Algebraic Geometry: Positivity and vanishing theorems

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Preface

These notes are based on my course "V5A2: Selected Topics in Algebra" in the Winter Semester 2013/14 at the University of Bonn. The course is meant to be an introduction to a more advanced course in birational geometry in the Summer Semester 2014.

The notes will grow non-linearly during the course. That means two things: first, I will try and update the material weekly as the course goes on, but the material will not be in 1-1 correspondence with what is actually said in the course. This is particularly true for Chapter 1: most of the stuff at the beginning of that chapter is a reference for later material, and will not be lectured. Second, it is quite possible that chapters will simultaneously grow: when new foundational material is encountered later in the course, the corresponding stuff will go to Chapter 1, and so on. I try to be pedagogical, and introduce new concepts only when/if needed.

Not everything is proved. I only prove things which I find particularly enlightening, or which are of particular importance for the later material. Most of the non-proved results are easily found in the standard reference [Har77] or in the book in the making [Vak13].

Many thanks to Nikolaos Tsakanikas for reading these notes carefully and for making many useful suggestions.

Chapter 1

Foundations

This chapter contains the background material which is mostly not lectured in the course. It should be taken as a reference for most of the results that come in the next chapters.

First I review some classical concepts as of the first half of XX century, and we will see how the field developed from commutative algebra by Hilbert. Then later, we will adopt the modern point of view of sheaves developed by Grothendieck and Serre - as we will see, these two viewpoints are equivalent, and the first one helps have the geometric intuition about problems. However, the latter is more versatile and will give us more tools to solve difficult and interesting problems.

We usually work with the field $\mathbb{C}$ of complex numbers; however most of the things in this course hold for any algebraically closed field k.

1.1 Classical theory

1.1.1 Affine theory

For an integer $n \geq 0$, denote $\mathbb{A}^n = \mathbb{C}^n$, the *affine n-space* over $\mathbb{C}$.

Definition 1.1. Let $I \subseteq \mathbb{C}[X_1, \ldots, X_n]$ be an ideal, and let $f_1, \ldots, f_k \in \mathbb{C}[X_1, \ldots, X_n]$ be finitely many polynomials such that $I = (f_1, \ldots, f_k)$ (note that this follows from Hilbert's basis theorem). The set of points $a \in \mathbb{A}^n$ such that $f_i(a) = 0$ for all i is an *algebraic set* corresponding to I, and is denoted by $Z(I)$, the zeroes of I.

It is easy to show that any finite union of algebraic sets is again algebraic, and that any intersection of a family of algebraic sets is also algebraic.

Example 1.2. The empty set and the affine n-space are algebraic sets: indeed, $\emptyset = Z(1)$ and $\mathbb{A}^n = Z(0)$. Algebraic sets on the line $\mathbb{A}^1$ are precisely finite subsets of $\mathbb{A}^1$.

This gives a topology on $\mathbb{A}^n$ called *Zariski topology*: we declare that closed subsets are precisely algebraic subsets of $\mathbb{A}^n$. This topology is nice – it is Noetherian, i.e. every descending sequence of closed subsets stabilises; this will follow from the I-Z correspondence (1.1)-(1.2).

We say that an algebraic set $V \subseteq \mathbb{A}^n$ is *irreducible* if it is not a union of two proper algebraic subsets.

Definition 1.3. An *affine algebraic subvariety of* $\mathbb{A}^n$ is an irreducible closed subset of $\mathbb{A}^n$. An open subset of an affine algebraic variety is an *quasi-affine variety.*

We have observed that there is a map Z which assigns an algebraic subset of $\mathbb{A}^n$ to every ideal in $\mathbb{C}[X_1, \dots, X_n]$. Conversely, to every subset $V \subseteq \mathbb{A}^n$ we can assign an ideal in $\mathbb{C}[X_1, \dots, X_n]$ as follows:

$$I(V) = \{f \in \mathbb{C}[X_1, \dots, X_n] \mid f(v) = 0 \text{ for every } v \in V\}.$$

It is now straightforward to check that the maps

$$Z\colon \{\text{ideals in } \mathbb{C}[X_1, \dots, X_n]\} \to \{\text{algebraic subsets of } \mathbb{A}^n\} \tag{1.1}$$

and

$$I\colon \{\text{subsets of } \mathbb{A}^n\} \to \{\text{ideals in } \mathbb{C}[X_1, \dots, X_n]\} \tag{1.2}$$

are inclusion-reversing, i.e. if $I_1 \subseteq I_2 \subseteq \mathbb{C}[X_1, \dots, X_n]$, then $\mathbb{A}^n \supseteq Z(I_1) \supseteq Z(I_2)$, and similarly for I. This also implies easily that for any subset $V \subseteq \mathbb{A}^n$, the set $Z(I(V))$ equals the closure of V (in the Zariski topology).

A natural question is for which ideals $\mathcal{I} \subseteq \mathbb{C}[X_1, \dots, X_n]$ we have $I(Z(\mathcal{I})) = \mathcal{I}$. This is the starting point of algebraic geometry, and the answer is given by Hilbert's Nullstellensatz below. First we must define radical ideals.

Definition 1.4. For an ideal $\mathcal{I} \subseteq \mathbb{C}[X_1, \dots, X_n]$, the *radical of* $\mathcal{I}$ is

$$\sqrt{\mathcal{I}} = \{f \in \mathbb{C}[X_1, \dots, X_n] \mid f^k \in \mathcal{I} \text{ for some } k > 0\}.$$

Observe that $\mathcal{I} \subseteq \sqrt{\mathcal{I}}$, $\sqrt{\sqrt{\mathcal{I}}} = \sqrt{\mathcal{I}}$, and $Z(\mathcal{I}) = Z(\sqrt{\mathcal{I}})$. So Hilbert's Nullstellensatz tells us exactly that the radical of an ideal $\mathcal{I}$ is the largest ideal containing $\mathcal{I}$ such that the corresponding algebraic set is equal to $Z(\mathcal{I})$.

Theorem 1.5 (Hilbert's Nullstellensatz)**.** *Let $\mathcal{I} \subseteq \mathbb{C}[X_1, \dots, X_n]$ be an ideal. Then*

$$I(Z(\mathcal{I})) = \sqrt{\mathcal{I}}.$$

The following beautiful argument is known as *Rabinowitsch trick.*

Proof. We will assume without proof the following statement, sometimes called "weak" Nullstellensatz:

Claim 1.6. *Let $\mathcal{I}$ be a proper ideal in $\mathbb{C}[X_1, \ldots, X_n]$. Then $Z(\mathcal{I}) \neq \emptyset$.*

The proof uses Noether's theorem and Artin-Tate's theorem and is not difficult if you know some Commutative Algebra.

Assuming the claim, let us prove the result. Let $g \in I(Z(\mathcal{I}))$, and assume $\mathcal{I}$ is generated by $f_1, \ldots, f_k \in \mathbb{C}[X_1, \ldots, X_n]$.

Consider the ring $\mathbb{C}[X_1, \ldots, X_n, X_{n+1}]$, and the ideal $\mathcal{J}$ in this ring generated by $f_1, \ldots, f_k$ and by $1-gX_{n+1}$. Assume that $\mathcal{J} \neq \mathbb{C}[X_1, \ldots, X_n, X_{n+1}]$. Then, by Claim 1.6, there exists $\alpha = (a_1, \ldots, a_n, a_{n+1}) \in Z(\mathcal{J}) \subseteq \mathbb{C}^{n+1}$. But then $(a_1, \ldots, a_n) \in Z(\mathcal{I})$, and therefore $g(a_1, \ldots, a_n) = 0$ since $g \in I(Z(\mathcal{I}))$. This implies that α is not a zero of $1 - gX_{n+1}$, a contradiction.

Therefore $\mathcal{J} = \mathbb{C}[X_1, \ldots, X_n, X_{n+1}]$, so there exist polynomials $p_1, \ldots, p_k, p \in \mathbb{C}[X_1, \ldots, X_n, X_{n+1}]$ such that

$$1 = \sum_{i=1}^{k} p_i f_i + p(1 - gX_{n+1}).$$

Substituting $X_{n+1} = 1/g$, we get

$$1 = \sum_{i=1}^{k} p_i(X_1, \ldots, X_n, 1/g) f_i. \tag{1.3}$$

Observe that for every i we have $p_i(X_1, \ldots, X_n, 1/g) = q_i/g^{N_i}$ for some polynomial $q_i \in \mathbb{C}[X_1, \ldots, X_n]$ and some $N_i \in \mathbb{N}$. Therefore, multiplying (1.3) by g^N for some large $N \in \mathbb{N}$ we have

$$g^N = \sum_{i=1}^{k} \hat{q}_i f_i$$

for some $\hat{q}_i \in \mathbb{C}[X_1, \ldots, X_n]$, which is exactly what we were supposed to prove. □

Therefore, there is a bijective correspondence between radical ideals and algebraic sets, and an algebraic set X is a variety iff $I(X)$ is a prime ideal. In particular, since $\mathbb{C}[X_1, \ldots, X_n]$ is a Noetherian ring, this proves the aforementioned claim that $\mathbb{A}^n$ is a Noetherian topological space.

Definition 1.7. Let $X \subseteq \mathbb{A}^n$ be an algebraic set. The *affine coordinate ring* of X is

$$\mathcal{O}(X) = \mathbb{C}[X_1, \ldots, X_n]/I(X).$$

This ring is a finitely generated integral domain if X is an algebraic variety.

Next we define the dimension of an algebraic set $X \subseteq \mathbb{A}^n$. There are several ways to do this (of course, all equivalent): if $K(X)$ denotes the field of fractions of $\mathcal{O}(X)$, then

$$\dim X := \operatorname{trdeg} K(X).$$

Equivalently, $\dim X$ equals the Krull dimension of $\mathcal{O}(X)$: that is the maximum of lengths of chains $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_k$ of prime ideals in $\mathcal{O}(X)$.

The following result is not surprising, and it follows from some commutative algebra.

Lemma 1.8. *We have that* $\dim \mathbb{A}^n = n$, *and that a subvariety* X *of* $\mathbb{A}^n$ *of dimension* $n-1$ *is the zero set of a single irreducible polynomial* f, *that is* $X = V(f)$.

1.1.2 Projective theory

The main reason why we are not completely happy with affine varieties is that they are not compact: for instance, one can easily find a cover of the line $\mathbb{A}^1$ which does not have a finite subcover. That is one of the main motivations for projective varieties.

Definition 1.9. The *projective n-space* is $\mathbb{P}^n = \mathbb{A}^{n+1}/\sim$, where $\sim$ is the equivalence relation given by

$$(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n)$$

for every $\lambda \in \mathbb{C}\backslash\{0\}$.

The projective n-space can be covered by a chart of affine varieties: indeed, for each $i = 0, \ldots, n$, we set U_i to be the set of all $(n+1)$-tuples $(a_0, \ldots, a_n)$ with $a_i = 1$. Then it is easy to see that each U_i is homeomorphic to $\mathbb{A}^n$ and that $\mathbb{P}^n = \bigcup_{i=0}^n U_i$. We also know that $\mathbb{P}^n$ can be given a structure of a complex manifold, and it can be shown that it is compact.

Now consider the ring $\mathbb{C}[X_0, \ldots, X_n]$. Recall that a polynomial f in this ring is *homogeneous of degree d* if $f(\lambda a_0, \ldots, \lambda a_n) = \lambda^d f(a_0, \ldots, a_n)$ for every $\lambda \in \mathbb{C}$. A homogeneous ideal $\mathcal{I} \subseteq \mathbb{C}[X_0, \ldots, X_n]$ is an ideal generated by (finitely many) homogeneous polynomials.

In particular, this means that it makes sense to ask whether $f(\alpha) = 0$ for $\alpha \in \mathbb{P}^n$ and a homogeneous polynomial $f \in \mathbb{C}[X_0, \ldots, X_n]$.

Therefore, as in the case of affine varieties, if $\mathcal{I}$ is a homogeneous ideal, we can define its set of zeroes $Z(\mathcal{I}) \subseteq \mathbb{P}^n$, and such a set is called a *projective algebraic set*. Conversely, for an algebraic set $X \subseteq \mathbb{P}^n$, one can define the homogeneous ideal $I(X) \subseteq \mathbb{C}[X_0, \ldots, X_n]$. Similar properties hold as in the case of affine varieties, i.e. $\emptyset$ and $\mathbb{P}^n$ itself are algebraic sets, and we can define the corresponding Zariski topology. As in the affine case, an algebraic subset of $\mathbb{P}^n$ is a *projective variety* if it is irreducible. We further say that an open subset of a projective variety is a *quasi-projective variety.*

Definition 1.10. We say that an algebraic subset X of $\mathbb{A}^n$ or $\mathbb{P}^n$ is a *(classical) variety* if it is a quasi-affine or a quasi-projective variety.

One can similarly as in the affine case prove the projective version of the Nullstellensatz: if $\mathcal{I} \subseteq \mathbb{C}[X_0, \dots, X_n]$ is a homogeneous ideal, and if $f \in I(Z(\mathcal{I}))$, then $f^k \in \mathcal{I}$ for some $k > 0$.

One can also define a dimension of a variety in general; this is an exercise. The *codimension* of a closed subvariety Y of a variety X is $\operatorname{codim}_X Y = \dim X - \dim Y$.

1.1.3 Morphisms

We want to make algebraic varieties into a category, and for this we have to define maps between varieties. We start with the case where the target is $\mathbb{A}^1 = \mathbb{C}$.

Definition 1.11. Let $X \subseteq \mathbb{A}^n$ (respectively $X \subseteq \mathbb{P}^n$) be a quasi-affine (respectively quasi-projective) variety, and consider a function $\varphi\colon X \to \mathbb{C}$. We say φ is *regular at a point* $x \in X$ if it is locally a well-defined rational function around x, i.e. if there is an open neighbourhood U of x, and there are polynomials $f, g \in \mathbb{C}[X_1, \dots, X_n]$ (respectively homogeneous polynomials $f, g \in \mathbb{C}[X_0, \dots, X_n]$ of the same degree) such that g is nowhere zero on U and $\varphi = f/g$ on U. Further, we say that φ is *regular* if it is regular at every point of X.

Now it is easy to check that any regular function is continuous in the Zariski topology.

It is not too difficult, but it is a bit involved, to show that a function on an affine variety $X \subseteq \mathbb{A}^n$ is regular iff it is in $\mathcal{O}(X)$. In other words, $\mathcal{O}(X)$ is the set of regular functions on X. For this reason, we denote by $\mathcal{O}(X)$ the set of regular functions on X, where X is either a quasi-affine or a quasi-projective variety.

In the projective case, the situation is quite the opposite: a function on a projective variety $X \subseteq \mathbb{P}^n$ is regular iff it is in $\mathbb{C}$, i.e. we always have $\mathcal{O}(X) = \mathbb{C}$. Note that this is very different from the *homogeneous coordinate ring of* X,

$$S(X) = \mathbb{C}[X_0, \dots, X_n]/I(X).$$

Now we define an important invariant of a variety X around a point $x \in X$:

Definition 1.12. A *local ring* at x, denoted by $\mathcal{O}_x$, is the set of equivalence classes of pairs of (U, f), where $U \subseteq X$ is an open neighbourhood of x and $f \in \mathcal{O}(U)$, such that two pairs (U, f) and (V, g) are equivalent if $f = g$ on $U \cap V$.

Equivalently, $\mathcal{O}_x = \varinjlim \mathcal{O}(U)$, where the limits is over all open sets $U \ni x$.

Elements of $\mathcal{O}_x$ are called *germs.*

It is immediate that there is an injective map

$$\mathcal{O}(V) \to \bigcap_{x \in V} \mathcal{O}_x$$

for every open subset V of X. We will see in Subsection 1.2.2 that this map is in fact bijective.

The local ring $\mathcal{O}_x$ at a point x is indeed a *local* ring: if $(U, f) \in \mathcal{O}_x$ and $f(x) \neq 0$, then $(V, 1/f) \in \mathcal{O}_x$ for some V. Therefore the set $\mathfrak{m}_x \subseteq \mathcal{O}_x$ of all germs of regular functions f which vanish at x is an ideal, and every element of $\mathcal{O}_x \backslash \mathfrak{m}_x$ is a unit. Note that $\mathcal{O}_x/\mathfrak{m}_x \simeq \mathbb{C}$.

Further, if X is an affine (respectively projective) variety, and x is a point in X, then $\mathcal{O}_x$ is the localisation of $\mathcal{O}(X)$ (respectively $S(X)$) at the ideal $\mathfrak{m}$ containing all $f \in \mathcal{O}(X)$ (respectively all homogeneous $f \in S(X)$) such that $f(x) = 0$.

The field of rational functions $k(X)$ of a variety is crucial in birational geometry, as we will see later in the course: it is invariant not only up to isomorphism, but even up to *birational maps*. It generalises $\mathbb{C}(X_1, \ldots, X_n)$.

Definition 1.13. Let X be a variety. The *field of rational functions* $k(X)$ is the set of equivalence classes of pairs of (U, f), where $U \subseteq X$ is an open subset and $f \in \mathcal{O}(U)$, such that two pairs (U, f) and (V, g) are equivalent if $f = g$ on $U \cap V$. In particular, if U is an open subset of X, then $k(X) \simeq k(U)$.

If X is an affine variety, then it is pretty straightforward to see that $k(X)$ is the field of fractions of $\mathcal{O}(X)$. This also implies that when X is a projective variety, $k(X)$ is the homogeneous localisation of $S(X)$ at the ideal (0), which is of course a field.

Finally, we can define morphisms between varieties.

Definition 1.14. Let X and Y be two varieties. A function $f\colon X \to Y$ is a *morphism* if it is continuous, and if for every open set $U \subseteq Y$ and every $\varphi \in \mathcal{O}(U)$, we have $\varphi \circ f \in \mathcal{O}(f^{-1}(U))$.

For example, it is easy to show (exercise!) the following: if X is a variety and $Y \subseteq \mathbb{A}^n$ is an affine variety, then a map $f\colon X \to Y$ is a morphism if and only if $f = (f_1, \ldots, f_n)$, where $f_i \in \mathcal{O}(X)$. In particular, if X is also affine, then f being a morphism is the same as f being a polynomial map.

This all makes varieties into a category. Further, from before we know that there is a bijection $X \mapsto \mathcal{O}(X)$ between affine varieties and integral domains finitely generated over $\mathbb{C}$, which we can strengthen as follows: this is an *arrow-reversing equivalence of categories*. Formally,

$$\mathrm{Hom}(X, Y) \simeq \mathrm{Hom}(\mathcal{O}(Y), \mathcal{O}(X)).$$

This can be used to give a new, much more general and more flexible, definition of varieties below.

1.2 Sheaf theory

1.2.1 Sheaves

It is clear that the rings $\mathcal{O}(U)$ defined above are related if U are open subsets of X, and that the functions in them satisfy certain gluing conditions, which are built-in the definitions. In particular, for every two open subsets $V \subseteq U$ of X, the restriction of regular functions gives a map $\mathcal{O}(U) \to \mathcal{O}(V)$. Also, if regular functions $s_1 \in \mathcal{O}(U_1)$ and $s_2 \in \mathcal{O}(U_2)$ on two open subsets U_1 and U_2 of X coincide on $U_1 \cap U_2$, then there is a (unique) regular function $s \in \mathcal{O}(U_1 \cup U_2)$ such that $s|_{U_1} = s_1$ and $s|_{U_2} = s_2$.

We will see below that this can be formalised to work for all topological spaces, although, naturally, we are interested in those which have algebro-geometric interpretations.

Note that one of the disadvantages of the classical definition of varieties is that they depend on an embedding in $\mathbb{A}^n$ or $\mathbb{P}^n$. This will be overcome by an introduction of *abstract varieties* below. Further, for some purposes it is important to work not necessarily over $\mathbb{C}$, but over any field, and even over any ring. All this justifies the passage to language of sheaves, which goes back to Leray, Grothendieck and Serre in 1950's and 1960's.

Definition 1.15. Let X be a topological space. A *sheaf* $\mathcal{F}$ on X is a collection $\{\mathcal{F}(U)\}$ of abelian groups for every open subset U of X, and morphisms $\rho_{UV} \colon \mathcal{F}(U) \to \mathcal{F}(V)$ for open sets $V \subseteq U \subseteq X$, such that:

(1) $\mathcal{F}(\emptyset) = 0$,

(2) $\rho_{UU} = \mathrm{id}_U$ for every open $U \subseteq X$,

(3) if $W \subseteq V \subseteq U$ are open subsets of X, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$,

(4) if U is an open subset of X, if $\{U_i\}$ is an open covering of U, and if $s_i \in \mathcal{F}(U_i)$ are such that $\rho_{U_i, U_i \cap U_j}(s_i) = \rho_{U_j, U_i \cap U_j}(s_j)$ for all i and j, then there exists a *unique* $s \in \mathcal{F}(U)$ such that $\rho_{UU_i}(s) = s_i$ for every i.

Elements of $\mathcal{F}(U)$ are called *sections of $\mathcal{F}$ over U*, and the maps ρ_{UV} are called *restriction maps*. If $s \in \mathcal{F}(U)$, then we usually denote $\rho_{UV}(s)$ by $s|_V$. We also use the notation $\Gamma(U, \mathcal{F})$ for $\mathcal{F}(U)$ (also denoted by $H^0(U, \mathcal{F})$; the reasons will become clear when we talk about cohomology below).

The condition (4) is a "gluing" condition: it means that if we have a collection of local sections on an open covering of an open set U in X satisfying compatibility conditions, we can "glue" them together to get a unique section on U.

Example 1.16. When X is a variety, it is obvious from the definition of the rings $\mathcal{O}(U)$ for open subsets $U \subseteq X$, that they form a sheaf called *the structure sheaf of* X and denoted by $\mathcal{O}_X$.

We continue with the notation from Definition 1.15. Similarly as in the case of varieties, we define the *stalk* of a sheaf $\mathcal{F}$ at a point $x \in X$ as

$$\mathcal{F}_x = \lim_{\rightarrow} \mathcal{F}(U),$$

where the limit is over all open subsets $x \in U \subseteq X$. Thus, if X is a variety, $\mathcal{O}_{X,x} = \mathcal{O}_x$.

1.2.2 Schemes

The category of sheaves on topological spaces is to big for algebraic geometry, and we want to enhance sheaves with some additional structure. The desired objects should model algebraic varieties, and there should exist a correspondence to affine/projective varieties.

First we define affine schemes. The definition is similar to affine varieties, though we work in larger generality as we deal with arbitrary (commutative, with unity) rings, and we do not care about an embedding into an affine space.

Recall that there is a bijective correspondence between points on an affine variety X and maximal ideals in $\mathcal{O}(X)$. Therefore, in a sense, to give points of X it is enough to list out maximal ideals in its ring of regular functions. However, to give information about the geometry of X, i.e. about all subvarieties it contains, it is equivalent to list all of its *prime* ideals (these are in bijective correspondence with closed subvarieties of X). And this is exactly what the definition of a spectrum is.

Definition 1.17. Let A be a ring. The space $\operatorname{Spec} A$ is the set of all prime ideals in A.

We want to put a topology on this space. Similarly as above, for any ideal $\mathcal{I} \subseteq A$, let $V(\mathcal{I})$ be the set of all prime ideals which contain $\mathcal{I}$, and we declare that these are the closed sets in this topology, called *Zariski topology.* This is indeed well-defined, since it is easy to check that then finite unions and arbitrary intersections of closed sets are again sets of this form.

This corresponds to the intuition that $V(\mathcal{I})$ should contain all the *zeroes* of $\mathcal{I}$, and if a subvariety belongs to this set, then its corresponding ideal should contain

$\mathcal{I}$. It is important to observe that this construction has Nullstellensatz built into it, since by an easy algebraic result, a radical of an ideal $\mathcal{I}$ is the intersection of all prime ideals containing $\mathcal{I}$.

To obtain a sheaf, we must define regular functions on $\operatorname{Spec} A$. Recall from the classical case above, if X is an affine variety and if $x \in X$ is a point, then $\mathcal{O}_x/\mathfrak{m}_x \simeq \mathbb{C}$, where $\mathfrak{m}_x$ is the maximal ideal in the local ring $\mathcal{O}_x$; more precisely, $\mathfrak{m}_x$ the set of all germs which vanish at x. Hence, if $f \in \mathcal{O}(U)$ for an open subset U of X, then $f(x)$ is exactly the image of f in $\mathcal{O}_x/\mathfrak{m}_x$.

Therefore, for a point $x \in \operatorname{Spec} A$, we want to view the localisation A_x as the *set of values of regular functions at* x (up to the maximal ideal). For every open subset $U \subseteq \operatorname{Spec} A$, we define the ring $\mathcal{O}_{\operatorname{Spec} A}(U)$ as the set of all functions

$$s\colon U \to \bigsqcup_{\mathfrak{p} \in U} A_{\mathfrak{p}}$$

such that s is a well-defined quotient of two elements locally around every point in U. More precisely, for every $\mathfrak{p} \in U$, there is a neighbourhood V of $\mathfrak{p}$ and $f, g \in A$, where $g \in A \backslash \mathfrak{q}$ for every $\mathfrak{q} \in V$, and $s(\mathfrak{q}) = f/g$ (in $A_{\mathfrak{q}}$).

Since the construction is local, it is obvious that all the conditions from Definition 1.15 are satisfied, and this indeed gives a structure of a sheaf on $\operatorname{Spec} A$, called the *spectrum of* A.

Now it is straightforward to show that $\mathcal{O}_{\operatorname{Spec} A, x} \simeq A_x$ for every point $x \in \operatorname{Spec} A$, and that $\Gamma(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}) \simeq A$, as expected.

It is important to note that $(\operatorname{Spec} A, \mathcal{O})$ is not only a sheaf, but it is a *locally ringed space*: these are sheaves $\mathcal{F}$ on topological spaces X where every group $\mathcal{F}(U)$ is a ring, and every stalk $\mathcal{F}_x$ is a local ring.

In order to define an affine scheme, we first have to introduce a notion of a morphism between locally ringed spaces $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$. Let $(\varphi, \varphi^\sharp)$ be a pair, where $\varphi\colon X \to Y$ is a continuous map and $\varphi^\sharp$ is a collection of morphisms $\varphi^\sharp_U\colon \mathcal{O}_Y(U) \to \mathcal{O}_X(\varphi^{-1}(U))$. Taking direct limits, this gives maps $\varphi^\sharp_x\colon \mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$ of local rings for every point $x \in X$. If the preimage of the maximal ideal of $\mathcal{O}_{X,x}$ equals the maximal ideal of $\mathcal{O}_{Y,\varphi(x)}$ for every such x, we say that $(\varphi, \varphi^\sharp)$ is a *morphism between locally ringed spaces*.

Definition 1.18. A locally ringed space is an *affine scheme* if it is isomorphic as a locally ringed space to $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A. A *scheme* is a locally ringed space which has a covering by affine schemes.

Now it should be obvious why this has advantages over the classical definition – we did not specify coordinates nor embedding of our (affine) scheme into any affine or projective space.

Example 1.19. Another way to give the affine space $\mathbb{A}^n$ is to define it as the locally ringed space $(\operatorname{Spec}\mathbb{C}[X_1,\ldots,X_n], \mathcal{O}_{\operatorname{Spec}\mathbb{C}[X_1,\ldots,X_n]})$. Indeed, everything so far has been modeled after this classical construction.

Moreover, for every ring A we have the *affine n-space over A*, defined as $\mathbb{A}^n_A = \operatorname{Spec} A[X_1,\ldots,X_n]$.

Now we turn to projective schemes. Recall that in the classical construction it was important that polynomials we deal with are homogeneous, and we used the notion of the degree. Another way to put it is to consider $\mathbb{C}[X_0,\ldots,X_n] = \bigoplus_{d\in\mathbb{N}} S_d$ as a *graded ring*, where the grading is by the degree of homogeneous polynomials.

It is an easy exercise (consequence of "weak" Nullstellensatz) to show that for a homogeneous ideal $\mathcal{I} \subseteq \mathbb{C}[X_0,\ldots,X_n]$, we have $Z(\mathcal{I}) = \emptyset$ iff $\bigoplus_{d>0} S_d \subseteq \sqrt{\mathcal{I}}$. Therefore, closed points in $\mathbb{P}^n$ are in bijective correspondence with maximal ideals of $\mathbb{C}[X_0,\ldots,X_n]$ other than $\bigoplus_{d>0} S_d$.

Definition 1.20. Consider any graded ring $S = \bigoplus_{d\in\mathbb{N}} S_d$, and the ideal $S_+ = \bigoplus_{d>0} S_d$. The space $\operatorname{Proj} S$ is the set of all homogeneous prime ideals in S other than S_+.

Similarly as before, to put the Zariski topology on this space, for any ideal $\mathcal{I} \subseteq S$, let $V(\mathcal{I})$ be the set of all homogeneous prime ideals in $\operatorname{Proj} S$ which contain $\mathcal{I}$, and we declare that these are the closed sets in this topology. This is again well-defined, since it is easy to check that then finite unions and arbitrary intersections of closed sets are again closed.

For each $\mathfrak{p} \in \operatorname{Proj} S$, we denote by $S_{(\mathfrak{p})}$ the *homogeneous localisation at* $\mathfrak{p}$: the set of all fractions f/g, where $f \in S$ and $g \in S\backslash\mathfrak{p}$ are homogeneous, and $\deg f = \deg g$. As in the case of affine schemes, for every open subset $U \subseteq \operatorname{Proj} S$, we define the ring $\mathcal{O}_{\operatorname{Proj} S}(U)$ as the set of all functions

$$s\colon U \to \bigsqcup_{\mathfrak{p}\in U} S_{(\mathfrak{p})}$$

such that s is a well-defined quotient of two homogeneous elements of the same degree locally around every point in U. More precisely, for every $\mathfrak{p} \in U$, there is a neighbourhood V of $\mathfrak{p}$ and homogeneous $f, g \in S$, where $g \in S\backslash\mathfrak{q}$ for every $\mathfrak{q} \in V$, $\deg f = \deg g$, and $s(\mathfrak{q}) = f/g$ (in $S_{(\mathfrak{q})}$).

Since the construction is local, it is obvious that all the conditions from the definition of a sheaf are satisfied, and this indeed gives a structure of a sheaf on $\operatorname{Proj} S$.

It is again straightforward to show that $\mathcal{O}_{\operatorname{Proj} S,x} \simeq S_{(x)}$ for every point $x \in \operatorname{Proj} S$. Also, similarly as in the classical case, $\operatorname{Proj} S$ is covered by affine schemes: it is an exercise to show that for every homogeneous $f \in S_+$, the ringed space on $\operatorname{Proj} S \setminus V(f)$ is isomorphic to $\operatorname{Spec} S_{(f)}$, and these sets cover $\operatorname{Proj} S$. Therefore, $(\operatorname{Proj} S, \mathcal{O}_{\operatorname{Proj} S})$ is a *projective scheme*.

Example 1.21. Another way to give the projective space $\mathbb{P}^n$ is to define it as the scheme $(\operatorname{Proj}\mathbb{C}[X_0,\ldots,X_n], \mathcal{O}_{\operatorname{Proj}\mathbb{C}[X_0,\ldots,X_n]})$.

Moreover, for every ring A we have the *projective n-space over A*, given as $\mathbb{P}^n_A = \operatorname{Proj} A[X_0,\ldots,X_n]$. Note that this is the same as $\mathbb{P}^n_{\mathbb{Z}} \times \operatorname{Spec} A$. We can extend this definition to any variety Y: we set $\mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times Y$, and we call this *the projective n-space over Y*. Another way to get this is to cover Y by open subsets $U_i = \operatorname{Spec} A_i$, form projective spaces $\mathbb{P}^n_{A_i}$, and then glue.

There are several properties of classical varieties that are desirable in the abstract context. First, notice that, tautologically, the section rings $\mathcal{O}(U)$ of classical varieties are integral domains, hence reduced rings (i.e. with no nilpotent elements) – this follows directly from Nullstellensatz. We would like to carry this property forward.

Definition 1.22. (1) A scheme X is *reduced* if $\mathcal{O}_X(U)$ has no nilpotents for every open set $U \subseteq X$.

(2) A scheme X is *integral* if $\mathcal{O}_X(U)$ is an integral domain for every open set $U \subseteq X$.

A scheme X is integral iff it is reduced and irreducible. One direction is trivial: if it is reducible, then $X = X_1 \cup X_2$, where X_i are proper closed subsets. Therefore, if $U_i = X \backslash X_i$, we have $U_1 \cap U_2 = \emptyset$, so $\mathcal{O}_X(U_1 \cup U_2) = \mathcal{O}_X(U_1) \times \mathcal{O}_X(U_2)$ which is clearly not an integral domain as $(u_1, 0)$ and $(0, u_2)$ multiply to zero for any $u_i \in \mathcal{O}_X(U_i)$. Converse is slightly more involved but not difficult.

Next, we can define the function field $k(X)$ of an integral scheme X as in the classical case, and we have the inclusion

$$\mathcal{O}_X(U) \subseteq \bigcap_{x \in U} \mathcal{O}_{X,x}$$

for every open subset U, where the intersection is taken inside $k(X)$. It is easy to see that this is an equality when X is integral – one only has to use the definition of a sheaf, and the fact that if a function vanishes on the whole U, then it is nilpotent, thus zero. In particular, this implies that on an integral scheme X, for every two open subsets $V \subseteq U \subseteq X$, the restriction map

$$\mathcal{O}_X(U) \to \mathcal{O}_X(V)$$

is *injective.*

Another property implicit from the classical picture is that a variety can be covered by finitely many affine varieties.

Definition 1.23. A scheme X is of *finite type* (over $\mathbb{C}$) if it can be covered by finitely many open subsets $U_i = \operatorname{Spec} A_i$, where each A_i is a finitely generated algebra over $\mathbb{C}$.

