*

Theorem A.3 ([10], Theorem 13.5)**.** *Let $\Phi := (\Phi_t)_{t \in [0,1]}$ be an $(\mathcal{S})^*$-valued process such that:*

(i) $t \mapsto S(\Phi_t)(\eta)$ *is measurable for every η in $\mathscr{S}(\mathbf{R})$.*

(ii) There exist p in $\mathbf{N}$, b in $\mathbf{R}^+$ and a function L in $L^1([0,1], dt)$ such that, for a.e. t in $[0,1]$, $|S(\Phi_t)(\eta)| \le L(t)\, e^{b|\eta|_p^2}$, for every η in $\mathscr{S}(\mathbf{R})$.

Then Φ is Bochner integrable on $[0,1]$ and $\int_0^1 \Phi(s)\, ds \in (\mathcal{S}_{-q})$ for every $q > p$ such that $2 \cdot b \cdot e^2 \cdot D(q-p) < 1$, where e denotes the base of the natural logarithm and where $D(r) := \frac{1}{2^{2r}} \sum_{n=1}^{+\infty} \frac{1}{n^{2r}}$, for any r in $(1/2, +\infty)$.

References

[1] A. Ayache and M. S. Taqqu. Multifractional processes with random exponent. *Publ. Mat.*, 49(2):459–486, 2005.

[2] A. Benassi, S. Jaffard, and D. Roux. Elliptic Gaussian random processes. *Rev. Mat. Iberoamericana*, 13(1):19–90, 1997.

[3] C. Bender. An Itô formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter. *Stochastic Process. Appl.*, 104(1):81–106, 2003.

[4] C. Bender. An S-transform approach to integration with respect to a fractional Brownian motion. *Bernoulli*, 9(6):955–983, 2003.

[5] R. Elliott and J. Van der Hoek. A general fractional white noise theory and applications to finance. *Mathematical Finance*, 13(2):301–330, 2003.

[6] E. Hille and R. Phillips. *Functional Analysis and Semi-Groups*, volume 31. American Mathematical Society, 1957.

[7] H. Holden, B. Oksendal, J. Ubøe, and T. Zhang. *Stochastic Partial Differential Equations, A Modeling, White Noise Functional Approach.* Springer, second edition, 2010.

[8] Y.-z. Hu and J.-a. Yan. Wick calculus for nonlinear Gaussian functionals. *Acta Math. Appl. Sin. Engl. Ser.*, 25(3):399–414, 2009.

[9] S. Janson. *Gaussian Hilbert Spaces*, volume 129 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1997.

[10] H.-H. Kuo. *White noise distribution theory.* Probability and Stochastics Series. CRC Press, Boca Raton, FL, 1996.

[11] J. Lebovits. From stochastic integral *w.r.t.* fractional Brownian motion to stochastic integral *w.r.t.* multifractional Brownian motion. *Ann. Univ. Buchar. Math. Ser.*, 4(LXII)(1):397–413, 2013.

[12] J. Lebovits. Stochastic calculus with respect to Gaussian processes. *Potential Anal.*, 50(1):1–42, 2019.

[13] J. Lebovits. Local Times of Gaussian Processes. *Preprint*, 2020. http://adsabs.harvard.edu/abs/2017arXiv170305006L.

[14] J. Lebovits and J. Lévy Véhel. White noise-based stochastic calculus with respect to multifractional Brownian motion. *Stochastics An International Journal of Probability and Stochastic Processes*, 86(1):87–124, 2014.

[15] J. Lebovits, J. Lévy Véhel, and E. Herbin. Stochastic integration with respect to multifractional Brownian motion *via* tangent fractional Brownian motions. *Stochastic Process. Appl.*, 124(1):678–708, 2014.

[16] R. Peltier and J. Lévy Véhel. Multifractional Brownian motion: definition and preliminary results, 1995. rapport de recherche de l'INRIA, n^0 2645.

[17] S. Stoev and M. Taqqu. How rich is the class of multifractional Brownian motions? *Stochastic Processes and their Applications*, 116:200–221, 2006.