In particular, one can easily show that a scheme of finite type satisfies Noetherian property as a topological set, i.e. every descending sequence of closed subsets stabilises.

We just briefly mention another crucial property: we say that a scheme X is *separated* if the image of X under the diagonal embedding in $X \times X$ is closed. This can (with some pain) be verified for classical varieties, and for quasi-projective schemes.

Finally we have all the ingredients to define an abstract variety.

Definition 1.24. An *abstract variety* is an integral separated scheme of finite type over $\mathbb{C}$.

This class of schemes is strictly larger than the class of quasi-projective varieties/schemes, which was showed by Nagata in 1956. In many applications, it is necessary to step out of the realm of quasi-projective schemes.

However, the following result shows that the classical picture corresponds fully to our new abstract quasi-projective setting.

Theorem 1.25. *There is a fully faithful functor between the category of (classical) varieties and the category of quasi-projective integral schemes over $\mathbb{C}$. A classical variety is homeomorphic to the set of closed points of its image under this functor.*

1.2.3 Morphisms of sheaves

Earlier we saw what a morphism between structure sheaves of two different varieties is. Now we will define a morphism between two sheaves on the same variety.

Definition 1.26. Let X be a topological space, and let $\mathcal{F}$ and $\mathcal{G}$ be two sheaves on X with restriction maps ρ and θ respectively. A *morphism* $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a collection of morphisms $\varphi(U)\colon \mathcal{F}(U) \to \mathcal{G}(U)$ of abelian groups, such that the following diagram is commutative for every open $V \subseteq U$: By taking direct limits, this introduces corresponding maps on stalks $\varphi_x\colon \mathcal{F}_x \to \mathcal{G}_x$ for every point $x \in X$.

The *kernel* of a morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ is the sheaf of groups $\ker \varphi(U)$ for each U (check that this is indeed a sheaf!). Then we say that φ is *injective* if $\ker \varphi = 0$.

Exercise 1.27. Show that φ is injective iff φ_x is injective for every $x \in X$.

The natural definition of the image of φ would be to say that it is the sheaf of groups $\operatorname{im} \varphi(U)$ for each U. However, this is in general not a sheaf!

Exercise 1.28. Show that there exists a sheaf $\operatorname{im} \varphi$ and group homomorphisms $\xi(U)\colon \operatorname{im}(\varphi(U)) \to (\operatorname{im} \varphi)(U)$ such that there is a unique morphism $\varphi^+\colon \operatorname{im} \varphi \to \mathcal{G}$ with $\varphi = \varphi^+ \circ \xi$. The construction is similar to that of the structure sheaf of a variety.

This construction generalises to similar contexts, and we call this process *sheafification.*

Then we say that φ is *surjective* if $\operatorname{im}\varphi = \mathcal{G}$. One can again check that φ is surjective iff φ_x is surjective for every $x \in X$. One can, alternatively, use this property as the definition of surjectivity of morphisms of sheaves.

Finally, we say that a morphism φ is an *isomorphism* is it is both injective and surjective. Equivalently, φ_x is an isomorphism for every $x \in X$.

Assume now that a sheaf $\mathcal{F}$ on a variety X has an additional structure of an $\mathcal{O}_X$*-module*, i.e. assume that for every open $U \subseteq X$, $\mathcal{F}(U)$ is an $\mathcal{O}_X(U)$-module, and that this is compatible with restriction maps on $\mathcal{O}_X$ and $\mathcal{F}$. The set of all morphisms from an $\mathcal{O}_X$-module $\mathcal{F}$ to an $\mathcal{O}_X$-module $\mathcal{G}$ is denoted by $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

Example 1.29. For two $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$, the collection of modules $\mathcal{F}(U) \oplus \mathcal{G}(U)$ forms a sheaf $\mathcal{F} \oplus \mathcal{G}$, a *direct sum of* $\mathcal{F}$ *and* $\mathcal{G}$.

Similarly, by sheafifying the collection of modules $\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$, we get a sheaf $\mathcal{F} \otimes \mathcal{G}$, the *tensor product of* $\mathcal{F}$ *and* $\mathcal{G}$.

For every open subset U of X, the *restriction* $\mathcal{F}|_U$, defined in the obvious way, is an $\mathcal{O}_X|_U$-module. The collection of modules $\operatorname{Hom}_{\mathcal{O}_X|_U}(\mathcal{F}|_U, \mathcal{G}|_U)$ forms a sheaf, denoted by $\underline{\operatorname{Hom}}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$.

We say that $\mathcal{F}$ is a *free* $\mathcal{O}_X$*-module of rank* r if $\mathcal{F} \simeq \mathcal{O}_X^{\oplus r}$ for some $r \geq 1$.

Even though much of what we say here holds for all modules, in practice we usually work with *quasi-coherent* and *coherent* sheaves.

In order to define what this means, let A be a ring and M an A-module. Then we can consider localisations $M_{\mathfrak{p}}$ for $\mathfrak{p} \in A$, and we can construct sets of regular functions on open sets $U \subseteq \operatorname{Spec} A$ with values in M as certain maps $U \to \coprod_{\mathfrak{p} \in U} M_{\mathfrak{p}}$ – this construction parallels that of regular functions, and is left as an easy (but important!) exercise.

In this way we construct sheaves $\widetilde{M}$ on $\operatorname{Spec} A$, and it is obvious that they are $\mathcal{O}_{\operatorname{Spec} A}$-modules. Note that $\mathcal{O}_{\operatorname{Spec} A} = \widetilde{A}$ in this new notation.

Definition 1.30. If $\mathcal{F}$ is an $\mathcal{O}_X$-module on a variety X, it is *quasi-coherent* if there is an open covering $\{U_i = \operatorname{Spec} A_i\}$ of X such that $\mathcal{F}_{|U_i} \simeq \widetilde{M_i}$ for some A_i-module M_i. If additionally each M_i is a finitely generated A_i-module, then we say that $\mathcal{F}$ is *coherent*.

Locally free sheaves of finite rank are obviously coherent sheaves.

We can run an analogous construction if we are given a graded ring S and a graded S-module M, to construct a sheaf $\widetilde{M}$ on $\operatorname{Proj} S$ (exercise!).

Next we want to define pullbacks and pushforwards of sheaves on varieties under morphisms.

Definition 1.31. Let $f\colon X \to Y$ be a morphism between two varieties, and let $\mathcal{F}$ be a sheaf on X. The *pushforward* of $\mathcal{F}$ (or *direct image* of $\mathcal{F}$) is a sheaf $f_*\mathcal{F}$, which is defined by

$$(f_*\mathcal{F})(U) = \mathcal{F}(f^{-1}(U))$$

for every open subset U of Y.

Now, if $\mathcal{F}$ is an $\mathcal{O}_X$-module, then there is an obvious $f_*\mathcal{O}_X$-module structure on $f_*\mathcal{F}$. Recall that there is a map $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$, and this turns $f_*\mathcal{F}$ into an $\mathcal{O}_Y$-module.

To get a feel for what this means, assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, and that $\mathcal{F} = \widetilde{M}$ for an A-module M. Recall that there is a ring homomorphism $B \to A$. Then it is easy to show (do it!) that $f_*\widetilde{M} = \widetilde{{}_BM}$, where ${}_BM$ is just M considered as a B-module via the map $B \to A$.

In general, if $\mathcal{F}$ is quasi-coherent, we can cover X and Y by open affines, and glue this construction.

Definition 1.32. Let $f\colon X \to Y$ be a morphism between two varieties, and let $\mathcal{F}$ be a sheaf on Y. The *pullback* of $\mathcal{F}$ (or *inverse image* of $\mathcal{F}$) is a sheaf $f^{-1}\mathcal{F}$, which is obtained by sheafifying the collection of groups

$$\lim_{f(U)\subseteq V} \mathcal{F}(V)$$

for every open subset U of X, where the injective limit runs through all open subsets V of Y which contain the set $f(U)$.

Even though involved, this definition is precisely what we want, since we want to relate open subsets of X and Y. Note that if f is an *open map*, then the limit in the definition is just $\mathcal{F}(f(U))$.

Therefore, if X is an open subvariety of Y with the inclusion map $i\colon X \hookrightarrow Y$, and if $\mathcal{F}$ is a sheaf on Y, then $\mathcal{F}_{|X}$ is just the pullback $i^{-1}\mathcal{F}$.

Similarly, if X is a closed subvariety of Y with the inclusion map (*closed immersion*) $i\colon X \hookrightarrow Y$, and if $\mathcal{F}$ is a sheaf on Y, we define the *restriction* $\mathcal{F}|_X$ *of* $\mathcal{F}$ as $i^{-1}\mathcal{F}$.

This is compatible with the situation when $\mathcal{F} = \mathcal{O}_Y$: indeed, it is easy to check that $\mathcal{O}_X = \mathcal{O}_Y|_X$ since both sides can be interpreted locally.

Exercise 1.33. If $f\colon X \to Y$ is a morphism between two varieties, and if $\mathcal{F}$ is a sheaf on X, show that there is a well-defined natural homomorphism $f^{-1}f_*\mathcal{F} \to \mathcal{F}$.

Now, this gives a composition of maps $f^{-1}\mathcal{O}_Y \to f^{-1}f_*\mathcal{O}_X \to \mathcal{O}_X$, and if $\mathcal{F}$ is an $\mathcal{O}_X$-module, then there is an obvious $f^{-1}\mathcal{O}_Y$-module structure on $f^{-1}\mathcal{F}$. Therefore, there is a well-defined $\mathcal{O}_X$-module

$$f^*\mathcal{F} = f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X$$

and we call it the *pullback of an $\mathcal{O}_Y$-module $\mathcal{F}$*. Now it is easy to check that

$$(f^*\mathcal{F})_x \simeq \mathcal{F}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$$

for every point $x \in X$.

To get a feel for what this means, assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, and that $\mathcal{F} = \widetilde{N}$ for a B-module N. It is again easy to show (do it!) that $f^*\widetilde{N} = (N \otimes_B A)\tilde{}$, where again recall that we are equipped with a map $B \to A$.

In general, if $\mathcal{F}$ is quasi-coherent, we can cover X and Y by open affines, and glue this construction.

Therefore, locally a section φ of $\mathcal{F}$ gets sent to $\varphi \otimes 1$.

Let again $f \colon X \to Y$ be a morphism between two varieties. The relation between functors f_* and f^* is that they are *adjoint*, i.e. for an $\mathcal{O}_X$-module $\mathcal{F}$ and an $\mathcal{O}_Y$-module $\mathcal{G}$, there is an isomorphism

$$\operatorname{Hom}(f^*\mathcal{G}, \mathcal{F}) \simeq \operatorname{Hom}(\mathcal{G}, f_*\mathcal{F}).$$

This follows easily from the above exercise.

As a side note, one can easily check that if $i \colon X \hookrightarrow Y$ is a closed inclusion, then for any sheaf $\mathcal{F}$ on Y we have $\mathcal{F}|_X = i^*\mathcal{F} = \mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_X$.

1.3 Cohomology

Similarly as in other branches of mathematics, cohomological methods are among the most powerful tools, and they enable us to solve hard problems which are unreachable by other methods. I start with some general facts about derived functors in the context of sheaves on varieties, and later we will see some important consequences regarding the cohomology of affine and projective varieties. The moral of the story is that cohomology is all about studying exact sequences, long and short.

Let X be a variety. Start with an $\mathcal{O}_X$-module $\mathcal{I}$ and a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

of $\mathcal{O}_X$-modules on X. We want to apply the $\operatorname{Hom}_{\mathcal{O}_X}(\cdot, \mathcal{I})$ functor to this sequence. It is an easy exercise (!) that then we have the exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{I}) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}).$$

This sequence is not always exact on the right. When it is exact on the right, i.e. when we have

$$0 \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{H}, \mathcal{I}) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{G}, \mathcal{I}) \to \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}) \to 0,$$

we say that the module $\mathcal{I}$ is *injective*. Now, the basic (but non-trivial) fact is the following.

Lemma 1.34. *Let X be a variety and let $\mathcal{F}$ be an $\mathcal{O}_X$-module on X. Then there is an injective $\mathcal{O}_X$-module $\mathcal{G}$ and an injection $i\colon \mathcal{F} \hookrightarrow \mathcal{G}$.*

In other words, any module can be extended to an injective module on X. The $\mathcal{G}$ as above is called an *injective extension of* $\mathcal{F}$.

Now, start with an $\mathcal{O}_X$-module $\mathcal{F}_0 = \mathcal{F}$ on X. Then, for $m \geq 0$ we construct inductively $\mathcal{O}_X$-modules $\mathcal{F}_m$ and $\mathcal{I}_m$ as follows. Assume we have constructed $\mathcal{F}_m$. Then, by the lemma, it has an injective extension $i_m\colon \mathcal{F}_m \hookrightarrow \mathcal{I}_m$, and set $\mathcal{F}_{m+1} = \mathcal{I}_m/\mathcal{F}_m$. Note that then we have the short exact sequence

$$0 \to \mathcal{F}_m \xrightarrow{i_m} \mathcal{I}_m \xrightarrow{\pi_m} \mathcal{F}_{m+1} \to 0$$

for every m, and this gives maps

$$\delta_m = i_{m+1} \circ \pi_m \colon \mathcal{I}_m \to \mathcal{I}_{m+1}.$$

Since $\ker \delta_m = \ker \pi_m = \operatorname{im} i_m = \operatorname{im} \delta_{m-1}$, this gives the long exact sequence (*resolution*)

$$0 \to \mathcal{F} \xrightarrow{i_0} \mathcal{I}_0 \xrightarrow{\delta_0} \mathcal{I}_1 \xrightarrow{\delta_1} \dots$$

Any such resolution by injective modules is called an *injective resolution of* $\mathcal{F}$. Any two injective resolutions are homotopy equivalent (in the sense of topology), and therefore if we set $\delta_{-1} = 0$ and

$$H^m(U, \mathcal{F}) = \ker \delta_{m,U} / \operatorname{im} \delta_{m-1,U}$$

for every open subset U of X, these groups do not depend on the choice of an injective resolution (up to isomorphism). These groups are the *cohomology groups of* $\mathcal{F}$. Note that $H^0(U, \mathcal{F}) = \Gamma(U, \mathcal{F})$ by definition.

Further, observe that if $\mathcal{I}$ is an injective module, then

$$0 \to \mathcal{I} \xrightarrow{\mathrm{id}} \mathcal{I} \to 0$$

is an injective resolution of $\mathcal{I}$, and therefore $H^q(U, \mathcal{I}) = 0$ for all $q > 0$.

Whenever we have a morphism $\theta\colon \mathcal{F} \to \mathcal{G}$ of sheaves on X, there are natural morphisms $H^q(U, \theta)\colon H^q(U, \mathcal{F}) \to H^q(U, \mathcal{G})$ for all open subsets U and all $q \geq 0$. The cohomology groups and these natural maps define covariant functors from the category of $\mathcal{O}_X$-modules to the category of abelian groups.

From the general theory of derived functors (Snake Lemma), we know that if we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

of $\mathcal{O}_X$-modules on X, then there is the associated long exact cohomology sequence

$$\begin{aligned} 0 &\to H^0(U,\mathcal{F}) \to H^0(U,\mathcal{G}) \to H^0(U,\mathcal{H}) \\ &\xrightarrow{d_0} H^1(U,\mathcal{F}) \to H^1(U,\mathcal{G}) \to H^1(U,\mathcal{H}) \\ &\xrightarrow{d_1} \dots \\ &\xrightarrow{d_{q-1}} H^q(U,\mathcal{F}) \to H^q(U,\mathcal{G}) \to H^q(U,\mathcal{H}) \\ &\xrightarrow{d_q} \dots \end{aligned}$$

where d_i are naturally defined *connecting morphisms* and other non-labeled maps are just natural maps defined above. This sequence is one of the basic tools in modern Algebraic Geometry.

The basic vanishing result is the following theorem of Grothendieck. It is analogous to a result in topology, where we have vanishing of (co)homology in degrees higher than the dimension of the topological space, and therefore it is a desirable and expected result.

Theorem 1.35. *Let X be a variety and let $\mathcal{F}$ be a sheaf on X. Then $H^q(X,\mathcal{F}) = 0$ for all $q > \dim X$.*

Thus, the long exact cohomology sequence introduced above has only finitely many non-zero terms. One of the main problems in geometry is to determine in which circumstances some other groups in the sequence vanish.

The picture is fairly easy when we are on an affine variety. The following result is due to Serre.

Theorem 1.36. *Let X be a variety. Then the following conditions are equivalent.*

(1) X is affine,

(2) $H^q(X,\mathcal{F}) = 0$ for every $q > 0$ and every quasi-coherent sheaf $\mathcal{F}$,

(3) $H^1(X,\mathcal{I}) = 0$ for every quasi-coherent ideal sheaf $\mathcal{I}$.

Therefore, if Y is a closed subvariety of an affine variety with the associated ideal sheaf $\mathcal{I}$, the long exact cohomology sequence degenerates to the following short exact sequence:

$$0 \to H^0(X,\mathcal{I}) \to H^0(X,\mathcal{O}_X) \to H^0(Y,\mathcal{O}_Y) \to 0,$$

which we already knew.

The following important result will be crucial when we discuss linear systems on projective varieties later in the course.

Theorem 1.37. *Let X be a projective variety and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then $H^q(X, \mathcal{F})$ is a finite-dimensional $\mathbb{C}$-vector space for every $q \geq 0$.*

The dimension $\dim_{\mathbb{C}} H^q(X, \mathcal{F})$ is usually denoted by $h^q(X, \mathcal{F})$. Therefore, this theorem and the vanishing theorem of Grothendieck allow to make the following definition, similarly as in topology.

Definition 1.38. Let X be a projective variety and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then the sum

$$\chi(X, \mathcal{F}) = \sum_{q=0}^{\infty} (-1)^q h^q(X, \mathcal{F})$$

is a finite sum of integers, and it is called the *Euler-Poincaré characteristic of $\mathcal{F}$*.

The basic fact about the Euler characteristic is that it is additive on short exact sequences.

Lemma 1.39. *Let X be a projective variety and let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be coherent $\mathcal{O}_X$-modules such that we have the short exact sequence*

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0.$$

Then $\chi(\mathcal{G}) = \chi(\mathcal{F}) + \chi(\mathcal{H})$.

The proof is very easy, and it follows from the long exact cohomology sequence (exercise!).

A less trivial property is the following:

Theorem 1.40 (Weak Riemann-Roch)**.** *Let X be a proper scheme of dimension n over a field and let $L_1, \dots, L_r$ be line bundles on X (cf. Definition 2.11). Then $\chi(X, k_1 L_1 + \cdots + k_r L_r)$ is a polynomial of degree at most n in $k_1, \dots, k_r$.*

The proof can be obtained by reducing first to the case where all L_i are very ample line bundles (cf. Definition 2.17), and then applying the strategy similar to [Har77, Exercise III.5.2(a)].

Now we go one step further, and we define a *relative version of cohomology*, i.e. cohomology attached to a morphism $f \colon X \to Y$. This cohomology is sometimes called *cohomology of the fibres*, but we will not touch upon this here.

A good thing to keep in mind is that the cohomology groups defined above are *global* objects, i.e. they tell us something about the behaviour of global sections of a variety under a derived functor. Cohomology in this lecture is *local* (over the base, i.e. over the target of a morphism), and it allows us to consider how various (global) cohomology groups behave locally.

So consider a morphism $f\colon X \to Y$ between two varieties, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Let

$$0 \to \mathcal{F} \xrightarrow{i_0} \mathcal{I}_0 \xrightarrow{\delta_0} \mathcal{I}_1 \xrightarrow{\delta_1} \dots$$

be an injective resolution of $\mathcal{F}$; as before, set $\delta_{-1} = 0$. Then it is easy to see that when we pushforward this sequence to Y, we get a complex of $\mathcal{O}_Y$-modules:

$$0 \to f_*\mathcal{F} \to f_*\mathcal{I}_0 \to f_*\mathcal{I}_1 \to \dots$$

Denote $(f_*\delta_m)_U = \delta_{m,f^{-1}(U)}$ for every m and for every open subset U of Y. Then the sheaves

$$R^m f_*(\mathcal{F}) = \ker(f_*\delta_m)/\operatorname{im}(f_*\delta_{m-1})$$

do not depend on an injective resolution (up to isomorphism). They are the *higher direct image sheaves of* $\mathcal{F}$. Note that

$$R^0 f_*(\mathcal{F}) = f_*\mathcal{F}$$

by definition. Also, if Y is a point, then

$$f_*\mathcal{F} = H^0(X, \mathcal{F}).$$

Further, observe that if $\mathcal{I}$ is an injective module, then $R^q f_*(\mathcal{I}) = 0$ for all $q > 0$.

Whenever we have a morphism $\theta\colon \mathcal{F} \to \mathcal{G}$ of $\mathcal{O}_X$-modules, there are natural morphisms $R^q f_*(\theta)\colon R^q f_*(\mathcal{F}) \to R^q f_*(\mathcal{G})$ for all $q \geq 0$. The higher direct image sheaves and these natural maps define covariant functors from the category of $\mathcal{O}_X$-modules to the category of $\mathcal{O}_Y$-modules.

Similarly as before, from the general theory of derived functors it follows that if we have a short exact sequence

$$0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$$

of $\mathcal{O}_X$-modules, then there is the associated long exact sequence of higher direct image sheaves

$$\begin{aligned} 0 &\to R^0 f_*(\mathcal{F}) \to R^0 f_*(\mathcal{G}) \to R^0 f_*(\mathcal{H}) \\ &\xrightarrow{d_0} R^1 f_*(\mathcal{F}) \to R^1 f_*(\mathcal{G}) \to R^1 f_*(\mathcal{H}) \\ &\xrightarrow{d_1} \dots \\ &\xrightarrow{d_{q-1}} R^q f_*(\mathcal{F}) \to R^q f_*(\mathcal{G}) \to R^q f_*(\mathcal{H}) \\ &\xrightarrow{d_q} \dots \end{aligned}$$

where d_i are naturally defined *connecting morphisms* and other non-labeled maps are just natural maps as above.

It is worth noting (and remembering!) that if $\mathcal{F}$ is a quasi-coherent sheaf on X, then every $R^q f_*(\mathcal{F})$ is a quasi-coherent sheaf on Y, which follows from Theorem 1.41 below. The analogous statement for coherent sheaves holds if f is a finite (cf. Definition 3.7) or projective morphism (cf. Definition 2.20).

We would like to relate cohomology groups and higher direct images of sheaves. Any relation between the two is very useful in algebraic geometry, as cohomology groups measure *global* properties of varieties, whereas higher direct images of sheaves measure *local* properties. The connection is summarised in the following result.

Theorem 1.41. *Let $f\colon X \to Y$ be a morphism and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then:*

(1) for each $q \geq 0$, the sheaf $R^q f_*(\mathcal{F})$ is the sheafification of the collection of groups $H^q(f^{-1}(U), \mathcal{F}_{|f^{-1}(U)})$ for all open subsets U of Y,

(2) if Y is affine and $\mathcal{F}$ is quasi-coherent, then $R^q f_*(\mathcal{F}) \simeq H^q(X, \mathcal{F})^\sim$.

1.4 Complexes of sheaves

Here I collect some facts about complexes of sheaves which we need in this course.

Let X be a variety and let $(\mathcal{F}^\bullet, d)$ be a complex of $\mathcal{O}_X$-modules, i.e. each $\mathcal{F}^i$ is an $\mathcal{O}_X$-module and we have maps $d_i\colon \mathcal{F}^i \to \mathcal{F}^{i+1}$, called *differentials* of $\mathcal{F}^\bullet$, such that $d_{i+1} \circ d_i = 0$ for all i; usually we omit the subscripts and write this relation just as $d^2 = 0$. The *cohomology* of the complex $\mathcal{F}^\bullet$ is

$$\mathcal{H}^i(\mathcal{F}) = \ker d_i / \operatorname{im} d_{i-1}.$$

A map of complexes $\sigma\colon \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a collection of morphisms $\sigma_i\colon \mathcal{F}^i \to \mathcal{G}^i$ which are compatible with the differentials of $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$. Any such map σ induces a map of cohomology sheaves

$$\mathcal{H}^i(\sigma)\colon \mathcal{H}^i(\mathcal{F}^\bullet) \to \mathcal{H}^i(\mathcal{G}^\bullet),$$

and σ is a *quasi-isomorphism* if $\mathcal{H}^i(\sigma)$ is an isomorphism for all i.

In this course we only consider complexes $\mathcal{F}^\bullet$ which are bounded from below and from above, i.e. $\mathcal{F}^i = 0$ for i sufficiently negative and sufficiently positive.

Definition 1.42. A map $\sigma\colon \mathcal{F}^\bullet \to \mathcal{I}^\bullet$ is an *injective resolution* of $\mathcal{F}^\bullet$ if $\mathcal{I}^\bullet$ is a complex of $\mathcal{O}_X$-modules bounded from below, σ is a quasi-isomorphism, and the sheaves $\mathcal{I}^i$ are injective for all i. It is well-known that every complex of $\mathcal{O}_X$-modules which is bounded from below admits an injective resolution. Then the i-th *hypercohomology group* is

$$\mathbb{H}^i(X, \mathcal{F}^\bullet) = \frac{\ker(\Gamma(X, \mathcal{I}^i) \to \Gamma(X, \mathcal{I}^{i+1}))}{\operatorname{im}(\Gamma(X, \mathcal{I}^{i-1}) \to \Gamma(X, \mathcal{I}^i))}.$$

This does not depend on the injective resolution, and it is easy to see that this generalises the definition of the cohomology of a sheaf $\mathcal{F}$, by taking a complex $\mathcal{F}^\bullet$ such that $\mathcal{F}^0 = \mathcal{F}$ and $\mathcal{F}^i = 0$ if $i \neq 0$.

In particular, if $\sigma\colon \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ is a quasi-isomorphism, then by taking an injective resolution of $\mathcal{G}^\bullet$ which is also an injective resolution of $\mathcal{F}^\bullet$, the map σ induces an isomorphism of the hypercohomology groups

$$\mathbb{H}^i(X, \mathcal{F}^\bullet) \simeq \mathbb{H}^i(X, \mathcal{G}^\bullet) \quad \text{for all } i.$$

Example 1.43. Let $\mathcal{F}^\bullet$ and $\mathcal{G}^\bullet$ be two complexes of locally free sheaves bounded from below, and denote by d both differentials of $\mathcal{F}^\bullet$ and of $\mathcal{G}^\bullet$. Assume that $d\mathcal{F}^{a-1}$ and $d\mathcal{G}^{b-1}$ are locally free for all a and b. Then the *tensor product* $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$ is defined as

$$(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet)^i = \bigoplus_a \mathcal{F}^a \otimes \mathcal{G}^{i-a},$$

with the differential sending $f_a \otimes g_{i-a} \in \mathcal{F}^a \otimes \mathcal{G}^{i-a}$ to

$$df_a \otimes g_{i-a} + (-1)^a f_a \otimes dg_{i-a} \in \mathcal{F}^{a+1} \otimes \mathcal{G}^{i-a} \oplus \mathcal{F}^a \otimes \mathcal{G}^{i+1-a}.$$

It is clear that $\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet$ is a complex of locally free sheaves bounded from below. Then it is easy to see that locally, one has complexes of free sheaves $\mathcal{F}'^\bullet$ and $\mathcal{G}'^\bullet$ such that

$$\mathcal{F}^a = d\mathcal{F}^{a-1} \oplus \mathcal{H}^a(\mathcal{F}^\bullet) \oplus \mathcal{F}'^a, \quad \mathcal{G}^b = d\mathcal{G}^{b-1} \oplus \mathcal{H}^b(\mathcal{G}^\bullet) \oplus \mathcal{G}'^a,$$

where $d\colon \mathcal{F}'^a \to d\mathcal{F}^a$ and $d\colon \mathcal{G}'^b \to d\mathcal{G}^b$ are isomorphisms. From here it is easy to deduce the *Künneth* formula:

$$\mathcal{H}^i(\mathcal{F}^\bullet \otimes \mathcal{G}^\bullet) = \bigoplus_a \mathcal{H}^a(\mathcal{F}^\bullet) \otimes \mathcal{H}^{i-a}(\mathcal{G}^\bullet) \quad \text{for all } i.$$

In particular, suppose we have a quasi-isomorphism $\sigma\colon \mathcal{A}^\bullet \to \mathcal{B}^\bullet$ of two complexes of locally free sheaves. Then for every $a \geq 1$, the induced map $\sigma^{\otimes a}\colon (\mathcal{A}^\bullet)^{\otimes a} \to (\mathcal{B}^\bullet)^{\otimes a}$ is a quasi-isomorphism.

Example 1.44. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be an open covering of a variety X, and let $(\mathcal{F}^\bullet, d)$ be a complex of $\mathcal{O}_X$-modules bounded from below. Denote $U_{i_0\dots i_a} = U_{i_0} \cap \cdots \cap U_{i_a}$, and let $j\colon U_{i_0\dots i_a} \to X$ be the inclusion. Then we have the associated *Čech complex* $\check{\mathcal{C}}^\bullet$ defined as

$$\check{\mathcal{C}}^i = \bigoplus_{a\geq 0} \check{\mathcal{C}}^a(\mathcal{U}, \mathcal{F}^{i-a}), \quad \text{where} \quad \check{\mathcal{C}}^a(\mathcal{U}, \mathcal{F}^{i-a}) = \prod_{i_0<i_1<\cdots<i_a} j_*\mathcal{F}^{i-a}|_{U_{i_0\cdots i_a}}.$$

This direct sum has finitely many summands. For $s \in \check{\mathcal{C}}^a(\mathcal{U}, \mathcal{F}^{i-a})$ we set

$$\delta(s)_{i_0 \dots i_{a+1}} = \sum_{\ell=0}^{a+1} (-1)^\ell s_{i_0 \dots \hat{i}_\ell \dots i_{a+1}}|_{U_{i_0 \cdots i_{a+1}}},$$

and define the differential Δ of $\check{\mathcal{C}}^\bullet$ by

$$\Delta(s) = (-1)^i \delta(s) + d(s).$$

Then the natural map $\sigma \colon \mathcal{F}^\bullet \to \check{\mathcal{C}}^\bullet$ defined by

$$\mathcal{F}^i \xrightarrow{j} \prod_{i \in I} j_* \mathcal{F}^i|_{U_i} = \check{\mathcal{C}}^0(\mathcal{U}, \mathcal{F}^i)$$

is a quasi-isomorphism.

1.4.1 Spectral sequences

Fix a ring R. Let $\mathcal{C}^{\bullet,\bullet}$ be a double complex of R-modules, i.e. a collection of R-modules with rightward morphisms $d_{\rightarrow}^{p,q} \colon \mathcal{C}^{p,q} \to \mathcal{C}^{p+1,q}$ and upward morphisms $d_{\uparrow}^{p,q} \colon \mathcal{C}^{p,q} \to \mathcal{C}^{p,q+1}$, and such that the following holds: (a) $d_{\rightarrow}^2 = 0$, (b) $d_{\uparrow}^2 = 0$, and (c) $d_{\rightarrow} d_{\uparrow} + d_{\uparrow} d_{\rightarrow} = 0$. The corresponding *total complex* $\mathcal{C}^\bullet$ is defined as $\mathcal{C}^i = \bigoplus_a \mathcal{C}^{a,i-a}$ and with the differential $d = d_{\rightarrow} + d_{\uparrow}$.

The *spectral sequence with rightward orientation* is a sequence of *pages* (indexed by integers $r \geq 0$) of R-modules ${}_{\rightarrow}E_r^{p,q}$ (indexed by $(p,q) \in \mathbb{Z}^2$) where ${}_{\rightarrow}E_0^{p,q} = \mathcal{C}^{p,q}$, with differentials

$${}_{\rightarrow}d_r^{p,q} \colon {}_{\rightarrow}E_r^{p,q} \to {}_{\rightarrow}E_r^{p-r+1,q+r}$$

which satisfy ${}_{\rightarrow}d_r^{p,q} \circ_{\rightarrow} d_r^{p+r-1,q-r} = 0$, and such that

$${}_{\rightarrow}E_{r+1}^{p,q} \simeq \ker {}_{\rightarrow}d_r^{p,q} / \operatorname{im} {}_{\rightarrow}d_r^{p+r-1,q-r}.$$

In particular, ${}_{\rightarrow}E_{r+1}^{p,q}$ is a sub-quotient of ${}_{\rightarrow}E_r^{p,q}$.

Now, if $\mathcal{C}^{\bullet,\bullet}$ is a first quadrant double complex, i.e. if $\mathcal{C}^{p,q} = 0$ for $p < 0$ or $q < 0$, then for any fixed (p,q), there exists $r_0 \gg 0$ such that

$${}_{\rightarrow}E_r^{p,q} \simeq {}_{\rightarrow}E_{r_0}^{p,q} \quad \text{for all } r \geq r_0,$$

and we denote this R-module by ${}_{\rightarrow}E_\infty^{p,q}$.

Then for every $k \geq 0$ there exists a filtration

$${}_{\rightarrow}E_\infty^{0,k} = \mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots \subseteq \mathcal{F}_{k-1} \subseteq \mathcal{F}_k = \mathcal{H}^k(\mathcal{C}^\bullet) \tag{1.4}$$

such that ${}_{\rightarrow}E_\infty^{i,k-i} \simeq \mathcal{F}_i/\mathcal{F}_{i-1}$ for $i = 1, \ldots, k$. Then for any $r \geq 0$, we say that the page ${}_{\rightarrow}E_r^{\bullet,\bullet}$ *converges* or *abuts* to $\mathcal{H}^\bullet(\mathcal{C}^\bullet)$, and we write

$$ {}_{\rightarrow}E_r^{p,q} \Longrightarrow \mathcal{H}^{p+q}(\mathcal{C}^\bullet). $$

Often we omit the direction of the differential and instead write

$$ E_r^{p,q} \underset{q}{\Longrightarrow} \mathcal{H}^{p+q}(\mathcal{C}^\bullet) $$

in order to stress that the differentials for $r = 0$ fix the second coordinate. If for some r we have $E_r^{p,q} \simeq E_\infty^{p,q}$ for all p and q, we say that the spectral sequence above *degenerates* at the page E_r.

One also has the *spectral sequence with upward orientation*, i.e. a sequence of pages of R-modules ${}_{\uparrow}E_r^{p,q}$ with differentials

$$ {}_{\uparrow}d_r^{p,q} \colon {}_{\uparrow}E_r^{p,q} \rightarrow {}_{\uparrow}E_r^{p+r,q-r+1}, $$

in which case we write

$$ E_r^{p,q} \underset{p}{\Longrightarrow} \mathcal{H}^{p+q}(\mathcal{C}^\bullet). $$

In general there are no isomorphisms between groups ${}_{\rightarrow}E_\infty^{p,q}$ and ${}_{\uparrow}E_\infty^{p,q}$.

Now, if $\mathcal{C}^{\bullet,\bullet}$ is a double complex of *vector spaces* over a field K, then from (1.4) we have an isomorphism of vector spaces

$$ \mathcal{H}^k(\mathcal{C}^\bullet) \simeq \bigoplus_i E_\infty^{i,k-i}. \tag{1.5} $$

Since $\dim_K E_\infty^{p,q} \leq \dim_K E_r^{p,q}$ for all p, q and r, then the spectral sequence degenerates at E_r if and only if

$$ \dim_K \mathcal{H}^k(\mathcal{C}^\bullet) = \sum_i \dim_K E_r^{i,k-i} \quad \text{for all } k. \tag{1.6} $$

1.4.2 Hodge-to-de Rham spectral sequence

Let X be a variety. Let $\mathcal{C}^\bullet$ be a complex of $\mathcal{O}_X$-modules which is bounded below (for instance, assume that $\mathcal{C}^p = 0$ for $p < 0$), and let $\mathcal{C}^\bullet \rightarrow \mathcal{I}^\bullet$ be an injective resolution. Then there exists a simultaneous injective resolution $\mathcal{I}^\bullet \rightarrow \mathcal{J}^{\bullet,\bullet}$: it can be built inductively from the lower left corner of the resolution upwards, or one can even construct a more precise version of the resolution called Cartan-Eilenberg resolution, see [Vak13, 23.3.7]. Thus, for each p, $\mathcal{I}^p \rightarrow \mathcal{J}^{p,\bullet}$ is an injective resolution of $\mathcal{I}^p$. Let $\mathcal{J}^\bullet$ be the corresponding total complex of the double complex $\mathcal{J}^{\bullet,\bullet}$.

Consider the spectral sequence with the upward orientation associated to $\mathcal{J}^{\bullet,\bullet}$. Then $E_1^{p,q}(\mathcal{J}^{\bullet,\bullet}) = 0$ for each $q > 0$, and $E_1^{p,0}(\mathcal{J}^{\bullet,\bullet}) = \mathcal{I}^p$. Hence $\mathcal{H}^p(\mathcal{J}^\bullet) \simeq E_2^{p,0}(\mathcal{J}^{\bullet,\bullet}) = \mathcal{H}^p(\mathcal{I}^\bullet)$, and therefore the complexes $\mathcal{I}^\bullet$ and $\mathcal{J}^\bullet$ are quasi-isomorphic.

Now apply the functor $\Gamma(X, \cdot)$ to $\mathcal{I}^\bullet$ and to $\mathcal{J}^{\bullet,\bullet}$. The *Hodge-to-de Rham spectral sequence* is the spectral sequence with the upward orientation associated to $\Gamma(X, \mathcal{J}^{\bullet,\bullet})$. It converges to $\mathcal{H}^\bullet(\Gamma(X, \mathcal{J}^\bullet)) \simeq \mathcal{H}^\bullet(\Gamma(X, \mathcal{I}^\bullet)) \simeq \mathbb{H}^\bullet(X, \mathcal{C}^\bullet)$, and the E_1-page gives the spectral sequence

$$E_1^{pq} = H^q(X, \mathcal{C}^p) \underset{p}{\Longrightarrow} \mathbb{H}^{p+q}(X, \mathcal{C}^\bullet).$$

1.4.3 Leray spectral sequence

Theorem 1.45. *Let $f\colon X \to Y$ be a morphism between two varieties. Then for any $\mathcal{O}_X$-module $\mathcal{F}$ there exists a spectral sequence*

$$E_2^{p,q} = H^q(Y, R^p f_* \mathcal{F}) \underset{q}{\Longrightarrow} H^{p+q}(X, \mathcal{F}).$$

The proof is similar to, but more involved than that of the Hodge-to-de Rham spectral sequence. Choose an injective resolution $\mathcal{F} \to \mathcal{I}^\bullet$, and consider a Cartan-Eilenberg resolution $f_*\mathcal{I}^\bullet \to \mathcal{J}^{\bullet,\bullet}$. Let $\mathcal{J}^\bullet$ be the corresponding total complex of the double complex $\mathcal{J}^{\bullet,\bullet}$. Then we calculate the cohomology of $\Gamma(X, \mathcal{J}^\bullet)$ using two spectral sequences associated to $\Gamma(Y, \mathcal{J}^{\bullet,\bullet})$.

Considering the upward orientation, we obtain $E_1^{p,q}(\Gamma(Y, \mathcal{J}^{\bullet,\bullet})) = 0$ for each $q > 0$, and $E_1^{p,0}(\Gamma(Y, \mathcal{J}^{\bullet,\bullet})) = \Gamma(Y, f_*\mathcal{I}^p) = \Gamma(X, \mathcal{I}^p)$. Hence

$$\mathcal{H}^p(\Gamma(Y, \mathcal{J}^\bullet)) \simeq E_2^{p,0}(\Gamma(Y, \mathcal{J}^{\bullet,\bullet})) = \mathcal{H}^p(\Gamma(X, \mathcal{I}^\bullet)) = H^p(X, \mathcal{F}).$$

On the other hand, by using carefully the construction of the Cartan-Eilenberg resolution, we obtain that the E_2-page of the rightward spectral sequence is $E_2^{p,q} = H^q(Y, R^p f_* \mathcal{F})$; the details are in [Vak13, 23.3.8, 23.4].

Chapter 2

Normal varieties

2.1 Weil divisors

One of the main aims of algebraic geometry is to study behaviour of subvarieties of algebraic varieties, and in particular two extreme cases are very important:

(1) the case of *curves*, that is subvarieties of dimension 1,

(2) the case of *prime divisors*, that is subvarieties of codimension 1.

We concentrate on the study of divisors in this course. More generally, a *Weil divisor* on a variety X is any formal $\mathbb{Z}$-linear combination of prime divisors on X.

On curves, divisors are just points. Recall that we have a rational function on a curve, this is (locally) just a polynomial, and it can have only finitely many zeroes along this curve, i.e. it can vanish only at finitely many points of this curve.

One of the features we would like to have on a variety is that any rational function vanishes along only finitely many prime divisors, in analogy with curves. We will pinpoint the exact class of varieties where this is the case. We will later also see why this class is important when we study line bundles.

First of all, recall that if X is a projective variety, the *value* of a rational function $f \in k(X)$ at a point $x \in X$ was the image of f in the local ring $\mathcal{O}_{X,x}$. We say that a rational function *vanishes along a subvariety* Y if it vanishes at its every point, that is if the germ of f belongs to the maximal ideal $\mathfrak{m}_x$ of $\mathcal{O}_{X,x}$ for every $x \in Y$.

If a subvariety Y has a *generic point*, that is a point $y \in Y$ such that $\overline{\{y\}} = Y$, then this condition is equivalent to saying that f vanishes at y.

Lemma 2.1. *Let X be a variety. Then X has a unique generic point.*

Proof. Let U be any affine open subset of X. Then $U = \operatorname{Spec} A$ for some ring A, and by definition the zero ideal $x \in \operatorname{Spec} A$ is the generic point of U. Since X is irreducible, X is the only closed subset of X which contains U, so x is the generic point of X.

Now assume that X has two generic points x_1 and x_2. As X is irreducible, there is an affine open subset of X which contains both x_1 and x_2. But then $x_1 = x_2$ since they are both the zero ideal in the corresponding ring. □

Next we want to make sense of the notion of *multiplicity* of $f \in k(X)$ at a point $x \in X$. For this, note first that multiplicities should add up, i.e. if $f \in k(X)$ has order m and $g \in k(X)$ has order n at x, where $m, n \in \mathbb{Z}$, then the multiplicity of fg should be $m + n$. Therefore, if f and g have the same order, then f/g is a rational function whose zeroes (or poles) at x cancel, i.e. f/g should not vanish at x. In other words, the image of f/g in the local ring $\mathcal{O}_{X,x}$ should be a unit. This naturally brings us to the following definition.

Definition 2.2. A Noetherian local ring A is a *regular local ring of dimension* 1 if its maximal ideal is principal, i.e. generated by one element.

It can be shown that if X is a classical variety, and if Y is a prime divisor in X with the generic point y, then $\mathcal{O}_{X,y}$ is a regular local ring. Therefore, every germ $g \in \mathcal{O}_{X,y}$ can be written as $g = ut^n$, where u is a *unit* in $\mathcal{O}_{X,y}$, t is a generator of the maximal ideal $\mathfrak{m}_y \subseteq \mathcal{O}_{X,y}$, and $n \in \mathbb{N}$. This n we call *the multiplicity of* g *at* x. Therefore, since the function field $k(X)$ is the field of fractions of $\mathcal{O}_{X,x}$, for every rational function $f \in k(X)$ we have the well-defined multiplicity $\operatorname{mult}_Y f \in \mathbb{Z}$.

Note that every regular local ring of dimension 1 is automatically a *discrete valuation ring*: the multiplicity function gives the valuation on it. One can prove the converse of this result, and also that in that case, that this is equivalent to the ring being integrally closed (in its field of fractions).

Therefore in order to get a proper definition of multiplicity along a divisor, we only need to require that the corresponding local rings are integrally closed. For our purposes, we require a bit more.

Definition 2.3. A variety X is *normal* if $\mathcal{O}_{X,x}$ is a normal ring for every point $x \in X$, i.e. if it is integrally closed in $k(X)$.

Since we know that $\Gamma(U, \mathcal{O}_X) = \bigcap_{x \in U} \mathcal{O}_{X,x}$, from here we get straight away that every $\Gamma(U, \mathcal{O}_X)$ is a normal ring. Conversely, if A is a normal ring, then the localisation $A_\mathfrak{p}$ is normal for every prime ideal $\mathfrak{p} \subseteq A$.

Next we show that a rational function cannot vanish along infinitely many prime divisors.

Lemma 2.4. *Let X be a normal variety and let f be a nonzero rational function. Then there are only finitely many prime divisors $Y \subseteq X$ such that* $\operatorname{mult}_Y f = 0$.

Proof. Let $U = \operatorname{Spec} A$ be an affine open subset of X where f is defined, and hence, we can view f as an element of A. Since $X \backslash U$ is a closed subset, and X is Noetherian, only finitely many divisors can avoid U (exercise!). Therefore, it is enough to show that there are finitely many divisors Y such that $Y \cap U \neq \emptyset$ and $\operatorname{mult}_Y f \neq 0$. Assume $\operatorname{mult}_Y f > 0$ (≥ 0 holds as f is regular on U). Therefore, Y is contained in the zero-set of f. But then again there finitely many such Y since X is Noetherian. $\square$

Thus it makes sense to talk about *the principal divisor associated to a rational function* $f \in k(X)$: it is defined as

$$\operatorname{div} f = \sum\nolimits_Y (\operatorname{mult}_Y f) Y,$$

where the sum runs over all prime divisors Y in X. According to Lemma 2.4, this is a finite integral sum, so this is a well-defined Weil divisor on X.

If D_1 and D_2 are two divisors on X, we say that they are *linearly equivalent*, and write $D_1 \sim D_2$, if there is a rational function $f \in k(X)$ such that

$$D_1 - D_2 = \operatorname{div} f.$$

This is obviously an equivalence relation.

If $D_1 = \sum d_i^1 D_i$ and $D_2 = \sum d_i^2 D_i$ are two divisors, we say that $D_1 \geq D_2$ if $d_i^1 \geq d_i^2$ for every i.

Definition 2.5. For a divisor D on a normal variety X, the set

$$|D| = \{D' \mid D' \sim D, D' \geq 0\}$$

is a *complete linear system associated to* D.

We will devote much time in this course to studying linear systems.

2.2 Smoothness

Normal varieties satisfy many good properties. Recall that on an integral scheme X we have $\Gamma(U, \mathcal{O}_X) = \bigcap_{x \in U} \mathcal{O}_{X,x}$ for every open subset $U \subset X$. However, on a normal variety we have something better.

Theorem 2.6. *Let X be a normal variety. Then*

$$\Gamma(U, \mathcal{O}_X) = \bigcap_{x \in U, \overline{\{x\}} \text{ is a divisor}} \mathcal{O}_{X,x}.$$

The proof is based on a following deep algebraic result, which I state without proof: if A is a normal ring, then

$$A = \bigcap_{\text{height of } \mathfrak{p} \subseteq A \text{ is } 1} A_{\mathfrak{p}}.$$

An immediate consequence is the following algebro-geometric version of *Hartogs principle.*

Corollary 2.7. *Let X be a normal variety, and let F be a closed subset of X such that* $\operatorname{codim}_X F \geq 2$. *Then*

$$\Gamma(X \backslash F, \mathcal{O}_X) = \Gamma(X, \mathcal{O}_X).$$

Note that a priori the LHS is larger than the RHS. Therefore, what this result says is that every regular function on some open set whose complement is *small* (that is, of codimension at least 2) can be extended *as a regular function* to the whole X. This is a crucial result which distinguishes normal varieties from other varieties. We will see later in the course how we can use this result when we study birational maps.

Even though the notion of normality might seem too strong, or possibly unnatural (if you are still not convinced by previous results), normal varieties include the class of varieties we all like - smooth varieties.

There are several equivalent definitions of smoothness.

Definition 2.8. A noetherian local ring R is a *regular local ring* if the dimension of the $R/\mathfrak{m}$-vector space $\mathfrak{m}/\mathfrak{m}^2$ is exactly $\dim R$, the Krull dimension of R. (We know by a result from Commutative Algebra that $\geq$ always holds.)

If X is a variety and x is a point in X, we say that X is *smooth* (or *nonsingular* or *regular*) at x if $\mathcal{O}_{X,x}$ is a regular local ring.

It can be shown that when $\dim R = 1$, then this definition is equivalent to the definition of regular local ring given before.

As we mentioned before, for rings of dimension 1, being a regular local ring is the same thing as being normal. Therefore, every point of a normal variety X of codimension 1 is automatically regular/smooth.

Also, it is trivial that the generic point of X is also regular. The set $\operatorname{Sing}(X)$ of *singular* (or nonregular) points on X forms a closed subset (this is proved below in Theorem 2.39), and we have just seen that its codimension is at least 2.

Therefore, to give a regular function on X, by Corollary 2.7 it is enough to define it on the set of regular points of X.

Now we show that the notion of regularity coincides with the usual notion of smoothness. So let X be a subvariety of $\mathbb{A}^n$ of dimension d, with the corresponding ideal $\mathcal{I} = (f_1, \dots, f_k)$. Let $P = (p_1, \dots, p_n)$ be a point in X with the maximal ideal $\mathfrak{m} = (X_1 - p_1, \dots, X_n - p_n)$. We have a well-defined map (check!)

$$\varphi\colon T_x = \mathfrak{m}/\mathfrak{m}^2 \to \mathbb{C}^n$$

given by

$$\varphi(f \bmod \mathfrak{m}^2) = \Big(\frac{\partial f}{\partial X_1}(P), \dots, \frac{\partial f}{\partial X_n}(P)\Big)$$

for $f \in \mathfrak{m}$. It is easy to check that this is an isomorphism.

Let $J = (\partial f_i/\partial X_j(P))_{ij}$ be the Jacobian of $\mathcal{I}$ at P. Then it is again straightforward that $\operatorname{rk} J = \dim \varphi(\mathcal{I} \bmod \mathfrak{m}^2)$. Also, one easily checks that if $\mathfrak{m}_P$ is the maximal ideal in $\mathcal{O}_{X,P} = (A/\mathcal{I})_{\mathfrak{m}}$, then $\mathfrak{m}_P/\mathfrak{m}_P^2 \simeq \mathfrak{m}/(\mathcal{I} + \mathfrak{m}^2)$. So by counting dimensions we get

$$\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 + \operatorname{rk} J = n.$$

On one hand, since the Krull dimension of $\mathcal{O}_{X,P}$ is d, this ring is regular if and only if $\dim_{\mathbb{C}} \mathfrak{m}/\mathfrak{m}^2 = r$. On the other hand, the classical definition (for example, from complex manifolds) of smoothness of X at P is that $\operatorname{rk} J = n - d$. Therefore, the two notions of smoothness are equivalent.

In particular, note that the classical definition of smoothness, which depends on an embedding $X \subseteq \mathbb{A}^n$, is by this result *intrinsic*, i.e. it only depends on X and not on the ambient space $\mathbb{A}^n$. This was proved by Zariski in 1947.

Finally, let us note the following deep result:

Theorem 2.9. *Every regular local ring is a UFD.*

Since every UFD is integrally closed, UFD sits somewhere between being normal and being regular. This justifies the following definition.

Definition 2.10. A variety is *locally factorial* if all of its local rings are UFDs.

2.3 Line bundles and Cartier divisors

In order to define line bundles, in analogy with topology, we want to say that a line bundle is locally isomorphic to the structure sheaf.

Definition 2.11. Let X be a variety. An $\mathcal{O}_X$-module $\mathcal{F}$ is a *locally free sheaf of rank* $r \geq 1$ if there is an open covering $\{U_i\}$ of X such that $\mathcal{F}|_{U_i}$ is a free $\mathcal{O}_X|_{U_i}$-module of rank r for every U_i. We also call $\mathcal{F}$ a *vector bundle of rank* r.

Locally free sheaves of rank 1 are also called *line bundles* on X.

Lemma 2.12. *Let X be a variety, let $\mathcal{L}$ and $\mathcal{M}$ be line bundles on X, and denote $\mathcal{L}^{-1} = \underline{\mathrm{Hom}}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$. Then $\mathcal{L} \otimes \mathcal{M}$ and $\mathcal{L}^{-1}$ are also line bundles, and $\mathcal{L} \otimes \mathcal{L}^{-1} \simeq \mathcal{O}_X$.*

Therefore line bundles, modulo isomorphisms, form an abelian group, called *the Picard group of X*, and denoted by $\mathrm{Pic}(X)$. The identity in this group is the isomorphism class of $\mathcal{O}_X$. This is why line bundles are also called *invertible sheaves*.

Example 2.13. On $\mathbb{P}^n$, let U_i, $i = 0, \dots, n$, be the standard affine chart. Then we can define a line bundle $\mathcal{O}_{\mathbb{P}^n}(1)$ as follows: $\mathcal{O}_{\mathbb{P}^n}(1)|_{U_i} \simeq \mathcal{O}_{U_i} \cdot X_i$. This is well defined as X_i/X_j is a regular function on $U_i \cap U_j$. Now one can easily show that $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ is isomorphic to the set of all homogeneous polynomials of degree 1. This sheaf is called *the twisting sheaf (of Serre)*. Similarly we can define sheaves $\mathcal{O}_{\mathbb{P}^n}(m)$ for every $m \geq 1$.

Next we want to make a bijective correspondence between line bundles and a certain subclass of Weil divisors.

Let X be a normal variety and $\mathcal{L}$ a line bundle on X. Denote by $\mathcal{K}$ a sheaf such that $\mathcal{K}(U) = k(X)$ for every nonempty open subset U of X, with identity restriction maps. Note that on every open subset V on which $\mathcal{L}|_V \simeq \mathcal{O}_V$, we have $\mathcal{L}|_V \otimes \mathcal{K}|_V \simeq \mathcal{K}|_V$, a constant sheaf on V. Since X is irreducible, this implies that globally $\mathcal{L} \otimes \mathcal{K} \simeq \mathcal{K}$, so the natural map $\mathcal{L} \to \mathcal{L} \otimes \mathcal{K}$ makes $\mathcal{L}$ into a subsheaf of $\mathcal{K}$.

Now let U_i be (finitely many) open subsets of X where $\mathcal{L}$ trivialises, and let f_i be the local generator of $\mathcal{L}|_{U_i}$, where by above we can assume that f_i is a rational function on X. Define the associated Weil divisor as follows: for each prime divisor Y on X such that $Y \cap U_i \neq \emptyset$, let

$$\nu_Y = \mathrm{mult}_Y\, f_i^{-1}.$$

This does not depend on the choice of f_i: assume f_j is another rational function on an open subset U_j. Then f_i/f_j is a rational function which is regular along $U_i \cap U_j$ (since $f_i|_{U_i \cap U_j}$ and $f_j|_{U_i \cap U_j}$ generate $\mathcal{L}|_{U_i \cap U_j}$ as a $\mathcal{O}_{U_i \cap U_j}$-module), and therefore $\mathrm{mult}_Y(f_i/f_j) = 0$, i.e. $\mathrm{mult}_Y\, f_i^{-1} = \mathrm{mult}_Y\, f_j^{-1}$.

Hence the sum

$$D = \sum \nu_Y Y$$

is well-defined and finite, and hence this is a Weil divisor on X. Note that, by construction, the restriction of the divisor D to each open subset U_i is a principal divisor: it is just the divisor associated to the rational function f_i^{-1}. This justifies the following definition.

Definition 2.14. A Weil divisor is called *locally principal*, or *Cartier*, if there is an open covering $\{U_i\}$ of X such that $D|_{U_i}$ is a principal divisor on U_i.

Now we can invert the story. Assume that we start from a Cartier divisor D on X. Then we have open sets U_i which cover X and rational functions f_i on U_i (and thus on X) which represent $D|_{U_i}$. Then we can define a line bundle, denoted by $\mathcal{O}_X(D)$, as follows: let $\mathcal{O}_X(D)|_{U_i}$ be the submodule of $\mathcal{K}|U_i$ which is generated by $1/f_i$. This is well-defined, since f_i/f_j is invertible in $\mathcal{O}_{U_i \cap U_j}$ by the definition of the multiplicity function

Therefore we have:

Theorem 2.15. *Let X be a normal variety. Then there is a bijective correspondence between Cartier divisors and line bundles on X, given by*

$$D \longmapsto \mathcal{O}_X(D).$$

Now it is easy to show that:

(1) $\mathcal{O}_X(\operatorname{div} f) \simeq \mathcal{O}_X$ for every rational function $f \in k(X)$,

(2) $\mathcal{O}_X(D_1) \simeq \mathcal{O}_X(D_2)$ for every two Cartier divisors $D_1 \sim D_2$,

(3) $\mathcal{O}_X(-D) \simeq \mathcal{O}_X(D)^{-1}$ for every Cartier divisor D,

(4) $\mathcal{O}_X(D_1 + D_2) \simeq \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)$ for every two Cartier divisors D_1 and D_2.

Therefore, there is a *bijection between the Picard group* $\operatorname{Pic}(X)$ *and the set of all Cartier divisors modulo linear equivalence.*

One might wonder if the set of Cartier divisors is too small compared to Weil divisors, since the way we defined them might look artificial. However, one can show that on a locally factorial variety, Weil and Cartier divisors coincide. Since smooth varieties are locally factorial, we see that on them, Cartier divisors are just the usual divisors.

2.3.1 Linear systems

Recall that we defined what a linear system is before. Now I give substance to that definition.

Let $\mathcal{L}$ be a line bundle, and assume there is a nonzero section $f \in \Gamma(X, \mathcal{L})$. Let D be a Cartier divisor corresponding to $\mathcal{L}$. For every open subset U of X where $\mathcal{L}$ trivialises, let $f_U \in \mathcal{O}_X(U)$ be the image of $f|_U$ under the isomorphism $\mathcal{L}_{|U} \simeq \mathcal{O}_U$. Since this isomorphism is determined up to an invertible element of $\mathcal{O}_U$, this gives a well-defined Cartier divisor D_0 by the construction above, and note that $D_0 \geq 0$ since f_U is regular on U for every such U.

We say D_0 is the *divisor of zeroes* of f. By the definition of $\mathcal{O}_X(D)$, if $\mathcal{O}_X(D)$ is locally generated on U by $1/g_U$, D_0 is locally defined by $f_U g_U$. Therefore $D_0 = D + \operatorname{div} f$, so $D \sim D_0$.

Conversely, if $D' \geq 0$ is a divisor linearly equivalent to D, and if $D' = D + \operatorname{div} f'$, then the same method shows that f' is a global section of $\mathcal{O}_X(D)$.

Finally, if f and f' define the same divisor, then $\operatorname{div} f = \operatorname{div} f'$, and hence $\operatorname{div}(f/f') = 0$. This means f/f' is regular everywhere, thus a global regular function on X. If X is projective, then $\Gamma(X, \mathcal{O}_X) = \mathbb{C}$, so $f = \lambda f'$ for $\lambda \in \mathbb{C}\backslash\{0\}$.

This all proves the following: if X is projective, a linear system $|D|$ is in bijection with the set $(\Gamma(X, \mathcal{O}_X(D))\backslash\{0\})/\mathbb{C}^*$. Since for any locally free sheaf $\mathcal{F}$, $\Gamma(X, \mathcal{F})$ is a finite-dimensional $\mathbb{C}$-vector space (this is a hard result), this makes a linear system into a projective space.

Definition 2.16. A line bundle $\mathcal{L}$ is *globally generated* if there exist sections $s_0, \dots, s_k \in \Gamma(X, \mathcal{L})$ whose germs generate $\mathcal{L}_x$ for every point $x \in X$.

If D is a divisor such that $\mathcal{O}_X(D)$ is globally generated, we say that D is *basepoint free.*

The meaning of this is that sections of $\Gamma(X, \mathcal{O}_X(D))$ do not have common zeroes, which follows from the definition. Equivalently (exercise!),

$$\bigcap_{D' \in |D|} \operatorname{Supp} D' = \emptyset,$$

where $\operatorname{Supp} D'$ denotes the set-theoretic union of the prime divisors in D'.

Now, if we are given such a globally generated line bundle $\mathcal{L}$, and such generating sections $s_0, \dots, s_k \in \Gamma(X, \mathcal{L})$, we can associate to it a map $\varphi\colon X \to \mathbb{P}^k$ as follows. Consider the sets $X_i = \{x \in X \mid s_i(x) \neq 0\}$. Let $\varphi_i\colon X_i \to U_i$ be the map given by $\varphi(x) = s_i(x)^{-1}(s_0(x), \dots, s_k(x))$, where U_i are the standard charts of $\mathbb{P}^k$. One can easily check that these are well-defined morphisms, and the definition of a line bundle shows that they glue to give a global morphism to $\mathbb{P}^k$.

Now from the construction it is easy to see that $s_i = \varphi_{|D|}^* X_i$, for each i. Therefore, since X_i generate $\Gamma(X, \mathcal{O}_{\mathbb{P}^k}(1))$, we have

$$\mathcal{L} \simeq \mathcal{O}_X(D) \simeq \varphi_{|D|}^* \mathcal{O}_{\mathbb{P}^k}(1).$$

Conversely, it is easy to see that any morphism $f\colon X \to \mathbb{P}^k$ is associated to the globally generated line bundle $f^*\mathcal{O}_{\mathbb{P}^k}(1)$.

Note that if we have a hyperplane $H = (X_0 = 0) \subseteq \mathbb{P}^n$, then $\mathcal{O}_{\mathbb{P}^n}(1) \simeq \mathcal{O}_{\mathbb{P}^n}(H)$, and sections $X_0, \dots, X_n$ generate (every stalk of) $\mathcal{O}_{\mathbb{P}^n}(1)$. Therefore, the map $\varphi_{\mathcal{O}_{\mathbb{P}^n}(1)}$ associated to these sections is an embedding (even isomorphism) of $\mathbb{P}^n$ into $\mathbb{P}^n$. Thus line bundles which realise a variety as a subvariety of some projective space play a special role in geometry. This motivates the following important definition.

Definition 2.17. Let X be a variety, let $\mathcal{L}$ be a globally generated line bundle on X, and fix sections $s_0, \dots, s_k \in \Gamma(X, \mathcal{L})$ which generate $\mathcal{L}$. If the corresponding map $\varphi_{\mathcal{L}}\colon X \to \mathbb{P}^k$ is a closed embedding, then we say that the line bundle $\mathcal{L}$ is *very ample*.

If $\mathcal{M}$ is a line bundle such that $\mathcal{M}^{\otimes m}$ is very ample for some $m > 0$, then we say that $\mathcal{M}$ is an *ample* line bundle.

We will see later that ample line bundles behave much better than very ample line bundles, in the sense that there are many practical characterisations of ampleness (cohomological, numerical) and very few of very ampleness.

I record here for later use the following important fact: that in a basepoint free linear system there are many smooth sections. The following *Bertini's theorem* makes this more precise.

Theorem 2.18. *Let X be a projective variety of dimension ≥ 2 and let D be a basepoint free divisor on X. Then a general element of the linear system $|D|$ is a smooth subvariety.*

The statement means the following: the linear system $|D|$ is naturally a projective space – the projectivisation of the vector space $H^0(X, \mathcal{O}_X(D))$. Then we say an element $D' \in |D|$ is general if there exists an open subset $U \subseteq \mathbb{P}(H^0(X, \mathcal{O}_X(D)))$ such that D' correspond to a point in U.

2.3.2 Morphisms with connected fibres

I next define a special class of morphisms, which behave much better than a random morphism in many situations. We start with the following easy result, known as the *projection formula* (the proof is an easy exercise).

Lemma 2.19. *Let $f\colon X \to Y$ be a morphism between two varieties, let $\mathcal{F}$ be an $\mathcal{O}_X$-module and let $\mathcal{L}$ be a locally free sheaf of finite rank on Y. Then*

$$f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{L} \simeq f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{L}).$$

Note that if we put $\mathcal{F} = \mathcal{O}_X$ in the previous lemma, we get

$$f_*\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{L} \simeq f_*f^*\mathcal{L}.$$

Often we want f_* and f^* to be "dual" to each other, i.e. to have a relation $\mathcal{L} \simeq f_*f^*\mathcal{L}$, which happens, for instance, when

$$f_*\mathcal{O}_X = \mathcal{O}_Y. \tag{2.1}$$

We will state a necessary and a sufficient condition for when this happens.

Definition 2.20. A morphism $f\colon X \to Y$ between varieties is *projective* if there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[dr, "f"'] & \mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times Y \arrow[d, "\pi"] \\ & Y \end{tikzcd}$$

where i is a closed immersion, and π is the projection on the second coordinate.

When $Y = \operatorname{Spec}\mathbb{C}$, this is the same as saying that X is projective.

Now we have the following necessary and sufficient condition for (2.1).

Theorem 2.21. *Let $f\colon X \to Y$ be a projective surjective morphism between normal varieties. Then the field $k(Y)$ is algebraically closed in $k(X)$ iff $f_*\mathcal{O}_X = \mathcal{O}_Y$.*

The proof is a consequence of the Stein factorisation which is our Theorem 3.31 below, see [Laz04, Example 2.1.12].

The following condition is much more difficult, and it is called *Zariski's Main Theorem*, or *Zariski's Connectedness Theorem.*

Theorem 2.22. *Let $f\colon X \to Y$ be a projective morphism between normal varieties. If $f_*\mathcal{O}_X = \mathcal{O}_Y$, then for every point $y \in Y$, the set $f^{-1}(y)$ is connected. The converse holds if f is surjective.*

Because of this result, we often say that a morphism $f\colon X \to Y$ has *connected fibres* when we actually mean that $f_*\mathcal{O}_X = \mathcal{O}_Y$. This class of maps is not small, and it includes projective *birational* morphisms (to be defined in Subsection 2.3.4).

Now, when we have a projective surjective morphism $f\colon X \to Y$, it is easy to see that $\dim X \geq \dim Y$. It is then very useful to investigate properties of divisors on X which are *contracted*, i.e. which do not map to divisors.

Definition 2.23. In this context, a divisor $E \geq 0$ on X is called *exceptional* if $\operatorname{codim}_Y f(E) \geq 2$.

The following result shows that exceptional divisors essentially do not introduce new sections.

Lemma 2.24. *Let $f\colon X \to Y$ be a morphism with connected fibres between normal varieties. Let $E \geq 0$ be an exceptional Cartier divisor on X. Then $f_*\mathcal{O}_X(E) = \mathcal{O}_Y$.*

Proof. First, from $\mathcal{O}_X \subseteq \mathcal{O}_X(E)$ (exercise!) we have $\mathcal{O}_Y = f_*\mathcal{O}_X \subseteq f_*\mathcal{O}_X(E)$.

For the reverse inclusion, note that for any open subset U of Y, we have

$$f_*\mathcal{O}_X(E)|_{U\setminus f(E)} = f_*(\mathcal{O}_X(E)|_{f^{-1}(U\setminus f(E))}) = f_*\mathcal{O}_X|_{U\setminus f(E)} = \mathcal{O}_Y|_{U\setminus f(E)},$$

and thus

$$\Gamma(U, f_*\mathcal{O}_X(E)) \subseteq \Gamma(U \backslash f(E), f_*\mathcal{O}_X(E)) = \Gamma(U \backslash f(E), \mathcal{O}_Y) = \Gamma(U, \mathcal{O}_Y).$$

This gives $f_*\mathcal{O}_X(E) \subseteq \mathcal{O}_Y$, and the proof is complete. $\square$

Note that we have used in this proof the full force of normality property, and the fact that the morphism has connected fibres.

An immediate consequence (of this lemma and of the projection formula) is the following pullback theorem which is very important in birational geometry.

Theorem 2.25. *Let $f\colon X \to Y$ be a morphism with connected fibres between normal varieties. Let $E \geq 0$ be an exceptional Cartier divisor on X, and let $\mathcal{L}$ be a locally free sheaf of finite rank on Y. Then*

$$\Gamma(X, f^*\mathcal{L} \otimes \mathcal{O}_X(E)) = \Gamma(Y, \mathcal{L}).$$

2.3.3 Rational maps

Next we turn to rational maps, which are a generalisation of rational functions. In the same way that that rational functions were not defined on the whole variety, so the domains of rational maps are only open subsets of varieties.

Definition 2.26. Let X and Y be varieties. A *rational map* f between X and Y, denoted by $f\colon X \dashrightarrow Y$, is an equivalence class of pairs (U, φ), where U is an open subset of X, and $\varphi\colon U \to Y$ is a morphism, and two pairs (U, φ) and (V, ψ) are equivalent iff $\varphi_{|U\cap V} = \psi_{|U\cap V}$.

The *domain* of f, denoted $\mathrm{dom}(f)$, is the union of all open sets U, where (U, φ) belongs to the equivalence class of f.

The *image* of f, denoted $\mathrm{im}(f)$, is the closure of the image of $\mathrm{dom}(f)$ under f. We say that f is *dominating* if $\mathrm{im}(f) = Y$.

Note that two rational maps $f\colon X \dashrightarrow Y$ and $g\colon Y \dashrightarrow Z$ can be composed to a rational map $g \circ f$ if $\mathrm{dom}(g) \cap \mathrm{im}(f) \neq \emptyset$.

The following result shows that rational maps behave well, i.e. that they glue in the same way as rational functions.

Lemma 2.27. *Let X and Y be varieties, and let $f, g\colon X \to Y$ be two morphisms. If there is an open subset U in X such that $f|_U = g|_U$, then $f = g$.*

If $h\colon X \dashrightarrow Y$ is a rational map, then there exists a morphism $\varphi\colon \mathrm{dom}(h) \to Y$ such that h is represented by $(\mathrm{dom}(h), \varphi)$.

Proof. Consider the set

$$K = \{x \in X \mid f(x) = g(x)\} \subseteq X,$$

and the morphism

$$F = (f, g)\colon X \to Y \times Y.$$

Note that $K = F^{-1}\Delta$, where Δ is the diagonal in $Y \times Y$. Now Δ is closed $Y \times Y$ (recall the definition of separatedness), so K is closed in X. But since it contains and open (and therefore dense) subset U of X, we have $K = X$.

The second claim follows from the first one, and is an exercise. $\square$

As in the case of morphisms, the most important examples of rational maps come from linear systems. First we make some preparation.

Definition 2.28. Let $D \geq 0$ be a divisor on a normal variety X. The set

$$\operatorname{Bs}|D| = \bigcap_{D' \in |D|} \operatorname{Supp} D'$$

is the *base locus of* $|D|$, where recall that $\operatorname{Supp} D'$ is the *support* of D', i.e. the set-theoretic union of all prime divisors in D'.

The *fixed part of the linear system* $|D|$ is defined as

$$\operatorname{Fix}|D| = \sum_Y \min_{D' \in |D|} \{\operatorname{mult}_Y D'\} \cdot Y,$$

where the sum runs over all prime divisors Y on X. In other words, it is the biggest divisor smaller than every element of the linear system $|D|$. Then for every $D' \in |D|$, we denote $\operatorname{Mob}(D') = D' - \operatorname{Fix}|D|$, and we call this divisor *the mobile part of* D'.

Therefore, by definition $\operatorname{Bs}|D| = \emptyset$ iff D is basepoint free.

Note that then $\operatorname{Mob}(D') \geq 0$, and if $C \geq 0$ is a divisor linearly equivalent to $\operatorname{Mob}(D')$, then $C + \operatorname{Fix}|D| \sim D'$. In other words,

$$|D| = |\operatorname{Mob}(D)| + \operatorname{Fix}|D|,$$

the sum of the complete linear system $|\operatorname{Mob}(D)|$ and the fixed divisor $\operatorname{Fix}(D)$.

I make several remarks. First, the support of $\operatorname{Fix}|D|$ is contained in $\operatorname{Bs}|D|$; moreover, it is exactly the *divisorial part of the base locus*. Second, $\operatorname{Bs}|\operatorname{Mob}(D)|$ does not contain divisors, i.e. its codimension in X is at least 2.

Example 2.29. Let X be a normal projective variety and let $|D|$ be a complete linear system on X. We can choose a basis $f_0, \dots, f_k$ of $\Gamma(X, \mathcal{O}_X(D))$, and let $D_0, \dots, D_k \in |D|$ be the corresponding divisors of zeroes of these sections. Then, for every i, $X \setminus \operatorname{Supp} D_i$ is the set where f_i does not vanish, and we can define the map from this set to $\mathbb{P}^k$ by $x \mapsto f_i(x)^{-1}(f_0(x), \ldots, f_k(x))$. These maps glue to give a morphism from $\bigcup_{i=0}^k X \setminus \operatorname{Supp} D_i = X \setminus \operatorname{Bs}|D|$ to $\mathbb{P}^k$.

In other words, every linear system $|D|$ yields a rational map

$$\varphi_{|D|} \colon X \dashrightarrow \mathbb{P}^k,$$

defined off the closed set $\operatorname{Bs}|D|$.

Now this suggests that the domain of $\varphi_{|D|}$ is equal to $X \setminus \operatorname{Bs}|D|$. However, observe that if f is a local generator of $\operatorname{Fix}|D|$ (on an open subset which is contained in the smooth locus of X), then (locally) f divides each of the sections f_i, i.e. locally f_i/f are sections of $\Gamma(X, \mathcal{O}_X(\operatorname{Mob}(D)))$. Since $f_i/f_j = (f_i/f)/(f_j/f)$, we see that the above construction for the linear system $|\operatorname{Mob}(D)|$ yields the same map, i.e. $\varphi_{|D|} = \varphi_{|\operatorname{Mob}(D)|}$, and hence this map is defined off the set $\operatorname{Bs}|\operatorname{Mob}(D)|$, which has codimension ≥ 2.

We might wonder whether we can continue to "cut down" the set where a rational map is not defined. However, the following result tells us that we have to stop when we remove divisorial components.

Lemma 2.30. *Let X be a normal projective variety and let $|D|$ be a complete linear system on X such that* $\operatorname{codim}_X \operatorname{Bs}|D| \geq 2$. *Then* $\operatorname{dom}(\varphi_{|D|}) = X \setminus \operatorname{Bs}|D|$.

In fact, the next result shows what happens in general.

Theorem 2.31. *Let X be a normal variety, and let $f \colon X \dashrightarrow \mathbb{P}^k$ be a rational map. Then* $\operatorname{codim}_X(X \setminus \operatorname{dom}(f)) \geq 2$.

Proof. Exercise! Use the fact that for every rational function φ and a prime divisor Y on X, the multiplicity $\operatorname{mult}_Y \varphi$ is well-defined. $\square$

2.3.4 Blowups and birational maps

An important class of rational maps is that of *birational maps*. We say that a rational map $f \colon X \dashrightarrow Y$ is birational if there is a rational map $g \colon Y \dashrightarrow X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$. Equivalently, f is birational iff there are nonempty open subsets $U \subseteq X$ and $V \subseteq Y$ such that $f_{|U} \colon U \to V$ is an isomorphism.

Note that any rational map $f \colon X \dashrightarrow Y$ induces a homomorphism $f^* \colon k(Y) \to k(X)$ by $f^*(\varphi) = \varphi \circ f$. Now it is trivial to show that if f is birational, then the induced map f^* is an isomorphism. In other words, varieties which are birational have isomorphic fields of rational functions. The converse also holds.

Now we will describe *projective birational morphisms*, i.e. projective morphisms $f \colon X \to Y$ which are birational maps. Note that the inverse rational map f^{-1} need not be a morphism.

We will see that there is a universal construction which describes every such map. First we need to make a special construction of the relative projective functor $\underline{\mathrm{Proj}}$, which generalises the standard Proj-construction.

To start with, let X be a variety, and let $\mathcal{S}$ be a sheaf of graded $\mathcal{O}_X$-algebras. In other words,

$$\mathcal{S} = \bigoplus_{d \geq 0} \mathcal{S}_d,$$

where for every open set $U \subseteq X$, $\mathcal{S}(U) = \bigoplus_{d \geq 0} \mathcal{S}_d(U)$ is a graded ring. We assume that $\mathcal{S}_0 = \mathcal{O}_X$, $\mathcal{S}_d$ are coherent $\mathcal{O}_X$-modules, and $\mathcal{S}$ is locally generated by $\mathcal{S}_1$ as an $\mathcal{O}_X$-algebra.

For every affine open subset $U = \operatorname{Spec} A$ of X, we have the scheme $\operatorname{Proj} \mathcal{S}(U)$, and the natural map $\operatorname{Proj} \mathcal{S}(U) \to U$ coming from the fact that $\mathcal{S}(U)$ is a finitely generated A-algebra. It is straightforward to see that these schemes and maps glue, so we obtain the scheme $\underline{\mathrm{Proj}}\, \mathcal{S}$ with a morphism

$$\pi \colon \underline{\mathrm{Proj}}\, \mathcal{S} \to X.$$

On every $\operatorname{Proj} \mathcal{S}(U)$, we have the *Serre twisting sheaf* $\mathcal{O}_{\operatorname{Proj} \mathcal{S}(U)}(1) = \widetilde{\bigoplus_{d \geq 1} \mathcal{S}_d(U)}$, and these sheaves also glue to give a sheaf $\mathcal{O}_{\underline{\mathrm{Proj}}\, \mathcal{S}}(1)$, also called the twisting sheaf. It can easily be shown that this sheaf is a line bundle.

A special case of this construction is when $\mathcal{S}_1$ is the ideal sheaf $\mathcal{I}$ of a closed subscheme Y of X – it is obtained by gluing local ideals along affine varieties in the usual way. Then we have the sheaf of graded algebras

$$\mathcal{S}_{\mathcal{I}} = \bigoplus_{d \geq 0} \mathcal{I}^d,$$

where by definition $\mathcal{I}^0 = \mathcal{O}_X$. The corresponding scheme

$$X_{\mathcal{I}} = \underline{\mathrm{Proj}}\, \mathcal{S}_{\mathcal{I}}$$

is called the *blowup of X along Y*, or the *blowup of X with respect to $\mathcal{I}$*. Recall that it comes with the structure map $\pi \colon X_{\mathcal{I}} \to X$.

Then we have the following structure theorem.

Theorem 2.32. *Let X be a variety, let $\mathcal{I}$ be the ideal sheaf of a closed subscheme Y of X, and let $\pi \colon X_{\mathcal{I}} \to X$ be the blowup of X along Y. Then:*

(1) $X_{\mathcal{I}}$ is a variety;

(2) if X is quasi-projective, respectively projective, then so is $X_{\mathcal{I}}$;

(3) π is a projective, surjective, birational morphism;

(4) if $U = X \backslash Y$, then $\pi_{|\pi^{-1}(U)} \colon \pi^{-1}(U) \to U$ is an isomorphism;

(5) the sheaf $\widetilde{\mathcal{I}} = \pi^{-1}\mathcal{I} \cdot \mathcal{O}_{X_{\mathcal{I}}}$ is a line bundle on $X_{\mathcal{I}}$, and it is equal to $\mathcal{O}_{X_{\mathcal{I}}}(1)$;

(6) the support of the closed subscheme corresponding to $\widetilde{\mathcal{I}}$ is equal to $X_{\mathcal{I}} \backslash \pi^{-1}(U)$.

Note that above, $\widetilde{\mathcal{I}}$ is (by definition) the ideal generated by the image of the ideal sheaf $\pi^{-1}\mathcal{I}$ under the natural map $\pi^{-1}\mathcal{O}_X \to \mathcal{O}_{X_{\mathcal{I}}}$.

Exercise 2.33. Show that for every Cartier divisor $D \geq 0$ on a variety X, if $\mathcal{I}$ is its ideal sheaf, then $\mathcal{I} = \mathcal{O}_X(-D)$. And conversely, for every line bundle $\mathcal{I}$ which is a sheaf of ideals in $\mathcal{O}_X$, there is a Cartier divisor $D \geq 0$ such that $\mathcal{I} = \mathcal{O}_X(-D)$.

Then the above theorem can be interpreted as follows. Denote by $E \geq 0$ the Cartier divisor on $X_{\mathcal{I}}$ such that $\widetilde{\mathcal{I}} = \mathcal{O}_{X_{\mathcal{I}}}(-E)$. Then $\operatorname{Supp} E = X_{\mathcal{I}} \backslash \pi^{-1}(U)$, and E is exceptional with respect to the map π.

The following theorem says that every projective birational morphism is a blowup.

Theorem 2.34. *Let $f \colon Y \to X$ be a projective birational morphism, where X and Y are varieties, and X is quasi-projective over $\mathbb{C}$. Then there exists a coherent sheaf of ideals $\mathcal{I}$ on X such that there is an isomorphism $\psi \colon Y \to X_{\mathcal{I}}$, and $f = \pi \circ \psi$, where $\pi \colon X_{\mathcal{I}} \to X$ is the structure morphism of the blowup.*

$$\begin{tikzcd} Y \arrow[r, "\psi", "\simeq"'] \arrow[dr, "f"'] & X_{\mathcal{I}} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

2.4 The canonical sheaf

We finally introduce one of the most important objects on a variety – its canonical sheaf. On a smooth variety, the canonical bundle is a special line bundle, which is very natural from the point of view of the geometry of the variety.

As its name says, it is *canonical*: its definition is intrinsic, and it is naturally defined on every (smooth or normal) variety. Indeed, it is in general quite difficult to come up with an interesting line bundle on a random smooth variety, and canonical bundles represent one obvious, but crucial way to come up with such examples.

Recall that on the affine variety $\mathbb{A}^n$, for every point P and the corresponding maximal ideal $\mathfrak{m}$ we have the isomorphism

$$\varphi \colon \mathfrak{m}/\mathfrak{m}^2 \to \mathbb{C}^n$$

given by

$$\varphi(f \bmod \mathfrak{m}^2) = \Big(\frac{\partial f}{\partial X_1}(P), \dots, \frac{\partial f}{\partial X_n}(P)\Big)$$

for $f \in \mathfrak{m}$. This indicates that the $\mathbb{C}$-module $\mathfrak{m}/\mathfrak{m}^2$ locally behaves like differentials around the point P. We will see that this holds in a more general setting of Kähler differentials.

Similarly as in the case of manifolds, on every algebraic variety we can define derivation, since derivations of polynomials can be given purely formally.

Definition 2.35. Let A be a ring, let B be an A-algebra, and let M be a B-module. An A-derivation of B into M is a map $d\colon B \to M$ such that:

(1) $d(b_1 + b_2) = d(b_1) + d(b_2)$ for all $b_1, b_2 \in B$,

(2) $d(b_1 b_2) = b_1 d(b_2) + b_2 d(b_1)$ for all $b_1, b_2 \in B$,

(3) $d(a) = 0$ for all $a \in A$.

Here, condition (3) should be understood as the standard condition that derivatives of constants are zero.

The we define the *module of relative differentials* $\Omega_{B/A}$ (or *module of regular differential forms*) as the free B-module generated by the symbols db for all $b \in B$, divided out by the relations:

(1) $d(b_1 + b_2) - d(b_1) - d(b_2)$ for all $b_1, b_2 \in B$,

(2) $d(b_1 b_2) - b_1 d(b_2) - b_2 d(b_1)$ for all $b_1, b_2 \in B$,

(3) $d(a)$ for all $a \in A$.

Then we have the obvious derivation $d\colon B \to \Omega_{B/A}$. This module satisfies the *universal property*: any A-derivation of B into some module factors through d.

Example 2.36. We define a derivation which will be a template for all other derivations. Let B be an A-algebra, $f\colon B \otimes_A B \to B$ be the homomorphism defined by $f(b_1 \otimes b_2) = b_1 b_2$, and let I be the kernel of f. Then I/I^2 is a B-module, where B acts by multiplication on the left. Then one can easily check that the map $d\colon B \to I/I^2$ given by

$$b \mapsto 1 \otimes b - b \otimes 1 \mod I^2$$

turns I/I^2 into the module of relative differentials, thus $\Omega_{B/A} \simeq I/I^2$. Compare this to the example on $\mathbb{A}^n$ above.

This derivation is called the *canonical derivation*.

Moreover, one can prove the following.

Lemma 2.37. *Let B be a local ring with the maximal ideal $\mathfrak{m}$ containing a field $k \simeq B/\mathfrak{m}$. Then $\mathfrak{m}/\mathfrak{m}^2 \simeq \Omega_{B/k} \otimes_B k$.*

Now we want to define the sheaf of regular forms on any variety. We follow the standard recipe: assume first that $X = \operatorname{Spec} A$ is an affine variety over $\mathbb{C}$. Then we have defined the A-module $\Omega_{A/\mathbb{C}}$ of regular forms on A, and we define the *sheaf of regular forms on* X as $\Omega_X = \widetilde{\Omega_{A/\mathbb{C}}}$ (recall the definition of modules of sheaves associated to modules from Subsection 1.2.3).

Furthermore, if S is any multiplicative set in B, then it is easy to show that $S^{-1}\Omega_{B/\mathbb{C}} \simeq \Omega_{S^{-1}B/\mathbb{C}}$. In particular, if $\mathfrak{p}$ is an ideal in A, this says that the local ring $\Omega_{X,\mathfrak{p}}$ is isomorphic to the localisation of $\Omega_{A/\mathbb{C}}$ at the point $\mathfrak{p}$.

Then in general, we cover a variety X by open affines, and the construction glues to give a global sheaf of regular differentials Ω_X.

There is also another way to give this construction "globally". Recall that in the example above we had a *diagonal map* $f\colon B \otimes_{\mathbb{C}} B \to B$ given by $f(b_1 \otimes b_2) = b_1 b_2$. One can easily check that this corresponds to the inclusion $\tilde{f}\colon X = \operatorname{Spec} B \to \operatorname{Spec}(B \otimes_{\mathbb{C}} B) = X \times X$ which maps X onto the diagonal $\Delta \subseteq X \times X$ (exercise!). Under this inclusion, $I = \ker f$ is precisely the ideal sheaf of Δ.

In general, if X is any variety, we can consider the diagonal inclusion $\pi\colon X \to X \times X$, i.e. the one where π maps X isomorphically to the diagonal Δ in $X \times X$. Let $\mathcal{I}$ be the ideal sheaf of Δ in $X \times X$, i.e. we have $\mathcal{O}_{X\times X}/\mathcal{I} \simeq \mathcal{O}_X$. Then we define the sheaf of regular differentials

$$\Omega_X = \pi^*(\mathcal{I}/\mathcal{I}^2).$$

Note that this is, by definition, isomorphic to the restriction of the module $\mathcal{I}/\mathcal{I}^2$ to the diagonal Δ. It is easy to see that this definition is equivalent to the one given above.

Now we will relate this sheaf to the question of whether a variety is smooth at a point. For this, we first need a lemma.

Lemma 2.38. *Let B be a local ring with the maximal ideal $\mathfrak{m}$ containing a field $k \simeq B/\mathfrak{m}$, which is a perfect field. Assume further that B is a localisation of a finitely generated k-algebra. Then B is a regular local ring iff $\Omega_{B/k}$ is a free B-module of rank $= \dim B$.*

Note that the rank of $\Omega_{B/k}$ is in general at least $\dim B$.

We can finally prove the following result which was stated in Section 2.2 without proof.

Theorem 2.39. *If X is a variety, then the set $\operatorname{Reg}(X)$ of smooth points is a non-empty open subset of X.*

Proof. If X is a variety and $x \in X$ is a closed point, Lemma 2.38 says that X is smooth at x iff its local ring $\Omega_{X,x}$ is a free $\mathcal{O}_{X,x}$-module of the minimal rank $\dim X$. To show that $\mathrm{Reg}(X)$ is open, it is enough to check this at closed points, since a local ring at a non-closed point is a localisation of a local ring at a closed point, and this preserves regularity.

Note that, by construction, Ω_X is a coherent $\mathcal{O}_X$-module. Therefore, $\Omega_{X,x}$ is a finitely generated $\mathcal{O}_{X,x}$-module for every point $x \in X$. It is enough to show that the function

$$\delta(x) = \dim_{k(x)} \Omega_{X,x} \otimes_{\mathcal{O}_{X,x}} k(x)$$

is upper semicontinuous: indeed, $\mathrm{Reg}(X)$ is exactly the set of all points $y \in X$ such that $\delta(y) = \dim X$.

For this, fix a point $x \in X$, and fix a minimal set of generators $\alpha_{1,x}, \dots, \alpha_{k,x}$ of $\Omega_{X,x}$ as an $\mathcal{O}_{X,x}$-module. By the definition of sheaves, there is an affine open subset U of X such that these germs "lift" to sections $\alpha_1, \dots, \alpha_k \in \Omega_X(U)$. Consider the coherent sheaf

$$\mathcal{M} = \Omega_U \Big/ \sum_{i=1}^{k} \mathcal{O}_U \cdot \alpha_i,$$

and observe that $\mathcal{M}_x = 0$. Since $\mathcal{M}$ is coherent and U is affine, there is a finitely generated $\mathcal{O}_X(U)$-module M such that $\mathcal{M} = \widetilde{M}$. For every section $m \in \mathcal{M}(U)$, the set $\{u \in U \mid m_u \neq 0 \text{ in } \mathcal{M}_u = M_u\}$ is equal to $V(\mathrm{Ann}(m))$, where $\mathrm{Ann}(m) \subseteq \mathcal{O}_X(U)$ is the annihilator of m (exercise!). Therefore, we have

$$\{u \in U \mid \mathcal{M}_u \neq 0\} = V(\mathrm{Ann}(M))$$

since $\mathcal{M}$ is coherent.

In other words, the set of points $y \in U$ where $\Omega_{U,y} = \sum_{i=1}^{k} \mathcal{O}_{U,y} \cdot \alpha_{i,y}$ is an open subset of U (and thus of X), so by shrinking U, we may assume that

$$\Omega_U = \sum_{i=1}^{k} \mathcal{O}_U \cdot \alpha_i.$$

In particular, for every point $y \in U$, we have

$$\delta(y) \leq k = \delta(x),$$

which proves what we wanted. □

Example 2.40. Let $X = \mathbb{A}^n_Y$ be an affine variety over a variety Y. Then it is easy to check that $\Omega_{X/Y} \simeq \mathcal{O}_X^{\oplus n}$, and if $X_1, \dots, X_n$ are coordinates of $\mathbb{A}^n_Y$ over Y, then $\Omega_{X/Y}$ is generated by $dX_1, \dots, dX_n$.

More generally, if $X = \operatorname{Spec} A$ is any affine variety of dimension n, and if $p \in X$ is a nonsingular point, then there exist an open neighbourhood U of x and sections $z_1, \dots, z_n \in \mathcal{O}_X(U)$ such that $\Omega_U = \sum_{i=1}^n \mathcal{O}_U \cdot dz_i \simeq \mathcal{O}_U^{\oplus n}$. Indeed, we have $A \simeq \mathbb{C}[X_1, \dots, X_N]/\mathfrak{p}$, where $\mathfrak{p}$ is an ideal in $\mathbb{C}[X_1, \dots, X_N]$, and set $x_i = X_i \mod \mathfrak{p}$ for every i. Without loss of generality, one may assume that $x_i(p) = 0$ for all i. By definition, after possibly relabelling X_i, there exist $f_1, \dots, f_{N-n} \in \mathfrak{p}$ such that the determinant $J = \det[\partial f_i/\partial X_j]_{1 \leq i,j \leq N-n}$ does not vanish at p. Setting $\mathfrak{j} = J \mod \mathfrak{p}$ and $\partial_j f_i = \partial f_i/\partial X_j \mod \mathfrak{p}$, we have $p \in X \setminus V((\mathfrak{j})) = \operatorname{Spec} A_{\mathfrak{j}}$ and $\sum_{j=1}^N (\partial_j f_i) dx_j = 0$ for all $1 \leq i \leq N-n$. Therefore, $\mathfrak{j} = \det[\partial_j f_i]_{1 \leq i,j \leq N-n}$. Then there exist $\beta_{ij} \in A_{\mathfrak{j}}$ with $dx_i = \sum_{j=N-n+1}^N \beta_{ij} dx_j$ for $1 \leq i \leq N-n$, and hence $\Omega_{A_{\mathfrak{j}}} = \sum_{j=N-n+1}^N A_{\mathfrak{j}} dx_j \simeq A_{\mathfrak{j}}^n$.

This all implies the following.

Theorem 2.41. *Let X be a variety. Then X is smooth iff Ω_X is a locally free sheaf of rank* $\dim X$.

Now, as in differential geometry, on any variety we can consider q-forms, given by wedging (locally) regular differentials.

Definition 2.42. Let X be a scheme and let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules on X. Then for every $q \geq 0$, the *q-th exterior power* $\bigwedge^q \mathcal{F}$ is the sheaf obtained by sheafifying the collection of groups $\bigwedge^q \mathcal{F}(U)$ for all open subsets U of X.

Exercise 2.43. (1) If $\mathcal{F}$ is a locally free sheaf of rank n on X, then $\bigwedge^q \mathcal{F}$ is a locally free sheaf of rank $\binom{n}{q}$.

(2) Assume there is an exact sequence of locally free sheaves

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$$

of ranks n_1, n_2, n_3 respectively. Then $\bigwedge^{n_2} \mathcal{F}_2 \simeq \bigwedge^{n_1} \mathcal{F}_1 \otimes \bigwedge^{n_3} \mathcal{F}_3$.

If X is a *nonsingular* variety of dimension n, we saw that Ω_X is a locally free sheaf of rank n. Therefore, for every $q \geq 0$, the exterior powers $\bigwedge^q \Omega_X$ are again locally free modules of ranks $\binom{n}{q}$, and we call them *sheafs of regular q-forms*, and denote them by Ω_X^q. Locally, these sheaves are generated by forms of the form $dx_{i_1} \wedge \cdots \wedge dx_{i_q}$, where $i_1 < i_2 < \cdots < i_q$.

In particular, for $q = n$, we have

Definition 2.44. The locally free sheaf $\omega_X = \Omega_X^n$ is called *the canonical bundle of* X.

This line bundle is locally generated by n-forms $dx_1 \wedge \cdots \wedge dx_n$. The global sections $\Gamma(X, \Omega_X^q)$ are the *regular q-forms on X*.

We know that there is a correspondence between divisors on X and line bundles on X. Therefore, if for a divisor K_X we have

$$\mathcal{O}_X(K_X) \simeq \omega_X,$$

then we say that K_X is a *canonical divisor of* X. Canonical divisors form a linear equivalence class, which is called the *canonical class*.

Now, if X is normal but not smooth, then the sheaf Ω_X is not locally free, so in particular $\bigwedge^n \Omega_X$ is not a line bundle. However, we can still define the canonical class on X. There are several different ways to proceed, but possibly the most natural approach is as follows.

We know that the set $U = X_{\text{smooth}}$ of smooth points on X is open, and that its complement in X has codimension at least 2. Therefore, the sheaf of regular 1-forms Ω_U is a locally free sheaf of rank $n = \dim X$, and thus ω_U is a locally free sheaf on U. Therefore, there is a canonical divisor K_U on U.

Since $X \backslash U$ is of codimension ≥ 2, there is a unique Weil divisor K_X on X such that $K_{X|U} = K_U$. We call the set of all such divisors *the canonical class on* X. By Hartogs principle, we can see that they form a linear equivalence class, since any rational function on U extends uniquely to a rational function on X.

Example 2.45. Recall that we have $\omega_{\mathbb{A}^n} \simeq \mathcal{O}_{\mathbb{A}^n}$, and therefore $K_{\mathbb{A}^n} = 0$.

Example 2.46. The situation on $\mathbb{P}^n$ is just a bit more complicated. Cover $\mathbb{P}^n$ by the standard open chart $U_i = (x_i = 1) \simeq \mathbb{A}^n$. On U_i, we have coordinates $x_{ji} = x_j/x_i$, and the transition function between coordinates of U_i and U_j is just $x_j/x_i = 1/x_{ij}$. Then we calculate:

$$dx_{ki} = d(x_{kj}/x_{ij}) = \frac{dx_{kj}}{x_{ij}} - \frac{x_{kj}}{x_{ij}^2} dx_{ij}$$

for $k \neq i, j$, and also

$$dx_{ji} = d(1/x_{ij}) = -\frac{dx_{ij}}{x_{ij}^2}.$$

By wedging, we get

$$dx_{0i} \wedge \cdots \wedge dx_{ni} = \frac{1}{x_{ij}^{n+1}} dx_{0j} \wedge \cdots \wedge dx_{nj},$$

where on the LHS we don't have x_{ii}, and on the RHS we don't have x_{jj}. In other words, we have that on U_i, the sheaf $\Omega_{\mathbb{P}^n}$ is isomorphic to the structure sheaf, and the transition functions are $x_{ij}^{-(n+1)}$. Recall that in the definition of the sheaf $\mathcal{O}_{\mathbb{P}^n}(1)$, the transition functions were precisely x_{ij}. Thus we obtain

$$\omega_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-n-1).$$

Moreover, if $H = (x_0 = 0)$, we have that $K_{\mathbb{P}^n} = (-n-1)H$ is a canonical divisor.

2.4.1 Adjunction formula

Next we want to find a relation between the canonical sheaf on a smooth variety X and the canonical sheaf on its smooth closed subvariety Y. If Y were open, then it is obvious by the construction that $\omega_{X|Y} = \omega_Y$, and we might think that the same holds when Y is closed, like in the case of the structure sheaf. However, we will see that we have to introduce an additional factor to make the formula work. We will do this only in the case of divisors.

Recall that the restriction of a sheaf is just pulling it back by the inclusion $i\colon Y \hookrightarrow X$. Since we are in general interested in the behaviour of the canonical sheaf under pullbacks, we first want to relate pullbacks in the sense of line bundles to pullbacks in the sense of divisors.

Definition 2.47. Let $f\colon X \to Y$ be a morphism, where X is a normal variety and Y is a locally factorial variety. Let D be a (Cartier) divisor on Y such that $D \cap f(X) \notin \{\emptyset, f(X)\}$. Then the *pullback* f^*D is a Cartier divisor on X given by the following data: if φ is the local generator of D on an open set U (recall that φ is a rational function), then the divisor f^*D is given by the local generator $\varphi \circ f$ on the set $f^{-1}(U)$.

Note that then $\operatorname{Supp} f^*D = f^{-1}(\operatorname{Supp} D)$ as sets. Further, if $i\colon X \hookrightarrow Y$ is an inclusion, and D is a Cartier divisor on Y, then $i^*D = D_{|X}$, the standard componentwise restriction.

The basic result about pullbacks is the following.

Lemma 2.48. *Let $f\colon X \to Y$ be a morphism, where X is a normal variety and Y is a locally factorial variety. Let D_1 and D_2 be divisors on Y such that their pullbacks on X can be defined, and assume that $D_1 - D_2 = \operatorname{div}\varphi$ for some $\varphi \in k(Y)$. Then*

$$f^*D_1 - f^*D_2 = \operatorname{div}(f^*\varphi),$$

where $f^*\varphi = \varphi \circ f$. In particular, $f^* \operatorname{div}\varphi = \operatorname{div}(f^*\varphi)$.

Therefore, pullback of divisors preserves linear equivalence. Using this result locally, one can show the basic relation between pullbacks of divisors and associated line bundles.

Lemma 2.49. *Let $f\colon X \to Y$ be a morphism, where X is a normal variety and Y is a locally factorial variety. Let D be a divisor on Y such that the pullback f^*D on X can be defined. Then*

$$f^*\mathcal{O}_Y(D) = \mathcal{O}_X(f^*D).$$

In particular, if $f\colon X \hookrightarrow Y$ is the inclusion, and D is a Cartier divisor on Y, then $\mathcal{O}_Y(D)|_X = \mathcal{O}_X(D|_X)$.

In this course, we usually work with dominant maps, such as birational morphisms and finite morphisms, so there are no technical difficulties to pull back divisors. In practice we usually do not distinguish between pullbacks of line bundles and pullbacks of associated Cartier divisors.

Definition 2.50. Let X be a smooth variety of dimension n and let p be a point on X. An affine open neighbourhood U of p in X is a *coordinate neighbourhood* of p if there exist $z_1, \dots, z_n \in \mathcal{O}_X(U)$ such that $\Omega^1_X|_U = \sum_{i=1}^n \mathcal{O}_U dz_i$. Moreover, $(z_1, \dots, z_n)$ is a *local coordinate system* at p if $z_i(p) = 0$ for all i. A local coordinate system at p exists by Example 2.40.

On a smooth variety X, K_X is a divisor associated to any rational n-form ω: if locally $\omega = f dx_1 \wedge \cdots \wedge dx_n$, where $(x_1, \dots, x_n)$ are local coordinate systems and f are rational functions compatible on overlaps, then the functions f give a Cartier divisor, and it is precisely K_X.

We can now prove the following important *adjunction formula.*

Theorem 2.51. *Let X be a smooth variety and let D be a smooth divisor on X. Then*

$$(K_X + D)_{|D} = K_D. \tag{2.2}$$

Note that I use loosely the equality here, since canonical divisors are determined only up to linear equivalence; this should really be understood as a statement about the corresponding line bundles, and be interpreted as a statement about pullbacks.

Proof. Let $n = \dim X$ and let $i\colon D \hookrightarrow X$ be the inclusion map. Let $\{U_\lambda\}$ be an open covering with local coordinate systems $(x_1^\lambda, \dots, x_n^\lambda)$, and we can assume, after reparametrising (changing coordinates by invertible Jacobians) that the equation of D on U_λ is x_n^λ, if $D \cap U_\lambda \neq \emptyset$. Denote $d\mathbf{x}^\lambda = dx_1^\lambda \wedge \cdots \wedge dx_n^\lambda$ for each λ.

Let ω be any rational $(n-1)$-form for which the pullback $i^*\omega$ is defined and not zero – for instance, take $\omega = dx_1^\nu \wedge \cdots \wedge dx_{n-1}^\nu$ for some ν. Then locally on U_λ, $\omega = f_\lambda dx_1^\lambda \wedge \cdots \wedge dx_{n-1}^\lambda + \psi_\lambda \wedge dx_n^\lambda$ for some rational $(n-2)$-form ψ_λ and for $f_\lambda \in k(X)$, and

$$\omega|_{D \cap U_\lambda} = f'_\lambda (dx_1^\lambda \wedge \cdots \wedge dx_{n-1}^\lambda)|_{D \cap U_\lambda}$$

(prime here denotes restriction to $D \cap U_\lambda$). Therefore, K_D is locally on $D \cap U_\lambda$ given as $\operatorname{div}(f'_\lambda)$ and we have $\mathcal{O}_{D\cap U_\lambda}(K_D) = \frac{1}{f'_\lambda}\mathcal{O}_{D\cap U_\lambda}$.

On the other hand, let $\varphi_{\lambda\mu} = x_n^\lambda / x_n^\mu$ be the transition functions of $\mathcal{O}_X(D)$. Consider the n-form $\omega \wedge dx_n^\lambda = f_\lambda d\mathbf{x}^\lambda$, and let $\theta_{\lambda\mu}$ be the transition function of $\mathcal{O}_X(K_X)$ on $U_\lambda \cap U_\mu$; in other words, $\theta_{\lambda\mu} = d\mathbf{x}^\mu / d\mathbf{x}^\lambda$. Then on $U_\lambda \cap U_\mu$:

$$\begin{aligned} f_\lambda d\mathbf{x}^\lambda = \omega \wedge dx_n^\lambda = (f_\mu dx_1^\mu \wedge \cdots \wedge dx_{n-1}^\mu + \psi_\mu \wedge dx_n^\mu) \wedge (\varphi_{\lambda\mu} dx_n^\mu + x_n^\mu d\varphi_{\lambda\mu}) \\ = \varphi_{\lambda\mu} f_\mu d\mathbf{x}^\mu + x_n^\mu(\cdots) = \varphi_{\lambda\mu} f_\mu \theta_{\lambda\mu} d\mathbf{x}^\lambda + x_n^\mu(\cdots) d\mathbf{x}^\lambda, \end{aligned}$$

hence

$$(f_\lambda / f_\mu)|_{D \cap U_\lambda \cap U_\mu} = (\varphi_{\lambda\mu}\theta_{\lambda\mu})|_{D \cap U_\lambda \cap U_\mu}.$$

Thus the transition functions of the two sheaves in (2.2) agree, which shows the result. □

Theorem 2.52. *Let $f\colon X \to Y$ be a dominating morphism between smooth varieties such that* $\dim X = \dim Y = n$. *Then we have the following* ramification formula*:*

$$K_X = f^* K_Y + R_f,$$

where $R_f \geq 0$ is the ramification divisor of f.

We omit the proof, but it is worthy to say that R_f should be understood as a Jacobian map which transforms local coordinate systems on Y to local coordinate systems on X.

Moreover, if f is a projective birational morphism, then the domain of the rational map f^{-1} is precisely $Y \backslash f(R_f)$. If $\operatorname{codim}_Y f(R_f) \geq 2$, then the divisor R_f is exceptional, and then by the pullback formula we have

$$\Gamma(X, K_X) = \Gamma(X, f^*K_Y + R_f) \simeq \Gamma(Y, K_Y).$$

This is just a manifestation of the following fact: if two smooth varieties X and Y are birational, then $\Gamma(X, K_X) = \Gamma(Y, K_Y)$. The "easiest" way to prove this is to invoke a hard theorem of Hironaka on resolution of singularities, but there are much more elementary ways, which we omit.

We will now calculate the ramification formula in a special, but very important case of a blowup. First we need a more precise statement about the blowup in a particular case.

Theorem 2.53. *Let X be a smooth variety, let $\mathcal{I}$ be the ideal sheaf of a smooth closed subvariety Y of X, and let $\pi\colon X_\mathcal{I} \to X$ be the blowup of X along Y. Let E be the corresponding (exceptional) divisor on $X_\mathcal{I}$. Then:*

(1) $X_\mathcal{I}$ is also smooth;

(2) $\mathcal{O}_{X_\mathcal{I}}(E)|_E \simeq \mathcal{O}_E(-1)$.

Let $f\colon \widetilde{X} \to X$ be the blowup of X at a smooth subvariety Y of codimension $r \geq 2$, and let E be the exceptional divisor of the blowup. Then it is easy to see (see it! – use the fact that f has connected fibres and the projection formula) that there is the equality

$$\operatorname{Pic}(\widetilde{X}) = f^* \operatorname{Pic}(X) \oplus \mathbb{Z}E.$$

Here we do not distinguish between divisors and line bundles, and use the additive notation.

In particular, $\omega_{\widetilde{X}} = f^*\mathcal{L} \otimes \mathcal{O}_{\widetilde{X}}(mE)$ for some $\mathcal{L} \in \operatorname{Pic}(X)$ and some $m \in \mathbb{Z}$. When we restrict to the set $\widetilde{X}\backslash E \simeq X\backslash Y$, we have

$$\mathcal{L}_{|X\backslash Y} \simeq \omega_{\widetilde{X}\backslash E} \simeq \omega_{X\backslash Y},$$

and since $r \geq 2$, we get $\mathcal{L} \simeq \omega_X$ by Hartogs principle.

Next, we tensor both sides by $\mathcal{O}_{\widetilde{X}}(E) \otimes \mathcal{O}_E$, and we get by adjunction formula and by the previous theorem

$$\omega_E = f^*\omega_X \otimes \mathcal{O}_E(-m-1).$$

Now when we restrict to any fibre $Z \simeq \mathbb{P}^{r-1}$ over a closed point $z \in Y$, it is easy to see that

$$\omega_Z = \mathcal{O}_Z(-m-1).$$

But we know that $\omega_{\mathbb{P}^{r-1}} = \mathcal{O}_{\mathbb{P}^{r-1}}(-r)$, so $m = r-1$. Finally, the ramification formula in this case is

$$K_{\widetilde{X}} = f^*K_X + (r-1)E.$$

2.5 Serre duality and Riemann-Roch

We saw that the cohomology on affine varieties is easy, and it does not give much information about its geometry; however, we saw that vanishing of higher cohomology distinguishes affine varieties from other varieties.

On projective varieties, the situation is much more complicated, and in particular the vanishing holds only in some cases, but these cases usually turn to be extremely important. When the vanishing of some sort holds, we will see that it almost always implies some remarkable consequences on the geometry of the variety at hand (or the sheaf on it). Results of that type are called *vanishing theorems*, and the most notable ones are the Kodaira vanishing and the Kawamata-Viehweg vanishing, which we will meet later in the course.

The following theorem is usually proved by using Čech cohomology (which is equivalent to the one in Chapter 1), but the proof is lengthy and we omit it.

Theorem 2.54. *Let A be a Noetherian ring, and let $X = \mathbb{P}^r_A$. Then:*

(1) $H^i(X, \mathcal{O}_X(n)) = 0$ *for all* $n \in \mathbb{Z}$ *and all* $i = 1, \dots, r-1$,

(2) $H^r(X, \mathcal{O}_X(-r-1)) \simeq A$,

(3) for each $n \in \mathbb{Z}$, *there is a perfect pairing of finitely generated free* A*-modules*

$$H^0(X, \mathcal{O}_X(n)) \times H^r(X, \mathcal{O}_X(-n-r-1)) \to H^r(X, \mathcal{O}_X(-r-1)).$$

In particular, recall that for $A = \mathbb{C}$ we have $\omega_{\mathbb{P}^r} \simeq \mathcal{O}(-r-1)$, hence this theorem says that $H^r(\mathbb{P}^r, \omega_{\mathbb{P}^r}) \simeq \mathbb{C}$, and that for each $n \in \mathbb{Z}$, we have an isomorphism

$$H^0(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(n)) \simeq H^r(\mathbb{P}^r, \mathcal{O}_{\mathbb{P}^r}(-n) \otimes \omega_{\mathbb{P}^r}).$$

This is just a manifestation (or a special case) of the following important *duality theorem* due to Serre, which further amplifies the importance of the canonical bundle on varieties.

Theorem 2.55. *Let* X *be a smooth projective variety of dimension* n *and let* $\mathcal{F}$ *be a locally free sheaf on* X*. Then for every* $0 \le i \le n$ *there is an isomorphism*

$$H^i(X, \mathcal{F}) \simeq H^{n-i}(X, \omega_X \otimes \mathcal{F}^{-1}).$$

In particular, since $\Omega_X^{n-p} \simeq (\Omega_X^p)^\vee \otimes \omega_X$, where $(\Omega_X^p)^\vee = \mathrm{Hom}_{\mathcal{O}_X}(\Omega_X^p, \mathcal{O}_X)$ is the *dual of* Ω_X^p (exercise!), we have the following corollary, which is prominent in Hodge theory.

Corollary 2.56. *Let* X *be a smooth projective variety of dimension* n*. Then for all* $0 \le p, q \le n$ *we have*

$$H^q(X, \Omega_X^p) \simeq H^{n-q}(X, \Omega_X^{n-p}).$$

Now, recall that, if $f \colon X \to Y$ is a morphism between two varieties, and $\mathcal{F}$ is an $\mathcal{O}_X$-module, then

$$H^0(X, \mathcal{F}) \simeq H^0(Y, f_*\mathcal{F})$$

by definition. We might wonder in which circumstances this continues to hold when we consider higher cohomology groups, that is, when do we have

$$H^p(X, \mathcal{F}) \simeq H^p(Y, f_*\mathcal{F})$$

for $p > 0$. The answer is given in the following important result, which gives another connection between global and local cohomology, and it is often used to pass from local to global questions.

Theorem 2.57. *Let $f\colon X \to Y$ be a morphism between two varieties, and let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Assume that*

$$R^q f_*(\mathcal{F}) = 0$$

for all $q > 0$. Then there are natural isomorphisms

$$H^p(X, \mathcal{F}) \simeq H^p(Y, f_*\mathcal{F})$$

for every $p \geq 0$.

This can be obtained as a consequence of the Leray spectral sequence (exercise!).

The projection formula for morphisms between two varieties gave a connection between functors f_* and f^* in certain cases. The following result is a natural generalisation of that result (exercise!).

Lemma 2.58. *Let $f\colon X \to Y$ be a morphism between two varieties, let $\mathcal{F}$ be an $\mathcal{O}_X$-module, and let $\mathcal{L}$ be a locally free $\mathcal{O}_Y$-module of finite rank. Then for every $q \geq 0$ we have*

$$R^q f_*(\mathcal{F} \otimes f^*\mathcal{L}) \simeq R^q f_*(\mathcal{F}) \otimes \mathcal{L}.$$

To prove this, you should cover Y by open affines where $\mathcal{L}$ trivialises, and prove the result locally, which is enough.

Now we have enough tools to give the proof of the Riemann-Roch theorem on curves. First a few preliminary comments. If X is a smooth projective curve, the sheaf

$$\Omega^1_X = \omega_X$$

is a line bundle on X, with the corresponding canonical divisor K_X. The number

$$g = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(K_X))$$

is the *genus* of X. If $D = \sum p_i P_i$ is a divisor on X, then $\deg D = \sum p_i$ is its degree.

Theorem 2.59. *Let X be a smooth projective curve, and let D be a divisor on X. Then we have the following Riemann-Roch formula:*

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(D)) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(K_X - D)) + \deg D - g + 1.$$

Proof. Let us first re-interpret the terms in this formula. By Serre duality we have

$$H^0(X, \mathcal{O}_X(K_X)) \simeq H^1(X, \mathcal{O}_X)$$

and

$$H^0(X, \mathcal{O}_X(K_X - D)) \simeq H^1(X, \mathcal{O}_X(D)).$$

Next, since we are on a projective curve, we must have $H^0(X, \mathcal{O}_X) \simeq \mathbb{C}$. Therefore, the formula re-reads as

$$\chi(X, \mathcal{O}_X(D)) = \deg D + \chi(X, \mathcal{O}_X).$$

This is what we will prove. The proof is by induction on the number of components of D. If $D = 0$, then the formula is trivially true.

Now assume that the formula holds for some D, and let P be any point on X.

Claim 2.60. *For every $m \in \mathbb{Z}$ we have $\chi(X, \mathcal{O}_X(D + mP)) = \chi(X, \mathcal{O}_X(D)) + m$.*

This claim immediately implies the theorem.

To prove the claim, note that it is trivial for $m = 0$. Also observe that, by symmetry, it is enough to prove it when $m < 0$. Further, by induction on m, it is enough to show it for $m = -1$, i.e. we will show that

$$\chi(X, \mathcal{O}_X(D - P)) = \chi(X, \mathcal{O}_X(D)) - 1. \tag{2.3}$$

Let $\mathcal{I}$ be the ideal sheaf of P in X, and recall that $\mathcal{I} = \mathcal{O}_X(-P)$. Also since $\chi(P, \mathcal{O}_P) = h^0(P, \mathcal{O}_P) = 1$, another way to write (2.3) is

$$\chi(X, \mathcal{I} \otimes \mathcal{O}_X(D)) = \chi(X, \mathcal{O}_X(D)) - \chi(P, \mathcal{O}_P).$$

This suggests that a more general result is true. The following lemma generalises (2.3) (by putting $Z = P$ and $\mathcal{L} = \mathcal{O}_X(D)$, and noting that $\mathcal{O}_X(D)|_P \simeq \mathcal{O}_P$). The result is interesting on its own right, and it demonstrates some of the standard techniques which we use when we work with exact sequences and cohomology.

Lemma 2.61. *Let X be a variety and let Z be a closed subvariety of X defined by a quasi-coherent ideal sheaf $\mathcal{I}$. Let $\mathcal{L}$ be a locally free $\mathcal{O}_X$-module of finite rank. Then*

$$\chi(X, \mathcal{I} \otimes \mathcal{L}) = \chi(X, \mathcal{L}) - \chi(Z, \mathcal{L}|_Z).$$

To prove the lemma, let $i\colon Z \hookrightarrow X$ be the inclusion. We have the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0.$$

Tensoring by $\mathcal{L}$, we obtain

$$0 \to \mathcal{I} \otimes \mathcal{L} \to \mathcal{L} \to i_*\mathcal{O}_Z \otimes \mathcal{L} \to 0$$

(this sequence is again exact since $\mathcal{L}$ is a locally free sheaf). Since we know that the Euler characteristic is additive on exact sequences, we get

$$\chi(X, \mathcal{I} \otimes \mathcal{L}) + \chi(X, i_*\mathcal{O}_Z \otimes \mathcal{L}) = \chi(X, \mathcal{L}).$$

Thus, we have to show that

$$\chi(X, i_*\mathcal{O}_Z \otimes \mathcal{L}) = \chi(Z, \mathcal{L}|_Z).$$

By the projection formula we have $i_*i^*\mathcal{L} = i_*\mathcal{O}_Z \otimes \mathcal{L}$, and we have $\chi(Z, \mathcal{L}_{|Z}) = \chi(Z, i^*\mathcal{L})$ by definition. So we should prove

$$\chi(X, i_*i^*\mathcal{L}) = \chi(Z, i^*\mathcal{L}).$$

By Theorem 2.57 it thus suffices to show that $R^q i_*(i^*\mathcal{L}) = 0$ for all $q > 0$. It is enough to show this locally over X, so we can replace X by an open affine subset U, and replace Z by $Z \cap U$. But then

$$R^q i_*(i^*\mathcal{L}) \simeq H^q(Z \cap U, \mathcal{L})^\sim,$$

and since $Z \cap U$ is affine, we have that this last sheaf is zero by Theorem 1.36. This completes the proof of the lemma, of the claim, and of the Riemann-Roch formula. $\square$

A morphism $f\colon X \to Y$ between varieties is *affine* if the inverse image of any open affine subvariety of Y is an open affine subvariety of X. Then we note for the future reference the following result which we showed at the end of the proof of Theorem 2.59, and which we often use in conjunction with Theorem 2.57.

Lemma 2.62. *Let $f\colon X \to Y$ be an affine morphism between varieties. Then for any quasi-coherent sheaf $\mathcal{F}$ on X and any $q > 0$ we have*

$$R^q f_*(\mathcal{F}) = 0.$$

Chapter 3

Positivity

In this chapter we explore various concepts of positivity in algebraic geometry. The basic notion of positivity is ampleness, for reasons which should be apparent from what follows below. We have already introduced ampleness before, but now we will see several criteria for ampleness, cohomological and numerical.

3.1 Cohomological characterisation of ampleness

As before, for any variety Y we can define the sheaf $\mathcal{O}_{\mathbb{P}^n_Y}(1)$ by gluing. Another way to put this, if

$$\pi\colon \mathbb{P}^n_Y = \mathbb{P}^n_{\mathbb{Z}} \times Y \to \mathbb{P}^n_{\mathbb{Z}}$$

is the projection, then $\mathcal{O}_{\mathbb{P}^n_Y}(1) = \pi^*\mathcal{O}_{\mathbb{P}^n_{\mathbb{Z}}}(1)$. If $X = \operatorname{Proj} S$ for a graded ring $S = \bigoplus_{d\in\mathbb{N}} S_d$, then we define $\mathcal{O}_X(d) = \widetilde{\bigoplus_{d\geq n} S_d}$. It's an easy exercise to see that all these definitions are compatible, and that all these are very ample sheaves.

A bit of notation. If we are on a projective variety X with the sheaf $\mathcal{O}_X(1)$, and if $\mathcal{F}$ is an $\mathcal{O}_X$-module, then $\mathcal{F}(n)$ denotes the sheaf $\mathcal{F} \otimes \mathcal{O}_X(n)$.

The following result due to Serre is not too difficult, especially part (1), which is left as an (advanced) exercise in sheaf theory. Part (2) requires a bit more calculation, in particular of the cohomology of twisting sheaves on $\mathbb{P}^n_A$ given in Theorem 2.54.

Theorem 3.1. *Let X be a projective variety and let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then:*

(1) there is a positive integer n_0 such that the sheaves $\mathcal{F}(n)$ are globally generated for all $n \geq n_0$,

(2) there is a positive integer m_0 such that $H^i(X, \mathcal{F}(n)) = 0$ for all $n \geq m_0$ and all $i > 0$.

This has an important generalisation.

Theorem 3.2 (Cartan-Serre-Grothendieck)**.** *Let X be a complete variety and let $\mathcal{L}$ be a line bundle on X. Then the following are equivalent:*

(1) $\mathcal{L}$ is ample,

(2) for any coherent sheaf $\mathcal{F}$ on X, there is a positive integer n_0 such that for all $n \geq n_0$ and all $i > 0$ we have

$$H^i(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0,$$

(3) for any coherent sheaf $\mathcal{F}$ on X, there is a positive integer m_0 such that for all $m \geq m_0$, the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes m}$ is generated by global sections,

(4) there is a positive integer p_0 such that for all $p \geq p_0$, the line bundle $\mathcal{L}^{\otimes p}$ is very ample.

The importance of this result lies in part (2): this gives a cohomological characterisation of ampleness; in particular this shows how ampleness depends on the geometry of a variety.

Proof. We note that (4) $\Rightarrow$ (1) is immediate. Here I only sketch the proofs of (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3), and omit the proof of the implication (3) $\Rightarrow$ (4).

We first show (1) $\Rightarrow$ (2). If $\mathcal{L}$ is ample, there is a positive integer k such that $\mathcal{L}^{\otimes k}$ is very ample, and thus defines an embedding i into some projective space $\mathbb{P}^N$. By considering pushforwards by i of sheaves onto $\mathbb{P}^N$ (since pushforward does not change cohomology thanks to Lemma 2.62 and Theorem 2.57), we can assume that $X = \mathbb{P}^N$. In particular, $\mathcal{L}^{\otimes k} = \mathcal{O}_{\mathbb{P}^N}(1)$.

Let $\mathcal{F}_\ell = \mathcal{F} \otimes \mathcal{L}^{\otimes \ell}$ for $\ell = 0, 1, \ldots, k-1$. By Serre's theorem above, there is a large integer n_0 such that $H^i(\mathbb{P}^N, \mathcal{F}_\ell(n)) = 0$ for all $i > 0$ and $n \geq n_0$, and $\ell = 0, 1, \ldots, k-1$ (since there are finitely many ℓ's, we can choose one n_0 to work for all the sheaves $\mathcal{F}_\ell$). But then $H^i(\mathbb{P}^N, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) = 0$ for all $i > 0$ and all $n \geq kn_0$.

For (2) $\Rightarrow$ (3), fix a closed point $x \in X$, let $\mathfrak{m}_x$ be the associated ideal sheaf, and consider the exact sequence

$$0 \to \mathfrak{m}_x\mathcal{F} \to \mathcal{F} \to \mathcal{F}/\mathfrak{m}_x\mathcal{F} = \mathcal{F} \otimes \mathbb{C}(x) \to 0,$$

where $\mathbb{C}(x)$ is the skyscraper sheaf at x. After tensoring the sequence by $\mathcal{L}^{\otimes n}$ and taking the long exact cohomology sequence, we get

$$\begin{aligned} 0 &\to H^0(X, \mathfrak{m}_x\mathcal{F} \otimes \mathcal{L}^{\otimes n}) \to H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \to H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n} \otimes \mathbb{C}(x)) \\ &\to H^1(X, \mathfrak{m}_x\mathcal{F} \otimes \mathcal{L}^{\otimes n}) \to \ldots \end{aligned}$$

By assumption, there is a positive integer n_0 such that this last group vanishes as soon as $n \geq n_0$, and therefore the map

$$H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n}) \to H^0(X, \mathcal{F}/\mathfrak{m}_x\mathcal{F} \otimes \mathcal{L}^{\otimes n})$$

is surjective. In particular, the global sections of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ generate the stalk of this sheaf at x. By using Nakayama's lemma, one can show that this implies that $H^0(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ generates the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ in an open neighbourhood of x (exercise!).

Caution: this neighbourhood depends on n. A way to overcome this is as follows. Applying the above argument for $\mathcal{F} = \mathcal{O}_X$, we get a positive integer n_1 such that $\mathcal{L}^{\otimes n_1}$ is globally generated in an open neighbourhood V of x. Then back to the original $\mathcal{F}$, there are neighbourhoods U_i for $i = 0, 1, \ldots, n_1 - 1$ such that the sheaves $\mathcal{F} \otimes \mathcal{L}^{\otimes(n_0+i)}$ are globally generated on them. Since any $n \geq n_0$ can be written as $(n_0 + i) + \ell n_1$ for some such i and some $\ell \geq 0$, we get that all sheaves $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ are globally generated in the neighbourhood $V \cap U_0 \cap U_1 \cap \ldots U_{n_1-1}$ of x.

Finally, since X is a Noetherian set, there is a finite open cover $\{W_j\}$ of X and positive integers k_j such that the sheaves $\mathcal{F} \otimes \mathcal{L}^{\otimes k}$ are globally generated on W_j for $k \geq k_j$, respectively. Now the sheaf $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ is globally generated for $n \geq \max_j\{k_j\}$. This finishes the proof. $\square$

The next criterion gives a geometric interpretation of ampleness in terms of separation of points and tangent vectors – the idea is that closed embeddings behave like immersions in differential geometry.

Theorem 3.3. *Let X be a projective variety and let $\mathcal{L}$ be a globally generated line bundle on X. Then $\mathcal{L}$ is very ample iff the following two conditions are satisfied.*

(1) (separation of points) For any two distinct closed points P and Q in X, there is a section $s \in H^0(X, \mathcal{L})$ such that $s(P) = 0$ and $s(Q) \neq 0$.

(2) (separation of tangent vectors) For every closed point P, the set $\{s \in H^0(X, \mathcal{L}) \mid s_P \in \mathfrak{m}_P\mathcal{L}_P\}$ spans the $\mathbb{C}$-vector space $\mathfrak{m}_P\mathcal{L}_P/\mathfrak{m}_P^2\mathcal{L}_P$.

Proof. We just prove necessity, sufficiency can be proved with some more effort using Nakayama's lemma.

If $\mathcal{L}$ is very ample, then its sections define a closed embedding $i \colon X \to \mathbb{P}^N$, so we can assume that $\mathcal{L}$ is the pullback of $\mathcal{O}_{\mathbb{P}^N}(1)$ under i, and thus $\mathcal{L} = \mathcal{O}_X(1)$ by definition. Then $\Gamma(X, \mathcal{O}_X(1))$ is spanned by the pullbacks of generators X_i of $\Gamma(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1))$. Given two closed points $P \neq Q$ on X, there are (many) hyperplanes $(\sum \alpha_i X_i = 0)$ passing through $i(P)$ and not through $i(Q)$. Now the pullbacks of the equations of those hyperplanes by i give desired sections of $\Gamma(X, \mathcal{O}_X(1))$, i.e. $s = \sum \alpha_i i^* X_i$. This proves (1).

To show (2), by changing coordinates we can assume that $P = (1, 0, \ldots, 0)$. Then P belongs to the open chart $U_0 = \operatorname{Spec} \mathbb{C}[Y_1, \ldots, Y_N]$ (where $Y_i = X_i/X_0$), and here $\mathcal{L}|_{U_0} \simeq \mathcal{O}_{U_0}$. Further, notice that

$$\mathfrak{m}_P \mathcal{L}_P / \mathfrak{m}_P^2 \mathcal{L}_P \simeq \mathfrak{m}_P / \mathfrak{m}_P^2,$$

and this is obviously spanned by the germs of $Y_1, \ldots, Y_N$. We are done. $\square$

We use this result to study ample and basepoint free divisors on curves. We will see that there is a simple criterion on any curve for a divisor to be ample.

Lemma 3.4. *Let X be a smooth projective curve and let D be a divisor on X. Then:*

(1) D is basepoint free iff for every point $P \in X$ we have

$$h^0(X, \mathcal{O}_X(D - P)) = h^0(X, \mathcal{O}_X(D)) - 1,$$

(2) D is very ample iff for every two (not necessarily distinct) points P and Q in X we have

$$h^0(X, \mathcal{O}_X(D - P - Q)) = h^0(X, \mathcal{O}_X(D)) - 2.$$

Proof. First we prove (1). Fix a point $P \in X$, consider the exact sequence

$$0 \to \mathcal{O}_X(-P) \to \mathcal{O}_X \to \mathcal{O}_P \to 0,$$

and note that $\mathcal{O}_P \simeq \mathcal{O}_{X,P}/\mathfrak{m}_P \simeq \mathbb{C}$. Tensoring this sequence by $\mathcal{O}_X(D)$ and taking global sections, we obtain

$$0 \to H^0(X, \mathcal{O}_X(D - P)) \to H^0(X, \mathcal{O}_X(D)) \to \mathbb{C},$$

so

$$0 \le h^0(X, \mathcal{O}_X(D)) - h^0(X, \mathcal{O}_X(D - P)) \le 1.$$

But if this difference is 0, then this is equivalent to saying that P is in the base locus of $|D|$ (exercise!).

Now we show (2). First of all, if D is very ample, then D is basepoint free by definition. Also, if the condition from (2) holds, then D is basepoint free by (1). So we can assume that D is basepoint free. Then the sections of $\mathcal{O}_X(D)$ define a morphism $\varphi\colon X \to \mathbb{P}^N$, and we will use the criterion from Theorem 3.3.

The separation of points tells precisely that for two points $P \neq Q$, $Q \notin \operatorname{Bs}|D - P|$, and this is equivalent to the equality in the statement.

Now we consider the case $P = Q$. We have $\dim_{\mathbb{C}} \mathfrak{m}_P/\mathfrak{m}_P^2 = 1$ since the curve X is smooth. The separation of tangent vectors then means that there is a section $s \in H^0(X, \mathcal{O}_X(D))$ whose germ belongs to $\mathfrak{m}_P$ and not to $\mathfrak{m}_P^2$; this means precisely that there is a divisor $F \in |D|$ such that $\operatorname{mult}_P F = 1$. But this is equivalent to saying that $P \notin \operatorname{Bs}|D - P|$, and thus $h^0(X, \mathcal{O}_X(D)) - h^0(X, \mathcal{O}_X(D - 2P)) = 2$. We are done. $\square$

Corollary 3.5. *Let X be a smooth projective curve of genus g and let D be a divisor on X. Then:*

(1) if $\deg D \geq 2g$, then D is basepoint free,

(2) if $\deg D \geq 2g+1$, then D is very ample.

In particular, D is ample iff $\deg D > 0$.

Proof. We have $\deg K_X = 2g-2$ by the Riemann-Roch theorem. Hence under the assumptions of both (1) and (2) we have

$$\deg(K_X - D) = \deg K_X - \deg D < 0,$$

and so $h^0(X, \mathcal{O}_X(K_X - D)) = 0$ since the degree is an invariant of a linear system. Similarly

$$h^0(X, \mathcal{O}_X(K_X - (D-P))) = 0$$

under the assumptions of (1), and

$$h^0(X, \mathcal{O}_X(K_X - (D-P-Q))) = 0$$

under the assumptions of (2), for any two points $P, Q \in X$. Therefore, Riemann-Roch gives

$$\begin{aligned} h^0(X, \mathcal{O}_X(D)) &= \deg D - g + 1, \\ h^0(X, \mathcal{O}_X(D-P)) &= \deg D - g, \\ h^0(X, \mathcal{O}_X(D-P-Q)) &= \deg D - g - 1, \end{aligned}$$

so we have (1) and (2) by the previous lemma. □

3.2 Numerical characterisation of ampleness

In this section I introduce the intersection theory on projective varieties and we relate it to positivity properties of ample divisors. This can be done in greater generality, but we only consider a special case which is of interest to us.

Assume that we have a projective variety X of dimension n, and a collection of Cartier divisors $D_1, \ldots, D_m$, where $m \geq n$. Then we would like to define the intersection product $D_1 \cdot \ldots \cdot D_m$ in such a way that some desired properties hold: we want this to be a symmetric multilinear map, i.e. linear in each variable and does not depend on the order of D_i; further we would want that it does not depend on the choice of D_i in its linear equivalence class for each of i; we also want it to behave similarly to set-theoretic intersection.

It turns out that there is essentially only one way to achieve these requirements.

Definition 3.6. Let X be a projective variety of dimension n, and let $D_1, \ldots, D_m$ be Cartier divisors on X, where $m \geq n$. Define

$$D_1 \cdot \ldots \cdot D_m = \sum_{j=0}^{m} (-1)^j \sum_{i_1 < \cdots < i_j} \chi(\mathcal{O}_X(-D_{i_1} - \ldots - D_{i_j})).$$

Note that, by definition, the summand on the RHS for $j = 0$ is $\chi(\mathcal{O}_X)$.

Then, with some pain, one can show that this map is indeed symmetric and multilinear. Further, it can be shown that if $m > n$ above, then $D_1 \cdot \ldots \cdot D_m = 0$, as expected.

I mention some other properties that the intersection product satisfies. Let Y be a Cartier divisor on X given by the coherent ideal sheaf $\mathcal{I} = \mathcal{O}_X(-Y)$. Then we have the exact sequence

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_Y \to 0.$$

From this and from Definition 3.6, it is easy to check that for Cartier divisors $D_1, \ldots, D_{n-1}$ we have

$$D_1 \cdot \ldots \cdot D_{n-1} \cdot Y = D_1|_Y \cdot \ldots \cdot D_{n-1}|_Y,$$

as expected. In general, if V is a closed subvariety of X of dimension $k < n$, and if $D_1, \ldots, D_k$ are Cartier divisors on X, we can define the intersection product by

$$D_1 \cdot \ldots \cdot D_k \cdot V := D_1|_V \cdot \ldots \cdot D_k|_V;$$

here we interpret $D_i|_V$ as $\mathcal{O}_X(D_i)|_V$. In particular, if X is a surface and C_1 and C_2 are two curves on X, then

$$C_1 \cdot C_2 = \deg(\mathcal{O}_X(C_1)|_{C_2}),$$

where this is defined as the degree of the linear system defined by the line bundle $\mathcal{O}_X(C_1)|_{C_2}$ on C_2.

If X is the projective space $\mathbb{P}^n$ and if H is a hyperplane in X, we want to show that $H^n = 1$. Indeed, let G be any other hyperplane; then $H \sim G$, so we can choose G so that $H|_G$ is defined. Note that as varieties, $H \simeq G \simeq \mathbb{P}^{n-1}$, so $H|_G$ defines a hyperplane in G. Then by the formula above:

$$H^n = H^{n-1} \cdot G = (H|_G)^{n-1}.$$

Therefore, by continuing this process, we get that H^n is equal to the degree of the line bundle $\mathcal{O}_{\mathbb{P}^1}(1)$, and this is just 1. We will use this result in the proof of the asymptotic Riemann-Roch formula below.

Definition 3.7. A morphism $f\colon X \to Y$ between varieties is *finite* if there is an affine open covering $\{U_i = \operatorname{Spec} A_i\}$ of Y such that each $f^{-1}(U_i)$ is affine, equal to $\operatorname{Spec} B_i$ for some finitely generated A_i-module B_i. It is *generically finite* if there is an open set $U \subseteq Y$ such that $f|_{f^{-1}(U)}$ is finite.

If $f\colon X \to Y$ is a generically finite morphism, then the degree $[k(X) : k(Y)]$ is finite, and we say it is the *degree of* f. If f is a finite morphism, it is easy to check that f is an affine morphism. One can show that then the preimage of every point of Y is a finite set of points of X. Embeddings are examples of finite maps, as it is easy to show, and birational morphisms are examples of generically finite maps.

One thing to remember is that if we have a generically finite projective surjective morphism $f\colon Z \to X$ of degree k from a projective variety Z, then for $m \geq \dim Z = \dim X$,

$$f^*D_1 \cdot \ldots \cdot f^*D_m = k(D_1 \cdot \ldots \cdot D_m).$$

This in particular holds when f is a birational morphism, where $k = 1$.

If $f\colon X \to Y$ is a proper morphism, if D is a Cartier divisor on Y and C is a curve on X, then we have the following projection formula

$$f^*D \cdot C = D \cdot f_*C, \tag{3.1}$$

where $f_*C = 0$ if $f(C)$ is a point, and $f_*C = [k(C) : k(f(C))] \cdot f(C)$ when $f(C)$ is a curve.

Now consider the situation where X is a projective variety and $\mathcal{L}$ is a line bundle on X. Then it is pretty easy, but long, to prove that $\chi(\mathcal{L}^{\otimes m})$ is a polynomial in m, of degree at most $\dim X$. By the Riemann-Roch on curves, we know that

$$\chi(\mathcal{O}_X(mD)) = \chi(\mathcal{O}_X) + \deg(mD),$$

so in particular the leading coefficient here is $\deg D$. For applications and in higher dimensions, it is often important to have a better understanding of what the leading coefficient of $\chi(\mathcal{O}(mD))$ is, and the answer is given in the following result, which is known as *asymptotic Riemann-Roch.*

Theorem 3.8. *Let X be a projective variety of dimension n and let D be a Cartier divisor on X. Then*

$$\chi(\mathcal{O}_X(mD)) = \frac{D^n}{n!}m^n + Q(m),$$

where Q is a complex polynomial of degree at most $n-1$.

Proof. Write $\chi(\mathcal{O}_X(mD)) = \alpha_{X,D}m^n + Q(m)$ with $\deg Q \leq n-1$. By definition and multilinearity of the intersection product we have

$$\begin{aligned}(-1)^n m^n D^n = (-mD)^n &= \sum_{i=0}^{n}(-1)^i\binom{n}{i}\chi(\mathcal{O}_X(miD)) \\ &= \left(\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^n\right)\alpha_{X,D}m^n + \sum_{i=0}^{n}(-1)^i\binom{n}{i}Q(im),\end{aligned}$$

hence

$$(-1)^n D^n = \left(\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^n\right)\alpha_{X,D}.$$

Thus it suffices to show that

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^n = (-1)^n n!. \tag{3.2}$$

Now, we specialise to the case when $X = \mathbb{P}^n$ and D is a hyperplane, so that $\mathcal{O}_X(D) = \mathcal{O}_{\mathbb{P}^n}(1)$. Note that $H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = 0$ for $i > 0$ when m is large by Serre's theorem. Therefore, for $m \gg 0$ we have

$$\chi(\mathcal{O}_{\mathbb{P}^n}(m)) = h^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(m)) = \binom{n+m}{n} = \frac{m^n}{n!} + \dots.$$

This gives by definition $\alpha_{\mathbb{P}^n,D} = 1/n!$, and since $D^n = 1$, we get (3.2), which completes the proof. □

Since, by Serre's theorem, we know that higher cohomology vanishes when we take a high enough power of an ample bundle, the Euler characteristic reduces just to h^0, so we have the following asymptotic statement.

Corollary 3.9. *Let X be a projective variety of dimension n and let D be an ample divisor on X. Then for $m \gg 0$ we have*

$$h^0(X, \mathcal{O}_X(mD)) = \frac{D^n}{n!}m^n + Q(m),$$

where Q is a complex polynomial of degree at most $n-1$.

This is a first hint that numerical and cohomological properties of ampleness are related.

This is also a convenient place to record, for later purpose, the Riemann-Roch for surfaces.

Theorem 3.10. *Let X be a smooth projective surface, and let D be a Cartier divisor on X. Then*
$$\chi(\mathcal{O}_X(D)) = \chi(\mathcal{O}_X) + \frac{1}{2} D \cdot (D - K_X).$$

Proof. From the definition of the intersection product and by the Serre duality we have
$$\begin{aligned} -K_X \cdot D &= \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-D)) - \chi(\mathcal{O}_X(-K_X)) + \chi(\mathcal{O}_X(-K_X - D)) \\ &= \chi(\mathcal{O}_X(D)) - \chi(\mathcal{O}_X(K_X + D)) \end{aligned}$$
and
$$\begin{aligned} -D \cdot D &= \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-D)) - \chi(\mathcal{O}_X(D)) + \chi(\mathcal{O}_X) \\ &= 2\chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(K_X + D)) - \chi(\mathcal{O}_X(D)), \end{aligned}$$
hence the desired formula easily follows. □

Lemma 3.11. *Let $f\colon X \to Y$ be a finite morphism of complete varieties. If $\mathcal{L}$ is an ample line bundle on Y, then $f^*\mathcal{L}$ is an ample line bundle on X.*

In particular, if X is a subvariety of Y, and if $\mathcal{L}$ is an ample line bundle on Y, then $\mathcal{L}|_X$ is an ample line bundle on X.

Proof. Let $\mathcal{F}$ be a coherent sheaf on X. Then $R^j f_*(\mathcal{F}) = 0$ for $j > 0$ by Lemma 2.62, so the projection formula gives
$$R^j f_*(\mathcal{F} \otimes f^*\mathcal{L}^{\otimes m}) = R^j f_*(\mathcal{F}) \otimes \mathcal{L}^{\otimes m} = 0$$
for every m. Therefore, by Theorem 2.57 we have
$$H^j(X, \mathcal{F} \otimes f^*\mathcal{L}^{\otimes m}) = H^j(Y, f_*\mathcal{F} \otimes \mathcal{L}^{\otimes m})$$
for $j \geq 0$, and these last groups vanish when $j \geq 1$ and $m \gg 0$ since $\mathcal{L}$ is ample. But then by Theorem 3.2, the divisor $f^*\mathcal{L}$ is ample. □

An easy consequence is the following characterisation of basepoint free ample divisors.

Corollary 3.12. *Let X be a complete variety and let D be a basepoint free Cartier divisor on X. Then D is ample iff $D \cdot C > 0$ for every irreducible curve C in X.*

Proof. Let $\varphi\colon X \to \mathbb{P}^N$ be the morphism associated to $\mathcal{O}_X(D)$. There are two cases.

If φ is not finite, then there is curve C on X which is contracted to a point. Therefore the bundle $\mathcal{O}_X(D)|_C \simeq (\varphi^*\mathcal{O}_{\mathbb{P}^N}(1))|_C$ is trivial, and so $D \cdot C = 0$. In particular, the bundle $\mathcal{O}_X(D)|_C$ is not ample by Corollary 3.5, but then D is not ample by the previous lemma.

If φ is finite, then D is ample by the previous lemma since $\mathcal{O}_X(D) = \varphi^*\mathcal{O}_{\mathbb{P}^N}(1)$, and so is $\mathcal{O}_X(D)|_C$ also by the lemma. But then $D \cdot C = \deg(\mathcal{O}_X(D)|_C) > 0$ by Corollary 3.5. □

We can now state the following Nakai-Moishezon-Kleiman criterion for ampleness. It tells us that in general, when D is not basepoint free, we should not test positivity only on curves, but on all subvarieties.

Theorem 3.13. *Let X be a projective variety and let D be Cartier divisor on X. Then D is ample iff for every closed subvariety Y of X with $\dim Y \geq 1$ we have*

$$D^{\dim Y} \cdot Y = (D|_Y)^{\dim Y} > 0.$$

Necessity is an easy application of Lemma 3.11. The converse is also not too difficult, but it requires us to work with non-reduced and reducible schemes, so we omit it.

Definition 3.14. We say that two Cartier divisors D_1 and D_2 on a projective variety X are *numerically equivalent*, and write $D_1 \equiv D_2$, if $D_1 \cdot C = D_2 \cdot C$ for every irreducible curve C on X. We denote

$$N^1(X) = \mathrm{Div}(X)/\equiv,$$

and we call this group *the Néron-Severi group of X*.

The basic result is

Theorem 3.15. *If X is a projective variety, then $N^1(X)$ is a free abelian group of finite rank.*

The proof follows from the fact that $N^1(X)$ can be realised as a subgroup of $H^2(X, \mathbb{Z})/(\text{torsion})$, where this is just the usual, singular cohomology of X. The number $\rho(X) = \mathrm{rk}\, N^1(X)$ is the *Picard number of X*.

3.3 Nefness

We saw that a divisor D on a projective variety X is ample iff for every subvariety Y of X of positive dimension we have

$$D^{\dim Y} \cdot Y > 0.$$

In other words, the *degree* of D on every subvariety of X is positive. Therefore, we might wonder what properties hold for divisors which satisfy a weaker condition, that their degree on every subvariety of X is non-negative. We will see that these divisors are, in some sense, limits of ample divisors, and they will be our object of investigation in this lecture.

First, we note the following easy result.

Lemma 3.16. *Let X be a projective variety and let D be an ample divisor on X. If E is a divisor on X such that $D \equiv E$, then E is ample. In other words, ampleness is a numerical property of divisors, so it makes sense to talk about* ample classes *in $N^1(X)$.*

The contents of this lemma is to prove that if F is a numerically trivial divisor, i.e. if $F \equiv 0$, and if $D_1, \ldots, D_k$ are Cartier divisors on X, where $k = \dim Y - 1$, then

$$F \cdot D_1 \cdot \cdots \cdot D_k \cdot Y = 0.$$

The proof of this is easy, but depends on a more general definition of the intersection product. The idea is that $D_1 \cdot \cdots \cdot D_k \cdot Y$ can be represented as a 1-*cycle*, that is a formal sum of irreducible curves, and then the statement is obvious from the definition.

We saw that the group $N^1(X)$ is a free group of finite rank, and therefore it is a discrete object. We will see that it is useful to consider its continuous partner, the group of *real classes*

$$N^1(X)_{\mathbb{R}} = N^1(X) \otimes \mathbb{R},$$

which we can view as a sub-vector space of the second homology $H_2(X, \mathbb{R})$ as before. Thus, it is useful to consider divisors with *real coefficients*.

Definition 3.17. Let X be a normal variety. The *group of* $\mathbb{R}$*-divisors* is the group of formal $\mathbb{R}$-linear combinations of integral Cartier divisors, i.e.

$$\operatorname{Div}_{\mathbb{R}}(X) = \operatorname{Div}(X) \otimes \mathbb{R},$$

and similarly for $\mathbb{Q}$-divisors.

The usual definitions of the support of a divisor, of its effectivity etc. carry forward to $\mathbb{Q}$- and $\mathbb{R}$-divisors.

Further, intersection theory can be extended to $\mathbb{R}$-divisors by tensoring with $\mathbb{R}$, i.e. if $D = \sum d_i D_i$ is a real divisor, where d_i are real numbers and D_i are integral divisors, then we intersect D with a curve C as

$$D \cdot C = \sum d_i (D_i \cdot C),$$

and so on (we extend the scalars from $\mathbb{Z}$ to $\mathbb{R}$ by multilinearity). Note that the intersection product is then a real number, or a rational number if all divisors involved are rational. Then we analogously have a notion of numerical equivalence of $\mathbb{R}$-divisors, the details are left as an exercise.

We define similarly pullbacks of $\mathbb{R}$-divisors via a morphism $f \colon Y \to X$, by pulling back all integral components, and extending scalars.

Definition 3.18. Let X be a projective variety. Two $\mathbb{R}$-divisors D_1 and D_2 are $\mathbb{R}$-*linearly equivalent*, and we write $D_1 \sim_{\mathbb{R}} D_2$, if there are rational functions $\varphi_1, \dots, \varphi_k \in k(X)$ and real numbers $r_1, \dots, r_k$ such that

$$D_1 - D_2 = \sum_{i=1}^{k} r_i \operatorname{div} \varphi_i.$$

Similarly for $\mathbb{Q}$-linear equivalence of $\mathbb{Q}$-divisors.

Note that two $\mathbb{Q}$-divisors D_1 and D_2 are $\mathbb{Q}$-linearly equivalent iff there is an integer p such that pD_1 and pD_2 are integral divisors, and $pD_1 \sim pD_2$ in the usual linear equivalence sense. *Caution*: it can happen that for two integral divisors D_1 and D_2 we have $D_1 \sim_{\mathbb{Q}} D_2$, but that they are not ($\mathbb{Z}$-)linearly equivalent!

Definition 3.19. Let X be a projective variety and let A be an $\mathbb{R}$-divisor on X. Then A is *ample* if it can be written as a positive linear combination of integral ample divisors, i.e. if there exist finitely many integral ample divisors A_i and real numbers $r_i > 0$ such that

$$A = \sum r_i A_i.$$

And similarly for ample $\mathbb{Q}$-divisors (note that this is equivalent to saying that there is a positive integer q such that qA is an integral ample divisor).

Now we have Nakai's criterion for ample $\mathbb{R}$-divisors.

Theorem 3.20. *Let X be a projective variety and let A be an $\mathbb{R}$-divisor on X. Then A is ample iff for every closed subvariety Y of X with $\dim Y \geq 1$ we have*

$$A^{\dim Y} \cdot Y = (A|_Y)^{\dim Y} > 0.$$

When A is a $\mathbb{Q}$-divisor, this follows trivially from Nakai's criterion for integral ampleness. Also, in the general setting of $\mathbb{R}$-divisors, one direction is clear – if A is ample, then the inequalities obviously hold. The true content is in the reverse implication, and that is a theorem of Campana and Peternell. The proof is not difficult, but it requires some knowledge of nef and big divisors.

As in the case of integral divisors, ampleness is a numerical property (i.e. we can talk about ample classes in $N^1(X)_{\mathbb{R}}$), and the proof is an exercise.

Now, a sum of two ample classes $N^1(X)_{\mathbb{R}}$ is again an ample class, and any positive multiple of an ample class is also ample. Therefore, ample classes in $N^1(X)_{\mathbb{R}}$ form a *cone*, denoted by $\operatorname{Amp}(X)$. The basic easy fact now is that this cone is *open*, and that is the content of the following result.

Lemma 3.21. *Let X be a projective variety and let A be an ample $\mathbb{R}$-divisor on X. Let $E_1, \dots, E_r$ be finitely many $\mathbb{R}$-divisors. Then the $\mathbb{R}$-divisor*

$$A + \varepsilon_1 E_1 + \cdots + \varepsilon_r E_r$$

is ample for all sufficiently small real numbers $0 \leq |\varepsilon_i| \ll 1$.

Proof. We only prove it here when all divisors involved and all numbers are rational, the rest is left as an exercise.

By clearing denominators, we may assume that all divisors are integral. We first claim that there exists a positive integer $m \gg 0$ such that all $mA \pm E_i$ are ample.

To see this, let n_0 be a positive integer such that nA is very ample, and such that $nA \pm E_i$ is basepoint free for all i and all $n \geq n_0$ (existence of n_0 follows from Cartan-Grothendieck-Serre criterion for ampleness). But then every

$$nA \pm E_i = (n - n_0)A + (n_0 A \pm E_i)$$

is basepoint free for every $n \geq 2n_0$, and for every irreducible curve C on X we have $A \cdot C > 0$ by Nakai's criterion, and $(n_0 A \pm E_i) \cdot C \geq 0$ since $n_0 A \pm E_i$ is basepoint free by (3.1). Therefore

$$(nA \pm E_i) \cdot C > 0$$

for every such a curve C and all $n \geq 2n_0$, so $nA \pm E_i$ is ample by Corollary 3.12. We set $m = 2n_0$.

Now that we know the claim, we fix such m, and let ε_i be rational numbers such that $|\varepsilon_i| < 1/mr$. Then

$$A + \varepsilon_1 E_1 + \cdots + \varepsilon_r E_r = (1 - m(|\varepsilon_1| + \cdots + |\varepsilon_r|))A + \sum_{i=1}^{r} |\varepsilon_i| \Big(mA + \frac{\varepsilon_i}{|\varepsilon_i|} E_i \Big),$$

and all the terms on the RHS are ample $\mathbb{Q}$-divisors. We are done. $\square$

Now that we know that the cone $\mathrm{Amp}(X)$ is open, we might wonder what happens on the boundary of this cone. First, for an $\mathbb{R}$-divisor D, denote by $[D]$ its class in $N^1(X)_{\mathbb{R}}$. Then if $[D]$ is in the closure of the ample cone, there are ample $\mathbb{R}$-divisors D_i such that $[D] = \lim_{i \to \infty} [D_i]$. Therefore, by Nakai's criterion, for every subvariety Y of X of positive dimension, by passing to the limit we must have

$$D^{\dim Y} \cdot Y \geq 0.$$

Definition 3.22. An $\mathbb{R}$-Cartier divisor D on a projective variety X is *nef* if

$$D^{\dim Y} \cdot Y \geq 0$$

for every subvariety $Y \subseteq X$ of dimension at least 1.

It is trivial to see that nefness is a numerical condition, and that a sum of two nef divisors is again nef, as is any positive multiple of a nef divisor. Therefore, numerical classes of nef divisors form a cone in $N^1(X)_{\mathbb{R}}$, denoted by $\mathrm{Nef}(X)$. It is obvious that $\mathrm{Amp}(X) \subseteq \mathrm{Nef}(X)$, and that the nef cone is closed, hence

$$\overline{\mathrm{Amp}(X)} \subseteq \mathrm{Nef}(X).$$

In fact, the following easy result gives the precise relationship between these two cones.

Corollary 3.23. *Let X be a projective variety and let D be a nef $\mathbb{R}$-divisor on X. If A is an ample $\mathbb{R}$-divisor on X, then $D + \varepsilon A$ is ample for every $\varepsilon > 0$. In particular,*

$$\mathrm{Nef}(X) = \overline{\mathrm{Amp}(X)} \quad \text{and} \quad \mathrm{Int}(\mathrm{Nef}(X)) = \mathrm{Amp}(X).$$

Now, a surprising fact is that nefness (unlike ampleness) can be tested only on curves. This is the famous *Kleiman's criterion.*

Theorem 3.24. *Let X be a projective variety and let D be an $\mathbb{R}$-divisor on X. Then D is nef iff for every irreducible curve C on X we have*

$$D \cdot C \geq 0. \tag{3.3}$$

Because of this result, the relations (3.3) are often taken as the definition of nefness.

Remark 3.25. It is useful to note that if $f\colon Z \to X$ is a projective morphism, and if N is a divisor on X satisfying (3.3) on X, then f^*N satisfies (3.3) on Z by the projection formula (3.1). The converse holds if additionally f is surjective. In particular, this holds when f is a closed embedding. Similarly, every basepoint free divisor on a complete variety is nef.

Proof. I give the proof in the special case when D is a $\mathbb{Q}$-divisor, the general case can be derived from this by a continuity argument (exercise!).

One direction is trivial; therefore, we prove that the relations (3.3) imply that D is nef. By induction on the dimension, we can assume that $D^{\dim Y} \cdot Y = (D|_Y)^{\dim Y} \geq 0$ for every proper subvariety Y of X. What then remains to show is $D^n \geq 0$, where $n = \dim X$.

Fix a very ample divisor A on X, and consider the polynomial

$$P(t) = (D + tA)^n = \sum_{i=0}^{n} \binom{n}{i} (D^{n-i} \cdot A^i) t^i$$

(considered as a polynomial by formal expansion of the RHS). We need to show that $P(0) \geq 0$. Assume for a contradiction that $P(0) < 0$.

Since A is very ample, we can view A as a subvariety of X, thus by induction we have

$$D^{n-i} \cdot A^i = (D|_A)^{n-i} \cdot (A|_A)^{i-1} \geq 0,$$

and the inequality is strict for $i = n$. Hence, $P'(t) > 0$ when $t > 0$, and the polynomial $P(t)$ is strictly increasing for $t > 0$. As $P(0) < 0$, this implies that there is a single $t_0 > 0$ such that $P(t_0) = 0$.

We claim that for every $t > t_0$, the divisor $D + tA$ is ample. Indeed, let Y be a subvariety of X of dimension $k \geq 1$. Then

$$(D + tA)^k \cdot Y = \sum_{i=0}^{k} \binom{k}{i} (D^{k-i} \cdot A^i \cdot Y) t^i.$$

Similarly as above, if $k < n$, then all the coefficients in this sum are non-negative, and the leading coefficient is positive, so the whole intersection number is positive. If $k = n$, then the intersection number in question equals $P(t) > P(t_0) = 0$. Therefore, the claim follows by Theorem 3.20.

Now we write $P(t) = Q(t) + R(t)$, with

$$Q(t) = D \cdot (D + tA)^{n-1}, \qquad R(t) = tA \cdot (D + tA)^{n-1}.$$

Since $D + tA$ is ample for rational $t > t_0$, similarly as above we have $Q(t) \geq 0$ by induction (here we use that the divisor $D+tA$ is rational in order to restrict to some multiple of $D + tA$). Thus, $Q(t_0) \geq 0$ by continuity. Further, we have $R(t_0) > 0$ since all the coefficients of $R(t)$ are non-negative and the leading coefficient A^n is positive. This implies $P(t_0) > 0$, a contradiction. □

Define *the closure of the cone of effective curves* $\overline{\mathrm{NE}}(X) \subseteq N_1(X)_{\mathbb{R}}$ as the closure of the cone of non-negative formal linear combinations of classes of irreducible curves on X. This enables us to give a numerical characterisation of ampleness – exercise!

Corollary 3.26. *Let X be a projective variety and let D be an $\mathbb{R}$-divisor on X. Then D is ample iff intersects every non-zero class in $\overline{\mathrm{NE}}(X)$ positively.*

We mention without a proof the following estimate on the higher cohomology of nef divisors; the proof uses Fujita's vanishing theorem.

Theorem 3.27. *Let X be a projective variety of dimension n, and let D be a nef divisor on X. Then we have*

$$h^i(X, \mathcal{O}_X(mD)) = O(m^{n-i}).$$

In particular, the asymptotic Riemann-Roch gives

$$h^0(X, \mathcal{O}_X(mD)) = \frac{D^n}{n!} m^n + O(m^{n-1}).$$

3.4 Iitaka fibration

We saw before that essentially all morphisms (or rational maps) to projective spaces come from divisors (with or without base points) on varieties. We will now see an important class of these maps, which are asymptotic versions of maps considered before. The idea is that when we take higher and higher multiples of divisors, properties of these become in some sense indistinguishable.

Let X be a normal projective variety and let $\mathcal{L}$ be a line bundle on X. Let $N(\mathcal{L})$ be the set of all integers $m \geq 0$ such that $H^0(X, \mathcal{L}^{\otimes m}) \neq 0$; it is easy to check that this is a monoid. For each $m \in N(\mathcal{L})$, by choosing a basis (or generating set) of $H^0(X, \mathcal{L}^{\otimes m})$, we have the associated rational map

$$\varphi_m = \varphi_{|\mathcal{L}^{\otimes m}|} \colon X \dashrightarrow \mathbb{P}^{N_m}.$$

It can be easily checked that these maps differ by an isomorphism if we choose a different basis, so we can talk about *the* rational map.

In what follows, I denote the variety $\overline{\varphi_m(X)}$ by X_m. Obviously $\dim X_m \leq \dim X$.

Definition 3.28. If $N(\mathcal{L}) \neq 0$, the *Iitaka dimension* of $\mathcal{L}$ is defined as

$$\kappa(X, \mathcal{L}) = \max_{m \in N(\mathcal{L})} \{\dim X_m\},$$

and we set $\kappa(X, \mathcal{L}) = -\infty$ otherwise. Therefore, $\kappa(X) \in \{-\infty, 0, 1, \ldots, \dim X\}$.

When X is smooth and $\mathcal{L} = \omega_X$, we call $\kappa(X, \omega_X)$ the *Kodaira dimension* of X, and denote it just by $\kappa(X)$.

If $\mathcal{L}$ is an ample line bundle, then $\mathcal{L}^{\otimes m}$ is very ample for all $m \gg 0$, and therefore $\kappa(X, \mathcal{L}) = \dim X$ since the corresponding maps φ_m are closed embeddings.

We defined ample line bundles as bundles which become very ample after passing to a high enough multiple. In particular, we can have line bundles which are not globally generated, but whose multiples are. They deserve a name:

Definition 3.29. A line bundle $\mathcal{L}$ on a complete variety X is *semiample* if there is a positive integer m such that $\mathcal{L}^{\otimes m}$ is globally generated. Similarly for semiample Cartier divisors.

Then we have the following special case of the Iitaka fibration.

Theorem 3.30 (Semiample fibration)**.** *Let X be a normal projective variety and let $\mathcal{L}$ be a semiample line bundle on X. Then there is a morphism with connected fibres*

$$f \colon X \to Y$$

such that Y is normal and for every sufficiently divisible m we have $f = \varphi_m$ and $Y = X_m$ (up to isomorphism). If F is a fibre of f over a general closed point, then $\kappa(F, \mathcal{L}|_F) = 0$.

We first make some preparation. Recall that we attached a sheaf $\underline{\mathrm{Proj}}$ to a quasi-coherent graded sheaf of algebras on a variety X. Similarly, if we have a quasi-coherent sheaf of algebras $\mathcal{A}$ on X, we can construct the associated sheaf $\underline{\mathrm{Spec}}\,\mathcal{A}$ together with the structure morphism to X as follows: take an open affine cover $\{U\}$ of X. For every such U we have the morphism $\mathrm{Spec}\,\mathcal{A}(U) \to U$, and these glue to give the desired object.

Theorem 3.31 (Stein factorisation)**.** *A projective morphism $f\colon X \to Y$ can be factorised as*

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[dr, "f"'] & Z \arrow[d, "h"] \\ & Y \end{tikzcd}$$

where g is a projective morphism with connected fibres and h is a finite morphism.

Proof. Since f is projective, the sheaf $f_*\mathcal{O}_X$ is coherent on Y. Define $Z := \underline{\mathrm{Spec}}\, f_*\mathcal{O}_X$. Then the structure morphism $h\colon Z \to Y$ is finite, and we have a factorisation as above. It remains to check that $\mathcal{O}_Z = g_*\mathcal{O}_X$, and it suffices to check it on an affine open cover. Let U be an affine subset of Y, and let $V = h^{-1}(U)$; this is again affine since h is finite. But then $g_*\mathcal{O}(X)(V) = \mathcal{O}_X(g^{-1}(V)) = \mathcal{O}_X(f^{-1}(U)) = \mathcal{O}_Z(V)$ by definition. $\square$

Proof of Theorem 3.30. Let m be an integer such that $\mathcal{L}^{\otimes m}$ is globally generated. Then $\mathcal{L}^{\otimes km}$ is also globally generated for every positive integer k, and note that we have the map

$$S^k H^0(X, \mathcal{L}^{\otimes m}) \to H^0(X, \mathcal{L}^{\otimes km}).$$

Let $s_1, \dots, s_\ell$ be a basis of $H^0(X, \mathcal{L}^{\otimes m})$. Monomials of degree k in these sections give a basis of $S^k H^0(X, \mathcal{L}^{\otimes m})$ (which have no common zeroes since s_i do not), and then we can in the usual way construct a morphism

$$\varphi_m^k = \varphi_{S^k H^0(X, \mathcal{L}^{\otimes m})}\colon X \to \mathbb{P}^N$$

associated to this basis. This is called a *Veronese embedding.* One can easily check that then the image of φ_m^k is isomorphic to X_m: indeed, we have a factorisation

$$\begin{tikzcd} X \arrow[r, "\varphi_m"] \arrow[dr, "\varphi_m^k"'] & X_m \arrow[d, "\theta"] \\ & \mathbb{P}^N \end{tikzcd}$$

and the map θ is given by the very ample line bundle $\mathcal{O}_{X_m}(k)$. Hence, we can think of the maps φ_m and φ_m^k as the same thing.

On the other hand, we can pick generating sets of $S^k H^0(X, \mathcal{L}^{\otimes m})$ and of $H^0(X, \mathcal{L}^{\otimes km})$ such that the former is a subset of the latter. In particular, by construction of these maps, this gives the projection

$$\pi_k \colon X_{km} \to X_m$$

which completes the diagram

$$\begin{tikzcd} X \arrow[r, "\varphi_{km}"] \arrow[dr, "\varphi_m"'] & X_{km} \arrow[d, "\pi_k"] \\ & X_m \end{tikzcd}$$

Further, on every X_m we have a very ample divisor A_m such that $\mathcal{L}^{\otimes m} = \varphi_m^* A_m$.

We claim that the above diagram is precisely the Stein factorisation of φ_m when $k \gg 0$, i.e. that φ_{km} is a morphism with connected fibres and that π_k is finite. The proof will show that, in particular, φ_{km} is independent of k for $k \gg 0$.

To prove this, let $X \overset{\psi}{\to} Z \overset{\mu}{\to} X_m$ be the Stein factorisation on φ_m. Since μ is finite, the line bundle $B = \mu^* A_m$ is ample by Lemma 3.11, and so $B^{\otimes k}$ is very ample for $k \gg 0$. Since $\psi^* B^{\otimes k} = \varphi_m^* A_m^{\otimes k} = \mathcal{L}^{\otimes km}$ and ψ has connected fibres, we have by the projection formula

$$H^0(X, \mathcal{L}^{\otimes km}) = H^0(Z, B^{\otimes k}).$$

But that means precisely that the global sections of $B^{\otimes k}$ give an isomorphism from Z to X_{km}.

Finally, if F is a fibre over a general closed point $p \in Y$, then

$$H^0(F, \mathcal{L}^{\otimes km}|_F) = H^0(p, f_* \mathcal{L}^{\otimes km} \otimes \mathbb{C}(p)) = \mathbb{C}$$

since $f_* \mathcal{L}^{\otimes km}$ is a line bundle on p, and the last claim follows. $\square$

One can prove the following strengthening of the previous result to the case of not necessarily semiample line bundles. I do not give a proof of it, since the main ideas are already contained in the proof of Theorem 3.30.

Theorem 3.32 (Iitaka fibration). *Let X be a normal projective variety and let $\mathcal{L}$ be a line bundle on X such that $\kappa(X, \mathcal{L}) > 0$. Let φ_k denote the associated rational maps for $k \in \mathbb{N}$. Then there exist varieties $\tilde{X}$ and X_∞ and a morphism with connected fibres $\varphi_\infty \colon \tilde{X} \to X_\infty$ such that* $\dim X_\infty = \kappa(X, \mathcal{L})$ *and for every k we have the commutative diagram*

$$\begin{tikzcd} \tilde{X} \arrow[r, "\varphi_\infty"] \arrow[d, "f_\infty"'] & X_\infty \arrow[d, dashed, "\pi_k"] \\ X \arrow[r, dashed, "\varphi_k"'] & X_k \end{tikzcd}$$

where f_∞ is a birational morphism and π_k is a birational map. Further, if F is a fibre of φ_∞ over a general closed point of X_∞, then

$$\kappa(F,(f_\infty^*\mathcal{L})|_F)=0.$$

This result is extremely useful in birational geometry. In order to understand what it says, assume that we are again in the situation where $\mathcal{L}$ is semiample. Assume further that $\kappa(X,\mathcal{L})<\dim X$. Then $X_\infty=X_k$, and we have $\dim X_\infty<\dim X$. So in order to study geometry of X, we can study the geometry of a lower-dimensional variety X_k, and of a general fibre F, for which we know that the Iitaka dimension is zero. This is good for inductive purposes.

Iitaka fibration (that is, the map φ_∞ in the theorem) satisfies the following universal property: if $\lambda\colon X\dashrightarrow W$ is a rational map of normal projective varieties such that $k(W)$ is algebraically closed in $k(X)$ (cf. Theorem 2.21), and if $\mathcal{L}$ is a line bundle on X with $\kappa(F,\mathcal{L}|_F)=0$ for the generic fibre F of λ, then λ factors through the Iitaka fibration of $\mathcal{L}$.

If X is a projective *scheme* of dimension n and if E is any divisor on X, then there is a constant $C>0$ such that

$$h^0(X,\mathcal{O}_X(mE))\leq Cm^n$$

for all m: indeed, fix an ample divisor A on X. Then $aA-E$ is effective for some $a\gg 0$, and consequently $h^0(X,\mathcal{O}_X(mE))\leq h^0(X,\mathcal{O}_X(maA))$. Now the result follows from the asymptotic Riemann-Roch for ample divisors.

One of the basic and important corollaries of the Iitaka fibration is the following generalisation of the result above, the proof of which I omit.

Corollary 3.33. *Let X be a normal projective variety and let $\mathcal{L}$ be a line bundle on X. Set $\kappa=\kappa(X,\mathcal{L})$. Then there are positive real numbers α and β such that for all sufficiently divisible positive integers m we have*

$$\alpha m^\kappa\leq h^0(X,\mathcal{L}^{\otimes m})\leq \beta m^\kappa.$$

We now apply the Iitaka fibration to study multiplication maps

$$H^0(X,\mathcal{L}^{\otimes a})\otimes H^0(X,\mathcal{L}^{\otimes b})\to H^0(X,\mathcal{L}^{\otimes a+b}),$$

for a globally generated line bundle $\mathcal{L}$ and for positive integers a and b. We claim that there exists a positive integer n_0 such that these maps are surjective for a,b divisible by n_0.

To see this, let $\varphi\colon X\to Y$ be the associated Iitaka fibration to $\mathcal{L}$, and let A be an ample line bundle on Y such that $\mathcal{L}^{\otimes n_0}=\varphi^*A$ for some positive integer n_0. Then by the projection formula we have

$$H^0(X,\mathcal{L}^{\otimes an_0})=H^0(Y,A^{\otimes a})$$

for all positive integers a. Therefore, we can assume that $\mathcal{L}$ is an ample line bundle to start with, and by passing to a multiple, we can assume it is very ample. But then again we have the associated embedding $\varphi_{\mathcal{L}}\colon X \to \mathbb{P}^N$ such that $\mathcal{L} = \varphi_{\mathcal{L}}^* \mathcal{O}_{\mathbb{P}^N}(1)$. If $\mathcal{I}$ is the homogeneous ideal of X in $\mathbb{P}^N$, then for $m \gg 0$, the space $H^0(X, \mathcal{O}_X(m))$ is isomorphic to the $\mathbb{C}$-vector space of monomials in $N+1$ indeterminates of degree m modulo $\mathcal{I}$, and the claim is obvious.

We use this to study one of the most important objects on a projective variety. If X is a normal projective variety and $\mathcal{L}$ a line bundle on X, the *section ring associated to* $\mathcal{L}$ is

$$R(X, \mathcal{L}) = \bigoplus_{n \in \mathbb{N}} H^0(X, \mathcal{L}^{\otimes n}).$$

Theorem 3.34. *If $\mathcal{L}$ is a semiample line bundle on a normal projective variety X, then the section ring $R(X, \mathcal{L})$ is finitely generated as a $\mathbb{C}$-algebra.*

For instance, if X is a projective variety $\operatorname{Proj} S$, where $S = \bigoplus_{n \in \mathbb{N}} S_n$ is a graded ring, and if $\mathcal{L} = \mathcal{O}_X(1)$ is a very ample line bundle on X, then it is easy to see that $S_n \simeq H^0(X, \mathcal{L}^{\otimes n})$ for every n, and therefore

$$X \simeq \operatorname{Proj} R(X, \mathcal{L}).$$

Proof. Consider the Veronese subring of $R(X, \mathcal{L})$ given by

$$R(X, \mathcal{L})^{(n_0)} = \bigoplus_{n \in n_0 \mathbb{N}} H^0(X, \mathcal{L}^{\otimes n}).$$

Then this ring is finitely generated by the claim above. But $R(X, \mathcal{L})$ is an integral extension of the ring $R(X, \mathcal{L})^{(n_0)}$ (exercise!), and hence $R(X, \mathcal{L})$ is finitely generated by Noether's theorem on the finite generation of integral closure. $\square$

The following result was one of the main outstanding conjectures in geometry until very recently, and it has crucial implications on the geometry of a smooth projective variety.

Theorem 3.35. *Let X be a smooth projective variety. Then its* canonical ring

$$R(X, \omega_X) = \bigoplus_{n \in \mathbb{N}} H^0(X, \omega_X^{\otimes n})$$

is finitely generated.

We will later see a quick proof of this theorem on surfaces.

3.5 Big line bundles

In this section all varieties are normal.

Now we introduce big divisors and line bundles. We will see that *nef and big* is a right generalisation of the concept of ampleness: ampleness does not behave well under pullbacks by birational maps, whereas nef and big divisors stay nef and big when pulled back by a birational map. This makes them particularly useful in birational geometry.

Big divisors have some of the nice features of ample divisors – their global sections grow maximally (asymptotically) like in the case of ample divisors, and their Iitaka fibrations are birational maps. This is precisely the definition of bigness.

Definition 3.36. Let X be a projective variety and let $\mathcal{L}$ be a line bundle on X. Then $\mathcal{L}$ is *big* if $\kappa(X, \mathcal{L}) = \dim X$. Similarly for Cartier divisors on X.

Combining this with Corollary 3.33, we have the following straightforward consequence of the definition.

Lemma 3.37. *Let X be a projective variety of dimension n. A line bundle $\mathcal{L}$ on X is big iff there are constants $C_1, C_2 > 0$ such that*

$$C_1 m^n \leq h^0(X, \mathcal{L}^{\otimes m}) \leq C_2 m^n$$

for all sufficiently divisible m.

In view of the Iitaka fibration theorem, when X is normal this is equivalent to ask that the map $\varphi_m \colon X \dashrightarrow \mathbb{P}^{N_m}$ associated to $\mathcal{L}^{\otimes m}$ is birational onto its image for some $m > 0$.

Definition 3.38. A smooth projective variety X is of *general type* if its canonical bundle ω_X is big.

The following Kodaira's lemma, also known as *Kodaira's trick*, is the basic tool in studying big divisors.

Theorem 3.39. *Let X be a projective variety, let D be a big Cartier divisor on X, and let F be an arbitrary effective Cartier divisor on X. Then*

$$H^0(X, \mathcal{O}_X(mD - F)) \neq 0$$

for all sufficiently divisible m. In other words, for every such m there is an effective divisor E_m such that $mD \sim F + E_m$.

Proof. Denote $n = \dim X$. Consider the exact sequence

$$0 \to \mathcal{O}_X(mD - F) \to \mathcal{O}_X(mD) \to \mathcal{O}_F(mD) \to 0.$$

The first few terms of the long cohomology sequence associated to this sequence are

$$0 \to H^0(X, \mathcal{O}_X(mD - F)) \to H^0(X, \mathcal{O}_X(mD)) \to H^0(F, \mathcal{O}_F(mD)).$$

It is enough to prove that this last map is not an injection.

Since D is big, there is a constant $C > 0$ such that $h^0(X, \mathcal{O}_X(mD)) \geq Cm^n$ for sufficiently divisible m. On the other hand, F is a scheme of dimension $n-1$, so we have $h^0(F, \mathcal{O}_F(mD)) = O(m^{n-1})$. Thus

$$h^0(X, \mathcal{O}_X(mD)) > h^0(F, \mathcal{O}_F(mD))$$

for sufficiently divisible m, and we are done. □

Kodaira's Lemma has several important consequences.

Corollary 3.40. *Let X be a projective variety and let D be a Cartier divisor on X. Then the following are equivalent:*

(1) D is big,

(2) for any ample divisor A on X, there exists a positive integer m and an effective divisor N on X such that $mD \sim A + N$,

(3) for some ample divisor A on X, there exists a positive integer m and an effective divisor N on X such that $mD \sim A + N$,

(4) there exists an ample divisor A, a positive integer m and an effective divisor N such that $mD \equiv A + N$.

Proof. We first show (1) ⇒ (2). Take $r \gg 0$ so that $rA \sim H_r$ and $(r+1)A \sim H_{r+1}$ are both effective. By Kodaira's lemma, there is a positive integer m and an effective divisor N_0 with

$$mD \sim H_{r+1} + N_0 \sim A + H_r + N_0.$$

Taking $N = H_r + N_0$ gives (2).

The implications (2) ⇒ (3) ⇒ (4) are trivial.

Finally, we prove (4) ⇒ (1). If $mD \equiv A + N$, then $mD - N$ is numerically equivalent to an ample divisor, and hence it is ample. So after possibly passing to an even larger multiple of D we can assume that $m'D \sim H + N'$, where H is very ample and $N' \geq 0$. But then $\kappa(X, D) = \kappa(X, m'D) \geq \kappa(X, H) = \dim X$, so D is big. □

Part (4) of Corollary 3.40 and the fact that ampleness is a numerical property imply the following.

Corollary 3.41. *The bigness of a divisor D depends only on its numerical equivalence class.*

Unlike in the case of ampleness, restrictions of big divisors to subvarieties are not necessarily big. However, after we shrink the variety, the same conclusion holds.

Corollary 3.42. *Let $\mathcal{L}$ be a big line bundle on a projective variety X. Then there is a proper Zariski closed subset $V \subseteq X$ having the property that if Y is a subvariety of X not contained in V, then $\mathcal{L}|_Y$ is a big line bundle on Y.*

Proof. Let D be a Cartier divisor such that $\mathcal{L} = \mathcal{O}_X(D)$. By Corollary 3.40 we can write $mD \sim H + N$, where $N \geq 0$ and H is very ample. Set $V = \operatorname{Supp} N$. If $Y \nsubseteq V$, then the restriction $mD|_Y$ is again the sum of a very ample and an effective divisor, and hence is big. □

Now we naturally extend the definition of bigness to $\mathbb{R}$-divisors.

Definition 3.43. An $\mathbb{R}$-divisor D is *big* if it can be written in the form

$$D = \sum d_i D_i,$$

where each D_i is a big integral divisor and d_i is a positive real number.

The formal properties of big divisors extend easily to this new setting – exercise.

Theorem 3.44. *Let D and D' be $\mathbb{R}$-divisors on a projective variety X.*

(1) If $D \equiv D'$, then D is big iff D' is big.

(2) D is big iff $D \equiv A + N$, where A is an ample and N is an effective $\mathbb{R}$-divisor.

Therefore, it makes sense to talk about big classes in $N^1(X)_{\mathbb{R}}$. Big classes form a cone, which we denote $\operatorname{Big}(X) \subseteq N^1(X)_{\mathbb{R}}$. The following result says that $\operatorname{Big}(X)$ is an open cone; the proof follows easily from Theorem 3.44(2) and Lemma 3.21.

Corollary 3.45. *Let X be a projective variety, let D be a big $\mathbb{R}$-divisor on X, and let $E_1, \ldots, E_m$ be arbitrary $\mathbb{R}$-divisors on X. Then $D + \varepsilon_1 E_1 + \cdots + \varepsilon_m E_m$ is big for all real numbers $0 < |\varepsilon_i| \ll 1$.*

The *pseudoeffective cone* $\overline{\operatorname{Eff}}(X) \subseteq N^1(X)_{\mathbb{R}}$ is the closure of the convex cone spanned by the classes of all effective $\mathbb{R}$-divisors. We say that an $\mathbb{R}$-divisor D on X is pseudoeffective if its class lies in $\overline{\operatorname{Eff}}(X)$. Then we have:

Theorem 3.46. *Let X be a projective variety. Then we have*

$$\overline{\mathrm{Big}(X)} = \overline{\mathrm{Eff}}(X).$$

Proof. The pseudoeffective cone is closed by definition and contains $\mathrm{Big}(X)$, hence $\overline{\mathrm{Big}(X)} \subseteq \overline{\mathrm{Eff}}(X)$.

Fix $\eta \in \overline{\mathrm{Eff}}(X)$ and an integral ample class $\alpha \in N^1(X)_{\mathbb{R}}$. Then there are classes of effective $\mathbb{R}$-divisors η_k such that $\eta = \lim_{k\to\infty} \eta_k$, and therefore obviously

$$\eta = \lim_{k\to\infty} (\eta_k + \tfrac{1}{k}\alpha).$$

But each of the classes in the parentheses is big by Theorem 3.44, so η is a limit of big classes. □

3.5.1 Nef and big

If we have a projective birational morphism $f\colon X \to Y$ between normal projective varieties, and an ample line bundle $\mathcal{L}$ on Y, then $f^*\mathcal{L}$ is nef and big on X: nefness follows from Remark 3.25, and bigness follows from the equality

$$H^0(X, f^*\mathcal{L}^{\otimes m}) \simeq H^0(Y, \mathcal{L}^{\otimes m})$$

(consequence of the projection formula, since f is a morphism with connected fibres). Similarly, if $\mathcal{L}$ is only nef and big, then its pullback is again nef and big. This shows that nef and big divisors behave well under birational modifications.

The following gives a numerical criterion for a nef divisor to be big.

Theorem 3.47. *Let D be a nef divisor on a projective variety X of dimension n. Then D is big iff $D^n > 0$.*

Proof. We know from the asymptotic Riemann-Roch for nef divisors that the leading coefficient of $h^0(X, mD)$ is $D^n/n!$, and the result follows from the definition of bigness. □

We can make the characterisation of big divisors in terms of ample and effective divisors more precise when the divisor is additionally nef.

Lemma 3.48. *Let D be a divisor on a projective variety X. Then D is nef and big iff there is an effective divisor N such that $D - \frac{1}{k}N$ is ample for all $k \gg 0$.*

Proof. Assume that D is big and nef. Then there exist a positive integer m, an effective divisor N, and an ample divisor A such that $mD \equiv A + N$. Thus for $k > m$:

$$kD \equiv ((k-m)D + A) + N,$$

and the term in parentheses, being the sum of a nef and an ample divisor, is ample.

The converse follows straight away from Kleiman's criterion of nefness (nef classes are limits of classes of ample divisors). □

Finally, we state a very useful criterion for when a nef and big divisor has a finitely generated section ring.

Theorem 3.49. *Let X be a normal projective variety and let D be a nef and big divisor on X. Then $R(X, D)$ is finitely generated iff D is semiample.*

Chapter 4

Vanishing theorems

4.1 GAGA principle

I start a general discussion of vanishing (and injectivity) results in birational geometry by a famous and extremely important correspondence between algebraic varieties and complex analytic spaces. This is usually referred to as the GAGA principle, according to the celebrated paper *Géométrie Algébrique et Géométrie Analytique* of J.-P. Serre from 1956. We will see that this correspondence actually blurs the difference between these two seemingly different concepts, and it allows to attack many problems either algebraically or analytically. At the end we will see examples of algebraic results that still do not have an algebraic proof, or that were first proved by analytic methods. These methods are sometimes called *transcendental methods*.

We start with the definition of a *complex analytic space*: we will see that it mimics the definition of an algebraic variety.

Definition 4.1. Let $\Delta = \{z = (z_1, \ldots, z_n) \in \mathbb{C}^n \mid |z_i| < 1 \text{ for } i = 1, \ldots, n\} \subseteq \mathbb{C}^n$ be the standard polydisc with the sheaf $\mathcal{O}_\Delta$ of germs of holomorphic functions on Δ. A topological space $\mathcal{X}$ together with a sheaf of rings $\mathcal{O}_\mathcal{X}$ is a *complex analytic space* if it can be covered by open sets U_i, such that each $(U_i, \mathcal{O}_{U_i})$ is isomorphic, as a ringed space, to the following data: there are holomorphic functions $f_1, \ldots, f_q \in \Gamma(\Delta, \mathcal{O}_\Delta)$ such that U_i is isomorphic to the set of common zeroes of f_i (this is closed in the Euclidean topology) and $\mathcal{O}_{U_i} \simeq \mathcal{O}_\Delta/(f_1, \ldots, f_q)$.

If we have a complex scheme X of finite type, we can construct an *associated complex analytic space* X^{an} as follows. We cover X by open affine sets $Y_i = \operatorname{Spec} A_i$, where each A_i is a finitely generated $\mathbb{C}$-algebra. In other words, $A_i \simeq \mathbb{C}[X_1, \ldots, X_n]/(f_1, \ldots, f_q)$ for some polynomials $f_i \in \mathbb{C}[X_1, \ldots, X_n]$. These are obviously holomorphic functions on $\mathbb{C}^n$, and we can construct the associated analytic spaces Y_i^{an}. Since X is obtained by gluing Y_i, we can glue Y_i^{an} to the analytic space

X^{an} by using the same data. This is the space we need. It is obvious from the construction that it is functorial.

Note that the Euclidean topology is finer than the Zariski topology, so if U is an open subset of a variety (in the Zariski topology), then U_h is an open subset of X_h (in the Euclidean topology).

Similarly, if we have a coherent sheaf $\mathcal{F}$ on a variety X, we can construct its analytification as follows. One can easily check(!) that we can cover X by open subsets U_i such that there are exact sequences

$$\mathcal{O}_{U_i}^{\oplus n} \xrightarrow{\varphi} \mathcal{O}_{U_i}^{\oplus m} \to \mathcal{F}|_{U_i} \to 0,$$

and φ is just a matrix of local sections of $\mathcal{O}_{U_i}$ (it is a linear map). Then these give a map of local sections of $\mathcal{O}_{U_i}^{an}$, and we define $\mathcal{F}^{an}$ to be locally the cokernel of the corresponding map.

It is obvious that there is a continuous map $\theta \colon X^{an} \to X$ of underlying topological spaces which is an inclusion: it sends points of X^{an} bijectively to closed points of X. This is also a map of ringed spaces: there is a natural map $\theta^{-1}\mathcal{O}_X \to \mathcal{O}_{X^{an}}$, and we have $\theta^*\mathcal{O}_X \simeq \mathcal{O}_{X^{an}}$.

Some basic facts to know are as follows:

(1) X is separated iff X^{an} is Hausdorff in the usual topology,

(2) X is connected in the Zariski topology iff X^{an} is connected in the usual topology,

(3) X is smooth iff X^{an} is complex manifold,

(4) if X is projective, then X^{an} is compact.

We can define cohomology of $\mathcal{F}^{an}$ on X^{an} similarly as for X, via injective resolutions. When $\mathcal{F}$ is a constant sheaf of coefficients (i.e. when $\mathcal{F}$ is one of the constant sheaves $\mathbb{Z}, \mathbb{R}, \mathbb{Q}, \mathbb{C}$), this coincides with the singular cohomology.

For the map θ as above, the isomorphism $\theta^*\mathcal{F} \simeq \mathcal{F}^{an}$ for a coherent sheaf $\mathcal{F}$ on X gives natural maps of cohomology groups

$$\alpha_i \colon H^i(X, \mathcal{F}) \to H^i(X^{an}, \mathcal{F}^{an}). \tag{4.1}$$

The basic result which is a bridge between algebraic and analytic categories, at least in the projective case, is the following Serre's GAGA principle.

Theorem 4.2. *Let X be a projective variety over $\mathbb{C}$. Then the analytification functor is an equivalence between the category of coherent sheaves on X and the category of coherent analytic sheaves on X^{an}. Furthermore, for every coherent sheaf $\mathcal{F}$ on X, the natural maps α_i in* (4.1) *are isomorphisms.*

This, together with some additional work, proves the following:

(1) If $\mathcal{X}$ is a complex analytic subspace of $\mathbb{P}^n_{\mathbb{C}}$, then there exists a subscheme $X \subseteq \mathbb{P}^n$ such that $X^{an} = \mathcal{X}$.

(2) Let X and Y be projective complex varieties and let $\varphi\colon X^{an} \to Y^{an}$ be a morphism of analytic spaces, then there is a unique morphism $f\colon X \to Y$ such that $f^{an} = \varphi$.

(3) If X is a projective variety and if $\mathfrak{F}$ is a coherent analytic sheaf on X^{an}, then there is a coherent sheaf $\mathcal{F}$ on X such that $\mathcal{F}^{an} = \mathfrak{F}$.

(4) If X is a projective variety and if $\mathcal{E}$ and $\mathcal{F}$ are two coherent sheaves on X such that $\mathcal{E}^{an} \simeq \mathcal{F}^{an}$, then $\mathcal{E} \simeq \mathcal{F}$.

4.1.1 Exponential sequence

To illustrate the use of complex methods in Algebraic Geometry, an easy example is the exponential sequence, which relates, among other things, cohomology and the structure of the group $\mathrm{Pic}(X)$.

Let $f\colon \mathbb{C} \to \mathbb{C}^*$ be the exponential map $f(z) = e^{2i\pi z}$, where on the left we have additive, and on the right multiplicative structure; this is a group homomorphism. Then we have the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{C} \xrightarrow{f} \mathbb{C}^* \to 0,$$

and by considering holomorphic functions with values in this sequence, we get the exact sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_{X^{an}} \xrightarrow{f} \mathcal{O}^*_{X^{an}} \to 0.$$

Now we consider the long exact cohomology sequence associated to this short exact sequence. Since global holomorphic functions are constants, on the H^0-level we just recover the exact sequence above. Starting from H^1 the long exact sequence becomes more interesting:

$$0 \to H^1(X^{an}, \mathbb{Z}) \to H^1(X^{an}, \mathcal{O}_{X^{an}}) \to H^1(X^{an}, \mathcal{O}^*_{X^{an}}) \to H^2(X^{an}, \mathbb{Z}).$$

Now, one can show that $H^1(X_h, \mathcal{O}^*_{X^{an}}) \simeq \mathrm{Pic}(X^{an})$ (this can be done using the Čech cohomology), and this is isomorphic to $\mathrm{Pic}(X)$ by the GAGA principle. Then we get the map

$$\mathrm{Pic}(X) \xrightarrow{c_1} H^2(X^{an}, \mathbb{Z})$$

(the Chern class map) with the kernel $H^1(X^{an}, \mathcal{O}_{X^{an}})/H^1(X^{an}, \mathbb{Z})$, which can be identified with the divisors which are numerically equivalent to zero (very roughly). This is a rough sketch of a proof that $N^1(X)$ embeds, modulo torsion, in $H^2(X, \mathbb{Z})$.

4.1.2 Kähler manifolds

An important class of complex manifolds is that of Kähler manifolds. We start with a complex manifold $\mathcal{X}$ with a hermitian metric h which in local holomorphic coordinates $z_1, \dots, z_n$ can be written as

$$h = \sum h_{\alpha\beta} dz_\alpha \otimes d\bar{z}_\beta;$$

here $(h_{\alpha\beta})$ is a (pointwise) positive definite Hermitian matrix of $\mathbb{C}$-valued $\mathcal{C}^\infty$ functions on $\mathcal{X}$. Then we have the associated $(1,1)$-form, which is a 2-form ω that can locally be written as

$$\omega = \frac{i}{2} \sum h_{\alpha\beta} dz_\alpha \wedge d\bar{z}_\beta.$$

We say that $\mathcal{X}$ is a *Kähler manifold* if ω is closed, i.e. if $d\omega = 0$.

Every projective manifold is Kähler: the Fubini-Study metric gives rise to a Kähler form on it.

Suppose now that we have a holomorphic line bundle $\mathcal{L} \to \mathcal{X}$ on a Kähler manifold $\mathcal{X}$ with a hermitian metric h. Write $|\cdot|_h$ for the corresponding length function on the fibres of $\mathcal{L}$. The hermitian line bundle $(\mathcal{L}, h)$ determines a curvature form $\Theta(\mathcal{L}, h) \in \mathcal{C}^\infty(X, \bigwedge^{1,1} T^*\mathcal{X}_\mathbb{R})$, which is locally given by

$$\Theta(\mathcal{L}, h) = -\partial\bar{\partial} \log |s|_h^2$$

(this does not depend on the choice of a local non-vanishing section s of $\mathcal{L}$).

Definition 4.3. A holomorphic line bundle $\mathcal{L}$ on a Kähler manifold $\mathcal{X}$ is *positive* (in the sense of Kodaira) if it carries a Hermitian metric h such that $\frac{i}{2\pi}\Theta(\mathcal{L}, h)$ is a Kähler form.

Now a fundamental result, *Kodaira's embedding theorem*, shows that this is the right notion to describe ampleness analytically: if a line bundle is positive, then it is algebraic and ample in the usual sense.

Theorem 4.4. *Let $\mathcal{X}$ be a compact Kähler manifold, and let $\mathcal{L}$ be a holomorphic line bundle on $\mathcal{X}$. Then $\mathcal{L}$ is positive iff there is a holomorphic embedding*

$$\varphi \colon \mathcal{X} \to \mathbb{P}^N$$

into some projective space such that $\varphi^*\mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{L}^{\otimes m}$ for some $m > 0$.

4.2 Lefschetz hyperplane section theorem

The Lefschetz hyperplane theorem compares the topology of a smooth projective variety X with the topology of (the subspace of X determined by the support of) a smooth effective ample divisor D on X.

We first state a result on the structure of the (co)homology of an affine variety; the proof uses some beautiful Morse theory and I omit it.

Theorem 4.5. *Let $V \subseteq \mathbb{C}^r$ be a closed connected complex submanifold of complex dimension n. Then*

$$H^i(V, \mathbb{Z}) = 0 \quad \textit{and} \quad H_i(V, \mathbb{Z}) = 0 \quad \textit{for } i > n.$$

If we have a closed submanifold D of a compact manifold X, we can form the exact sequence of singular complexes

$$0 \to \mathcal{C}_\bullet(D) \to \mathcal{C}_\bullet(X) \to \mathcal{C}_\bullet(X)/\mathcal{C}_\bullet(D) \to 0,$$

and to it we can attach a long cohomology sequence

$$\cdots \to H^i(X, D; \mathbb{Z}) \to H^i(X, \mathbb{Z}) \to H^i(D, \mathbb{Z}) \to \cdots \tag{4.2}$$

Then we have the Lefschetz hyperplane theorem.

Theorem 4.6. *Let X be a smooth complex projective variety of dimension n, and let D be a smooth effective ample divisor on X. Then the restriction*

$$H^i(X, \mathbb{Z}) \to H^i(D, \mathbb{Z})$$

is an isomorphism for $i \leq n-2$, and an injection when $i = n-1$.

Proof. Since mD is very ample for some $m \gg 0$, it gives an embedding $X \subseteq \mathbb{P}^r$ such that, if $H \subseteq \mathbb{P}^r$ is a hyperplane, then $X \cap H = mD$. In particular, $X \backslash D = X \backslash \operatorname{Supp}(mD)$ is affine. Therefore $H_j(X \backslash D, \mathbb{Z}) = 0$ for $j \geq n+1$ by Theorem 4.5, and the theorem follows from (4.2) by the Lefschetz duality $H^i(X, D; \mathbb{Z}) \simeq H_{2n-i}(X \backslash D, \mathbb{Z})$. □

An important corollary is:

Corollary 4.7. *Let X be a smooth complex projective variety of dimension n, and let D be a smooth effective ample divisor. Let*

$$r_{p,q} \colon H^q(X, \Omega^p_X) \to H^q(D, \Omega^p_D)$$

be the natural maps given by restriction. Then $r_{p,q}$ is bijective for $p+q \leq n-2$, and injective for $p+q = n-1$.

Proof. The Lefschetz hyperplane theorem gives that the restriction maps

$$r_j\colon H^j(X,\mathbb{C}) \to H^j(D,\mathbb{C})$$

are isomorphisms when $j \le n-2$ and injections when $j = n-1$. The Hodge theorem and the Dolbeaut isomorphisms give functorial decompositions

$$H^j(X,\mathbb{C}) = \bigoplus_{p+q=j} H^q(X,\Omega_X^p), \qquad H^j(D,\mathbb{C}) = \bigoplus_{p+q=j} H^q(D,\Omega_D^p),$$

and we have the splitting $r_j = \bigoplus_{p+q=j} r_{p,q}$. This implies the lemma. $\square$

4.3 Kodaira vanishing: statement and first consequences

One of the important applications of the Lefschetz hyperplane section theorem is the following fundamental result, again due to Kodaira; it is the Kodaira vanishing theorem.

Theorem 4.8. *Let X be a smooth complex projective variety of dimension n, and let A be an ample divisor on X. Then*

$$H^i(X,\mathcal{O}_X(K_X+A)) = 0 \quad \textit{for } i > 0.$$

Equivalently, by Serre duality,

$$H^i(X,\mathcal{O}_X(-A)) = 0 \quad \textit{for } i < n.$$

It is important to stress that the Kodaira vanishing is true over any algebraically closed field of characteristic zero, but fails in positive characteristic.

I will give three proofs of this very important result; one of them will involve the Lefschetz hyperplane section theorem, and I give a proof in a special case at the end of this section. First we see why this result is very important by discussing a couple of immediate consequences.

Definition 4.9. A smooth projective variety X is a *Fano manifold* if $-K_X$ is ample.

Example 4.10. The projective n-space $\mathbb{P}^n$ is a Fano. Furthermore, any degree d hypersurface D in $\mathbb{P}^n$ is Fano when $d \le n$: indeed, by adjunction formula we have

$$K_D = (K_{\mathbb{P}^n} + D)|_D = \mathcal{O}_D(d-n-1).$$

Now, if we have a Fano manifold X, it is immediate from the Kodaira vanishing that
$$H^i(X, \mathcal{O}_X) = 0 \quad \text{for } i > 0.$$
Now the exponential sequence gives us the isomorphism
$$\operatorname{Pic}(X) \simeq H^2(X, \mathbb{Z}).$$
This implies $\operatorname{Pic}(X) \simeq N^1(X)$: in other words, on a Fano manifold, linear and numerical equivalence of divisors coincide. This is far from being true in general.

As another application, we show that on a smooth projective variety the Euler characteristic is a numerical invariant. This is true in general by Hirzebruch-Riemann-Roch, but we will see that it is an immediate consequence of the Kodaira vanishing when the variety is smooth.

Theorem 4.11. *Let X be a smooth complex projective variety and let D and D' be Cartier divisors on X such that $D \equiv D'$. Then $\chi(X, D) = \chi(X, D')$.*

Proof. Fix an ample divisor H on X. Set
$$Q(u, v) = \chi(X, D + u(D' - D) + vH) \quad \text{and} \quad P(v) = Q(0, v);$$
these are polynomials in u and v by Theorem 1.40. Fix $v_0 \gg 0$ such that $D - K_X + v_0H$ is ample. Since $D' - D \equiv 0$, the divisors $D - K_X + u(D' - D) + v_0H$ are also ample for all u. By the Kodaira vanishing we have
$$H^i(X, D + u(D' - D) + v_0H) = 0 \quad \text{for all } u \text{ and } i > 0.$$
Hence
$$Q(u, v_0) = h^0(X, D + u(D' - D) + v_0H) \quad \text{for all } u.$$
I claim that then $Q(u, v_0)$ is bounded as a function in u. This immediately implies the theorem: indeed, a bounded polynomial is constant, hence $Q(u, v_0) = P(v_0)$ for $v_0 \gg 0$, and it is easy to see that this implies $Q(u, v) = P(v)$ for all u and v. Therefore
$$\chi(X, D) = P(0) = Q(1, 0) = \chi(X, D').$$

It remains to show the claim, and we do it by induction on $n = \dim X$. It is true on curves by the Riemann-Roch, and assume it holds in dimension $n - 1$. Pick a large positive integer m such that
$$(D + v_0H - mH) \cdot H^{n-1} < 0. \tag{4.3}$$
This implies
$$H^0(X, D + u(D' - D) + v_0H - mH) = 0 \quad \text{for all } u. \tag{4.4}$$

Indeed, otherwise there would exist an effective divisor $S_u \sim D + u(D' - D) + v_0H - mH$, and then $S_u \cdot H^{n-1} < 0$ by (4.3), which contradicts the (easy direction of) Nakai-Moishezon-Kleiman criterion for ampleness.

Let $Y \in |mH|$ be a smooth divisor. Then we have a short exact sequence

$$\begin{aligned} 0 &\to \mathcal{O}_X(D + u(D' - D) + v_0H - Y) \to \mathcal{O}_X(D + u(D' - D) + v_0H) \\ &\to \mathcal{O}_Y((D + u(D' - D) + v_0H)|_Y) \to 0, \end{aligned}$$

hence from the long exact sequence in cohomology, by (4.4) we have

$$h^0(X, D + u(D' - D) + v_0H) \le h^0(Y, (D + u(D' - D) + v_0H)|_Y).$$

By induction hypothesis, this finishes the proof of the claim and of the theorem. $\square$

Now let us see an easy proof of the Kodaira vanishing in a special case.

Lemma 4.12. *The Kodaira vanishing holds when A is a smooth effective ample divisor on X.*

Proof. By the case $p = 0$ and $q = j$ in Corollary 4.7, we have that the restriction map

$$H^j(X, \mathcal{O}_X) \to H^j(D, \mathcal{O}_A)$$

is an isomorphism when $j \le n - 2$ and an injection when $j = n - 1$. But then the long cohomology sequence associated to the short exact sequence

$$0 \to \mathcal{O}_X(-A) \to \mathcal{O}_X \to \mathcal{O}_A \to 0$$

gives $H^j(X, \mathcal{O}_X(-A)) = 0$ when $j \le n - 1$, which proves the theorem. $\square$

So our goal is to reduce the Kodaira vanishing to this lemma. We start by showing a simple result that suggests that we should construct a suitable finite morphism. First, recall that if we have a Galois extension of fields K/k with the Galois group G, there exists a trace map $\mathrm{Tr}_{K/k}\colon K \to k$ by sending an element $a \in K$ to $\sum_{g \in G} ga \in k$. Then we have:

Lemma 4.13. *Let $f\colon Y \to X$ be a finite surjective Galois morphism of normal projective varieties and let $\mathcal{E}$ be a vector bundle on X. Then the natural homomorphism*

$$H^i(X, \mathcal{E}) \to H^i(Y, f^*\mathcal{E})$$

induced by f is injective for every $i \ge 0$.

Proof. We first show that the trace map $\mathrm{Tr}_{k(Y)/k(X)}$ induces a map $\mathrm{Tr}\colon f_*\mathcal{O}_Y \to \mathcal{O}_X$ which splits the natural injection $i\colon \mathcal{O}_X \to f_*\mathcal{O}_Y$. Indeed, the statement is local, so we may assume that $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, where B is an integral extension of A. Then for $b \in B$ we have $\mathrm{Tr}_{k(Y)/k(X)}(b) \in k(X)$, and $gb \in B$ for every $g \in G$ since they share the same minimal polynomial. Hence $\mathrm{Tr}_{k(Y)/k(X)}(b) \in B$, and therefore $\mathrm{Tr}_{k(Y)/k(X)}(b) \in A$ as A is integrally closed in $k(X)$. Now set $\mathrm{Tr} = \frac{1}{|G|}\mathrm{Tr}_{k(Y)/k(X)}$.

In particular, if $\mathcal{F} = \operatorname{coker} i$, we have $f_*\mathcal{O}_Y = \mathcal{O}_X \oplus \mathcal{F}$, and thus

$$H^i(Y, f^*\mathcal{E}) = H^i(X, f_*f^*E) = H^i(X, \mathcal{E} \otimes f_*\mathcal{O}_Y) = H^i(X, \mathcal{E}) \oplus H^i(X, \mathcal{E} \otimes \mathcal{F})$$

since f is finite, which proves the lemma. $\square$

4.4 Cyclic coverings

In this section we construct a finite map which makes a line bundle effective, assuming that some power of the line bundle is effective. The process is often, rightly, called *taking a root of an effective line bundle.* First an important definition.

Definition 4.14. Let X be a smooth variety of dimension n and let $D = \sum D_i$ be a reduced divisor on X with prime components D_i. The divisor D has *simple normal crossings* if for every point $p \in X$ there is an open neighbourhood U of p and local coordinates $(x_1, \dots, x_n)$ such that D is defined in U by the equation $x_1 \cdots x_r = 0$ for some $r \le n$. In other words, all D_i are smooth and intersect transversally.

The *singularities of* D are the locus in X where the components of D_i intersect.

The following result is difficult to digest when first seen; a good thing to do is to see what it says when D is just a smooth prime divisor.

Proposition 4.15 (m-fold cyclic cover)**.** *Let X be a variety and let L be a line bundle on X. Let m be a positive integer and let $s \in H^0(X, L^m)$ be a non-zero section which defines a divisor $D \subseteq X$. Then there exists a finite map $\mu\colon Y \to X$, where Y is a normal scheme, such that there exists a section $s' \in H^0(Y, \mu^*L)$ with $\mu^*s = (s')^m$. In other words, if D' is a divisor of s', then $\pi^*D = mD'$.*

If additionally X is smooth and if D is a reduced simple normal crossings divisor, then Y is a normal variety which is smooth outside of the singularities of D, the divisor D' maps isomorphically to D outside of the singularities of D, and we have

$$\mu_*\mathcal{O}_Y \simeq \bigoplus_{i=0}^{m-1} L^{-i}. \tag{4.5}$$

If Γ is a divisor on X such that $D+\Gamma$ has simple normal crossings, then $\mu^*(D+\Gamma)$ has simple normal crossings outside of the singularities of D.

Proof. There are several ways to define Y. The easiest description is the following: L is a subbundle of the sheaf $\mathcal{K}(X)$ of rational functions on X, and take a rational function $f \in k(X)$ to which s corresponds. Then let Y be the normalisation of X in the field $k(X)(\sqrt[m]{f})$, and let μ be the corresponding finite map.

Another way to do the same is to do it locally. Suppose that $U = \operatorname{Spec} A$ is an affine open subset of X on which we have a trivialisation $L|_U \simeq \mathcal{O}_U$. Then $L^m|_U \simeq \mathcal{O}_U$, the section $s|_U$ corresponds to $f \in H^0(U, \mathcal{O}_U)$ and by introducing a new variable T, set $\mathcal{A}(U) = A[T]/(T^m - f) = \bigoplus_{i=0}^{m-1} A[(\sqrt[m]{f})^i]$. It is easy to see that these algebras glue to $Z = \underline{\operatorname{Spec}}\,\mathcal{A}$ with the projection $\pi\colon Z \to X$, where $\mathcal{A} = \bigoplus_{i=0}^{m-1} L^{-i}$, and hence $\pi_*\mathcal{O}_Z = \mathcal{A}$. Now Y is the normalisation of Z and $\mu\colon Y \to X$ the induced finite morphism.

Finally, in coordinates, consider the product $X \times \mathbb{A}^1$, and let T be the coordinate on $\mathbb{A}^1$. Then the variety Z is a subvariety of $X \times \mathbb{A}^1$ defined by the equation $T^m - s = 0$.

On each $\pi^{-1}(U)$ consider the section T. Then it is clear that these glue to give an effective Cartier divisor D_Z on Z, and let D' be the pullback of D_Z on Y and s' the corresponding section. We obviously have $mD_Z = \pi^*D$, and D_Z is isomorphic to D via π.

I claim that Z is smooth outside of the singularities of D. This then immediately implies (4.5): indeed, then Y and Z are isomorphic outside of the singularities of D, hence $\mu_*\mathcal{O}_Y$ and $\pi_*\mathcal{O}_Z$ coincide outside of the singularities of D, hence everywhere as X is normal and the singularities of D have codimension ≥ 2.

To show the claim, we use the Jacobian criterion. By replacing X by an open subset of X which avoids the singularities of D, we may assume that $X \subseteq \mathbb{A}^n$ is the zero set of polynomials $f_1, \dots, f_r \in \mathbb{C}[X_1, \dots, X_n]$, and that D is a smooth prime divisor given by a regular function $f = p/q$ on X, where $p, q \in \mathbb{C}[X_1, \dots, X_n]$ and q does not vanish along X. Denote by $J = (a_{ij})$ the Jacobian matrix of X in $\mathbb{A}^n$, where $1 \leq i \leq r$ and $1 \leq j \leq n$. Then $Z \subseteq \mathbb{A}^{n+1}$ is cut out by the polynomials $f_1, \dots, f_r$ and $qT^n - p$ in $\mathbb{C}[X_1, \dots, X_n, T]$, and the corresponding Jacobian is

$$J_Z = (b_{ij}), \quad \text{where } b_{ij} = \begin{cases} \frac{\partial q}{\partial X_j} T^m - \frac{\partial p}{\partial X_j} & \text{if } i = r+1,\ j \leq n, \\ 0 & \text{if } j = n+1,\ i \leq r, \\ nqT^{n-1} & \text{if } i = r+1,\ j = n+1. \end{cases} \tag{4.6}$$

Fix a point $P = (\underline{x}, t) = (x_1, \dots, x_n, t) \in Z$. There are two cases. If $t \neq 0$, then $nq(\underline{x})t^{n-1} \neq 0$ and the corank of J_Z is the corank of J plus 1. Therefore, Z is smooth at P since X is smooth at $\underline{x}$. Now assume that $t = 0$. Since D is cut out of $\mathbb{A}^n$ by $f_1, \dots, f_r, p$, its Jacobian matrix is

$$J_D = (d_{ij}), \quad \text{where } d_{ij} = \frac{\partial p}{\partial X_i} \text{ if } i = r+1. \tag{4.7}$$

Now Z is smooth at P by (4.6) and (4.7) since D is smooth.

Finally, the last assertion of the proposition follows easily from the description of Z in the local coordinates. $\square$

Now we can finally give the first proof of the Kodaira vanishing.

Proof of Theorem 4.8. Let m be a positive integer such that mA is very ample, and let $D \in |mA|$ be a general section. Then D is smooth by Bertini's theorem, and let $\pi\colon Y \to X$ be the m-fold cyclic covering over D. By Proposition 4.15 the variety Y is smooth, and there exists a smooth prime divisor A' on Y such that $A' \sim \pi^*A$. Note that A' is ample since A is ample and π is finite. By Lemma 4.12 we have $H^i(Y, \mathcal{O}_Y(-A')) = 0$ for $i < n$, hence the result follows by Lemma 4.13. $\square$

4.5 Differentials with log poles

In this section we introduce a very useful construction, which generalises the construction of the sheaves of (regular) differentials. Note first that the usual exterior differentiation induces the *de Rham complex* $\Omega_X^\bullet$ on X:

$$0 \to \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \cdots \xrightarrow{d} \Omega_X^n \to 0,$$

which induces the de Rham complex $\Omega_X^\bullet \otimes k(X)$ of meromorphic forms on X.

Definition 4.16. Let X be a smooth variety of dimension n, and let D be a reduced divisor on X with simple normal crossings. The *sheaf* $\Omega_X^1(\log D)$ *of* 1*-forms on* X *with log poles along* D is a subsheaf of $\Omega_X^1 \otimes k(X)$ described locally as follows. Let U be an affine open subset of X and let $x_1, \dots, x_n$ be local coordinates on U such that D is defined in U by $x_1 \cdots x_r$ for some $r \leq n$. Then $\Omega_X^1(\log D)$ is a sheaf generated on U by

$$\frac{dx_1}{x_1}, \dots, \frac{dx_r}{x_r}, dx_{r+1}, \dots, dx_n.$$

For every integer $p \geq 0$, set $\Omega_X^p(\log D) = \bigwedge^p \Omega_X^1(\log D)$.

This definition does not depend on the choice of coordinates: indeed, if h is an invertible local section of $\mathcal{O}(U)$, then

$$\frac{d(hx_i)}{hx_i} = \frac{dh}{h} + \frac{dx_i}{x_i} \in \sum_{j=1}^n \mathcal{O}(U)dx_j + \mathcal{O}(U)\frac{dx_i}{x_i}.$$

It is clear that $\Omega_X^1(\log D)$ is a locally free sheaf of rank n containing Ω_X^1, and it is easy to see that $\Omega_X^0(\log D) = \mathcal{O}_X$ and $\Omega_X^n(\log D) = \mathcal{O}_X(K_X + D)$.

Since $d(\frac{dx_i}{x_i}) = 0$, it follows immediately that the de Rham differential preserves the forms with log poles along D, hence we get the de Rham complex with log poles $\Omega_X^\bullet(\log D)$.

Proposition 4.17. *Let X be a smooth variety of dimension n, let L be a line bundle on X, let m be a positive integer and let $s \in H^0(X, L^m)$ be a section which defines a reduced simple normal crossings divisor D. If $\pi\colon Y \to X$ is the m-cyclic cover corresponding to s, and if D' is the effective divisor on Y such that $\pi^*D = mD'$, then for every integer $p \geq 0$ we have*

$$\pi^*\Omega^p_X(\log D) \simeq \Omega^p_Y(\log D'), \tag{4.8}$$

and in particular, $p = n$ yields

$$\pi^*\mathcal{O}_X(K_X + D) \simeq \mathcal{O}_Y(K_Y + D'). \tag{4.9}$$

Moreover,

$$\pi_*\Omega^p_Y(\log D') \simeq \bigoplus_{i=0}^{m-1} \Omega^p_X(\log D) \otimes L^{-i}. \tag{4.10}$$

Proof. It is enough to prove (4.8) for $p = 1$, since the general case follows by taking exterior powers, and it suffices to prove it locally on Y. Since the singularities of D form a subset of codimension ≥ 2 in X, by shrinking X and by Hartogs principle we may assume that the components of D are disjoint. Let $U = \operatorname{Spec} A$ be an affine open subset in X such that $x_1, \dots, x_n$ are local coordinates of X on U and such that U intersects at most one component of D. If $D \cap U = \emptyset$, then it is clear that $\pi^*\Omega^1_X = \Omega^1_Y$ on $\pi^{-1}(U)$. Otherwise, we may assume that $D|_U$ is defined by x_1. On $\pi^{-1}(U) \simeq \operatorname{Spec} A[T]/(T^m - x_1)$ we have algebraic coordinates $T, x_2, \dots, x_n$ which satisfy

$$\pi^*(dx_i) = dx_i \text{ for } i \geq 2 \text{ and } \pi^*\Big(\frac{dT}{T}\Big) = \frac{d(T^m)}{T^m} = m\frac{dT}{T},$$

hence (4.8) follows. The relation (4.10) follows immediately from (4.8) and (4.5) by the projection formula. □

Corollary 4.18. *With the assumptions from Proposition 4.17, we have*

$$\pi_*\Omega^p_Y \simeq \Omega^p_X \oplus \bigoplus_{i=1}^{m-1} \Omega^p_X(\log D) \otimes L^{-i}. \tag{4.11}$$

Proof. As in the previous proof, we may assume that the components of D are disjoint. Let $U = \operatorname{Spec} A$ be an affine open subset in X such that $x_1, \dots, x_n$ are local coordinates of X on U, such that U intersects at most one component of D, and that $L^{-1}|_U = f\mathcal{O}_U$ for some local section f. If $D \cap U = \emptyset$, then the assertion is clear from (4.10). Otherwise, we may assume that $D|_U$ is defined by x_1. On $\pi^{-1}(U) \simeq \operatorname{Spec} A[T]/(T^m - x_1)$ we have algebraic coordinates $T, x_2, \dots, x_n$, and we have to check when the pullback of the local differential $\frac{dx_1}{x_1} f^i$ to Y gives a regular differential on Y for $i \geq 0$. However, this pullback is, up to an invertible function, equal to

$$\frac{d(T^m)}{T^m}T^i = mT^{i-1}dT.$$

Now it is clear that this differential is regular if and only if $i \geq 1$, which proves the claim. $\square$

We are now ready to give the second proof of the Kodaira vanishing.

Proof of Theorem 4.8. Set $L = \mathcal{O}(A)$. Let m be a positive integer such that mA is basepoint free and that

$$H^j(X, K_X + (m+1)A) = 0 \quad \text{for } j > 0, \tag{4.12}$$

which is possible by the cohomological criterion for ampleness. If $D \in |mA|$ is a general section, then D is smooth by Bertini's theorem, and let $\pi\colon Y \to X$ be the m-fold cyclic covering over D.

The exterior differential $d\colon \mathcal{O}_Y \to \Omega^1_Y$ induces the map $\pi_* d\colon \pi_*\mathcal{O}_Y \to \pi_*\Omega^1_Y$. By (4.10) and (4.11), this is the map

$$\pi_* d\colon \bigoplus_{i=0}^{m-1} L^{-i} \to \Omega^p_X \oplus \bigoplus_{i=1}^{m-1} \Omega^p_X(\log D) \otimes L^{-i}$$

which factors through the maps of the corresponding summands. Denote the map for $i = 1$ by

$$\nabla^1\colon L^{-1} \to \Omega^p_X(\log D) \otimes L^{-1},$$

and this map induces maps on cohomology

$$\delta_j\colon H^j(X, L^{-1}) \to H^j(X, \Omega^p_X(\log D) \otimes L^{-1}).$$

Consider the maps $d_j\colon H^j(Y, \mathcal{O}_Y) \to H^j(Y, \Omega^1_Y)$ induced by d. By the projection formula and since the map π is finite, these maps give

$$\pi_* d_j\colon H^j\Big(X, \bigoplus_{i=0}^{m-1} L^{-i}\Big) \to H^j\Big(X, \Omega^p_X \oplus \bigoplus_{i=1}^{m-1} \Omega^p_X(\log D) \otimes L^{-i}\Big),$$

hence the map δ_j is a direct summand of $\pi_* d_j$. But the maps d_j are zero: indeed, by Hodge theory, $H^j(Y, \mathcal{O}_Y)$ is the vector space of harmonic $(0, j)$-forms, and these are d-closed. Hence the maps δ_j are all zero.

Let $x_1, \dots, x_n$ be a local coordinate system on an open affine subset $U \subseteq X$ such that $D|_U$ is given by x_1. There is the residue map $\mathrm{Res}\colon \Omega^1_X(\log D) \to \mathcal{O}_D$ which sends a local form $\alpha_1 \frac{dx_1}{x_1} + \sum_{i\geq 2} \alpha_i dx_i$ to $\alpha_1|_D$, and consider the composite map

$$\theta\colon L^{-1} \xrightarrow{\nabla^1} \Omega^1_X(\log D) \otimes L^{-1} \xrightarrow{\mathrm{Res}\otimes\mathrm{id}} \mathcal{O}_D \otimes L^{-1}.$$

If f is a local generator of L^{-1} on U and r is a local section of $\mathcal{O}_U$, then $f^m = x_1$, and hence $\frac{dx_1}{x_1} = m\frac{df}{f}$. Therefore,

$$\nabla^1(rf) = fdr + rdf = f\Big(dr + \frac{1}{m} r \frac{dx_1}{x_1}\Big),$$

which implies $\theta(rf) = \frac{1}{m}(rf)|_D$. In other words, the map $m\theta$ is the restriction map, and the induced maps

$$\theta_j \colon H^j(X, L^{-1}) \to H^j(X, \mathcal{O}_D \otimes L^{-1})$$

are zero since they factor through δ_j. From the long exact sequence in cohomology associated to the short exact sequence

$$0 \to L^{-1} \otimes \mathcal{O}_X(-D) \to L^{-1} \to L^{-1} \otimes \mathcal{O}_D \to 0$$

we obtain that the maps

$$H^j(X, L^{-1} \otimes \mathcal{O}_X(-D)) \to H^j(X, L^{-1})$$

are surjective for all j, or equivalently by Serre duality and since $D \sim mA$, that the maps

$$H^j(X, \mathcal{O}_X(K_X + A)) \to H^j(X, \mathcal{O}_X(K_X + (m+1)A))$$

are injective for all j. Now we conclude from (4.12). $\square$

4.6 Kawamata coverings

In this section we will prove an important generalisation of cyclic coverings, which we will then use to prove far-reaching extensions of the Kodaira vanishing.

Theorem 4.19. *Let X be a smooth quasiprojective variety, let $D = \sum_{i=1}^t D_i$ be a simple normal crossing divisor on X, and fix positive integers $m_1, \ldots, m_t$. Then there exists a smooth variety Y and a finite covering $f \colon Y \to X$ such that $f^*D_i = m_i D_i'$ for some smooth divisors D_i' on Y, where $\sum_{i=1}^t D_i'$ has simple normal crossings.*

Proof. For the purpose of induction, we replace the condition that D_i are smooth prime divisors by the condition that D_i is a disjoint union of smooth divisors. Hence we may assume that $m_2 = \cdots = m_t = 1$.

Fix a very ample divisor H such that $m_1 H - D_1$ is basepoint free. For $n = \dim X$, pick general elements $H_1, \ldots, H_{n+1}$ in the linear system $|m_1 H - D_1|$. Then the divisor $D + \sum_{i=1}^{n+1} H_i$ has simple normal crossings. We construct a tower of cyclic coverings

$$Y = Y_{n+1} \xrightarrow{f_{n+1}} \ldots \xrightarrow{f_2} Y_1 \xrightarrow{f_1} X = Y_0$$

as follows. Assume that Y_{i-1} has already been constructed and denote $\pi_i = f_i \circ \cdots \circ f_1$ for $i \geq 1$ and $\pi_0 = \mathrm{id}$. Let f_i be the m_1-fold cyclic covering of Y_{i-1} branched along $\pi_{i-1}^*(H_i + D_1)$. Then Y_i is smooth outside of the singularities of Y_{i-1} and outside of the singularities of the divisor $\pi_{i-1}^*(H_i + D_1)$. In particular, Y_1 is smooth outside of $H_1 \cap D_1$. However, for $i \geq 2$, the cover branched along $\pi_{i-1}^*(H_i + D_1) = \pi_{i-1}^* H_i + m_1(\pi_{i-1}^* D_1)_{\mathrm{red}}$ is the same as the cover branched along $\pi_{i-1}^* H_i$ (exercise!), which has no singularities, hence inductively, the singularities of Y_i lie over $H_1 \cap D_1$.

An alternative way to define Y is as follows: let $s_1, \ldots, s_{n+1}$ be global sections of $\mathcal{O}_X(m_1 H)$ whose corresponding divisors are $H_1 + D_1, \ldots, H_{n+1} + D_1$. Then Y is the normalisation of X in the field $k(X)(\sqrt[m_1]{s_1}, \ldots, \sqrt[m_1]{s_{n+1}})$. Thus, it is clear that the above construction of the tower is commutative: we could start with any divisor $H_i + D_1$. Therefore, Y is smooth away from

$$\bigcap_{i=1}^{n+1} (H_i \cap D_1) = \Big(\bigcap_{i=1}^{n+1} H_i \Big) \cap D_1 = \emptyset,$$

which finishes the proof. □

The following easy corollary is usually called the Bloch-Gieseker coverings.

Corollary 4.20. *Let X be a smooth quasiprojective variety, let $\mathcal{M}$ be a line bundle on X, and fix a positive integer m. Then there exist a smooth variety Y, a finite surjective map $f\colon Y \to X$, and a line bundle $\mathcal{L}$ on Y such that $f^*\mathcal{M} = \mathcal{L}^m$. Further, given a simple normal crossing divisor D on X, we can arrange that its pullback f^*D is a simple normal crossings divisor on Y.*

Proof. Let $\mathcal{N}$ be any very ample line bundle on X. Then there exist a positive integer n such that $\mathcal{A} = \mathcal{M} \otimes \mathcal{N}^n$ is globally generated line bundle. By Bertini's theorem, there exists smooth prime divisors A and B such that $\mathcal{A} \simeq \mathcal{O}_X(A)$ and $\mathcal{N}^n \simeq \mathcal{O}_X(B)$, hence $\mathcal{M} = \mathcal{O}_X(A - B)$. By Theorem 4.19 there exists a finite map $f\colon Y \to X$ such that $f^*A = mA'$ and $f^*B = mB'$ for some smooth divisors A' and B' on Y. Now set $\mathcal{L} = \mathcal{O}_Y(A' - B')$. □

4.7 Esnault-Viehweg-Ambro injectivity theorem

If you analyse the previous two proofs of the Kodaira vanishing, you can see that we did not use the ampleness assumption almost at all. Especially in the second proof, we used mostly that some multiple of the divisor A is basepoint free, i.e. that A is semiample. Furthermore, in most applications of the Kodaira vanishing, one only uses the vanishing of the group H^1, in order to conclude that some restriction map in cohomology is surjective. More precisely, if we have a short exact sequence of coherent sheaves on a variety X,

$$0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0,$$

in order to conclude that the map $H^0(X, \mathcal{F}_2) \to H^0(X, \mathcal{F}_3)$ is surjective, it suffices to have $H^1(X, \mathcal{F}_1) = 0$. If $\mathcal{F}_1$ is a line bundle of the form $\mathcal{O}_X(K_X + A)$, where A is an ample line bundle, then the Kodaira vanishing can be applied. A more elaborate version of this principle will appear later when we prove the finite generation of the canonical ring on surfaces.

However, to prove the desired surjectivity, it is in fact equivalent to show a weaker result than the above vanishing, that the map $H^1(X, \mathcal{F}_1) \to H^1(X, \mathcal{F}_2)$ is injective. Results of this type are called injectivity theorems. The most general result in this direction thus far is the following theorem of Esnault-Viehweg-Ambro.

Theorem 4.21. *Let X be a smooth projective variety and let $\Delta = \sum_{i=1}^r \Delta_i$ be a simple normal crossings divisor on X, where Δ_i are distinct prime divisors. Assume that B is a Cartier divisor on X such that $B \sim_{\mathbb{Q}} \sum_{i=1}^r b_i \Delta_i$, where $0 < b_i \leq 1$ for all i. Then for every effective divisor D with $\operatorname{Supp} D \subseteq \operatorname{Supp} \Delta$, the maps*

$$H^q(X, \mathcal{O}_X(K_X + B)) \to H^q(X, \mathcal{O}_X(K_X + B + D)),$$

coming from the short exact sequence

$$0 \to \mathcal{O}_X(K_X + B) \to \mathcal{O}_X(K_X + B + D) \to \mathcal{O}_D \otimes \mathcal{O}_X(K_X + B + D) \to 0,$$

are injective for all q. Equivalently, by Serre duality, the maps

$$H^q(X, \mathcal{O}_X(-B - D)) \to H^q(X, \mathcal{O}_X(-B))$$

are surjective for all q.

The remainder of this course is dedicated to the proof of this important result. This theorem is a strong generalisation of the Kodaira vanishing – indeed, we can now give a third proof, which is very similar to a step of the second proof we gave before.

Proof of Theorem 4.8. Let m be a positive integer such that mA is basepoint free and that

$$H^j(X, K_X + (m+1)A) = 0 \quad \text{for } j > 0,$$

which is possible by the cohomological criterion for ampleness. If $D \in |mA|$ is a general section, then D is smooth by Bertini's theorem, and we have $A \sim_{\mathbb{Q}} \frac{1}{m} D$. By Theorem 4.21, the map

$$H^j(X, \mathcal{O}_X(K_X + A)) \to H^j(X, \mathcal{O}_X(K_X + A + D))$$

is injective for every $j \geq 0$, and we conclude since $A + D \sim (m+1)A$. $\square$

The proof of Theorem 4.21 proceeds in two logical steps. The first is to show the result when all the coefficients b_i are equal to 1, and in the second step the general result is deduced from this special case.

Proposition 4.22. *Let X be a smooth projective variety and let $\Delta = \sum_{i=1}^r \Delta_i$ be a simple normal crossings divisor on X, where Δ_i are distinct prime divisors. Then for every effective divisor D with* $\operatorname{Supp} D \subseteq \operatorname{Supp} \Delta$*, the maps*

$$H^q(X, \mathcal{O}_X(K_X + \Delta)) \to H^q(X, \mathcal{O}_X(K_X + \Delta + D))$$

are injective for all $q \geq 0$. Equivalently, by Serre duality, the maps

$$H^q(X, \mathcal{O}_X(-\Delta - D)) \to H^q(X, \mathcal{O}_X(-\Delta))$$

are surjective for all q.

We will prove this result later, since the techniques involved in its proof are unlike anything we have seen thus far. However, assuming this technical step, we are now able to prove Theorem 4.21.

Proof of Theorem 4.21. For the purpose of induction, we replace the condition that Δ_i are smooth prime divisors by the condition that Δ_i is a disjoint union of smooth divisors. We argue by induction on the cardinality of the set $I = \{i \mid b_i < 1\}$.

If $\#I = 0$, then $B \sim_{\mathbb{Q}} \Delta$, and let n be the smallest positive integer such that $nB \sim n\Delta$. Denoting $\mathcal{M} = \mathcal{O}_X(B - \Delta)$, we have $\mathcal{M}^n \simeq \mathcal{O}_X$. Let $\mu \colon Y \to X$ be the n-cyclic covering associated to a nowhere vanishing global section of $\mathcal{O}_X$, and denote $B_Y = \mu^* B$ and $\Delta_Y = \mu^*\Delta$ so that $\mu^*\mathcal{M} = \mathcal{O}_Y(B_Y - \Delta_Y)$. Then $\mu^*\mathcal{M} \simeq \mathcal{O}_Y$ by Proposition 4.15, hence $B_Y \sim \Delta_Y$. Therefore, by Proposition 4.22 the maps

$$H^q(Y, \mathcal{O}_Y(-B_Y - D_Y)) \to H^q(Y, \mathcal{O}_Y(-B_Y)) \tag{4.13}$$

are surjective. By Proposition 4.15 we have $\mu_*\mathcal{O}_Y \simeq \bigoplus_{i=0}^{n-1} \mathcal{M}^{-i}$, hence by the projection formula and since μ is finite, we have

$$H^q(Y, \mathcal{O}_Y(-B_Y - D_Y)) \simeq \bigoplus_{i=0}^{n-1} H^q(X, \mathcal{O}_X(-B - D) \otimes \mathcal{M}^{-i})$$

and

$$H^q(Y, \mathcal{O}_Y(-B_Y)) \simeq \bigoplus_{i=0}^{n-1} H^q(X, \mathcal{O}_X(-B) \otimes \mathcal{M}^{-i})$$

The maps (4.13) split through the summands, and taking the component corresponding to $i = 0$ shows that the maps

$$H^q(X, \mathcal{O}_X(-B - D)) \to H^q(X, \mathcal{O}_X(-B))$$

are surjective.

Now assume that $\#I > 0$, and without loss of generality we can assume that $b_1 < 1$. Let m be a positive integer such that $a_1 = mb_1$ is an integer $\leq m-1$, for example, let m be the denominator of b_1. By Corollary 4.20 there exist a smooth variety Y, a finite surjective morphism $f\colon Y \to X$ and a line bundle $\mathcal{L}$ on Y such that $\mathcal{O}_Y(f^*\Delta_1) \simeq \mathcal{L}^m$ and $f^*\Delta$ is a reduced divisor with simple normal crossings. Denoting $B_Y = f^*B$ and $D_Y = f^*D$, we have $B_Y \sim_{\mathbb{Q}} \sum_{i=1}^r b_i f^*\Delta_i$, hence Y, $f^*\Delta$, B_Y and D_Y satisfy the assumptions of the theorem. We claim that it suffices to show that the maps $H^q(Y, \mathcal{O}_Y(-B_Y - D_Y)) \to H^q(Y, \mathcal{O}_Y(-B_Y))$ are surjective: indeed, this follows from the commutative diagram

$$\begin{array}{ccc} H^q(Y,\mathcal{O}_Y(-B_Y-D_Y)) & \longrightarrow & H^q(Y,\mathcal{O}_Y(-B_Y)) \\ \downarrow & & \downarrow \\ H^q(X,\mathcal{O}_X(-B-D)) & \longrightarrow & H^q(X,\mathcal{O}_X(-B)) \end{array}$$

where the vertical maps are the surjective maps induced by $\mathrm{Tr}_{k(Y)/k(X)}$, which are dual to the maps from the proof of Lemma 4.13. Therefore, after replacing X by Y, Δ by $f^*\Delta$, B by B_Y and D by D_Y, we may assume that there is a line bundle $\mathcal{L}$ on X such that $\mathcal{O}_X(\Delta_1) \simeq \mathcal{L}^m$.

Let $g\colon Z \to X$ be the m-cyclic covering corresponding to a section of $\mathcal{L}^m$ which defines Δ_1. Then Z is a smooth variety, and we have $g^*\Delta_1 = m\Delta'_1$ and $g^*\Delta_i = \Delta'_i$ for all $i \geq 2$, where all Δ'_i are disjoint unions of smooth prime divisors and $\Delta_Z = \sum_{i=1}^r \Delta'_i$ has simple normal crossings. Defining $D_Z = g^*D$ and $B_Z = g^*B + (1 - a_1)\Delta'_1$, we have $B_Z \sim_{\mathbb{Q}} \Delta'_1 + \sum_{i=2}^r b_i\Delta'_i$. By induction, the maps

$$H^q(Z, \mathcal{O}_Z(-B_Z - D_Z)) \to H^q(Z, \mathcal{O}_Z(-B_Z)) \tag{4.14}$$

are surjective for all q. We have $g^*\mathcal{L} \simeq \mathcal{O}_Z(\Delta'_1)$ by Proposition 4.15, hence by the projection formula and by (4.5):

$$\begin{aligned} g_*\mathcal{O}_Z(-B_Z) &= g_*\mathcal{O}_Z(-g^*B + (a_1-1)\Delta'_1) \simeq \mathcal{O}_X(-B) \otimes g_*g^*\mathcal{L}^{a_1-1} \\ &= \bigoplus_{j=0}^{m-1} \mathcal{O}_X(-B) \otimes \mathcal{L}^{a_1-1-j}. \end{aligned}$$

Since g is finite, this yields

$$H^q(Z, \mathcal{O}_Z(-B_Z)) \simeq \bigoplus_{j=0}^{m-1} H^q(X, \mathcal{O}_X(-B) \otimes \mathcal{L}^{a_1-1-j}),$$

and similarly

$$H^q(Z, \mathcal{O}_Z(-B_Z - D_Z)) \simeq \bigoplus_{j=0}^{m-1} H^q(X, \mathcal{O}_X(-B-D) \otimes \mathcal{L}^{a_1-1-j}).$$

By assumption we have $1 \leq a_1 \leq m-1$, and the map (4.14) splits through the summands. Taking the component corresponding to $j = a_1 - 1$, we obtain that the maps

$$H^q(X, \mathcal{O}_X(-B-D)) \to H^q(X, \mathcal{O}_X(-B))$$

are surjective for all q. $\square$

4.8 Kawamata-Viehweg vanishing

Another consequence of Theorem 4.21 is the following strong generalisation of the Kodaira vanishing, in which ampleness is replaced by its birationally stable version – nef and big. We start first with the following important and difficult theorem on the resolution of singularities by Hironaka, which we, of course, do not prove.

Theorem 4.23. *Let X be a complex algebraic variety, and let D be an effective Cartier divisor on X. Then there exist a smooth variety Y and a projective birational morphism $\mu\colon Y \to X$ such that the exceptional locus $\mathrm{Exc}(\mu)$ is a divisor, and the support of the divisor $\mu^*D + \mathrm{Exc}(\mu)$ has simple normal crossings.*

Furthermore, if A is an ample $\mathbb{Q}$-divisor on X, then for every $\varepsilon > 0$ there exists an effective μ-exceptional divisor F on Y such that the coefficients of F are smaller than ε and such that $\mu^*A - F$ is ample.

The second part of the previous theorem actually holds under more general conditions and is not too difficult to prove, but it requires knowing properties of ampleness relative to a morphism, so we take it on faith.

Now we are ready to prove the Kawamata-Viehweg vanishing.

Theorem 4.24. *Let X be a smooth complex projective variety of dimension n, and let B be a Cartier divisor on X such that $B \sim_{\mathbb{Q}} N + \Delta$, where N is a nef and big $\mathbb{Q}$-divisor and $\Delta = \sum \delta_i \Delta_i$ is a $\mathbb{Q}$-divisor with simple normal crossings support such that $0 < \delta_i < 1$ for all i, and all Δ_i are prime divisors. Then*

$$H^i(X, \mathcal{O}_X(K_X + B)) = 0 \quad \textit{for } i > 0.$$

Equivalently, by Serre duality,

$$H^i(X, \mathcal{O}_X(-B)) = 0 \quad \textit{for } i < n.$$

Proof. We proceed in three steps.

Step 1. Assume that N is ample. By the cohomological criterion for ampleness, we can choose a large integer m such that mN is very ample and that

$$H^i(X, \mathcal{O}_X(K_X + B + mN)) = 0 \quad \text{for } i > 0.$$

By Bertini's theorem, pick a general smooth element $D \in |mN|$. Then the divisor $\Delta + \frac{1}{m}D$ has simple normal crossings support, and $B \sim_{\mathbb{Q}} \Delta + \frac{1}{m}D$. By Theorem 4.21, the map

$$H^i(X, \mathcal{O}_X(K_X + B)) \to H^i(X, \mathcal{O}_X(K_X + B + D))$$

is injective for all i, and the theorem follows in this case.

Step 2. With assumptions from the theorem, assume that $\Delta = 0$. By Lemma 3.48, there exists an effective divisor E such that $N - \frac{1}{k}E$ is ample for $k \gg 0$. By Theorem 4.23, there exist a smooth projective variety Y and a birational morphism $\mu\colon Y \to X$ such that $\operatorname{Exc}(\mu)$ is a divisor and such that $\operatorname{Exc}(\mu) + \mu^*E$ has simple normal crossings support. Pick $k \gg 0$ such that the coefficients of $\frac{1}{k}\mu^*E$ are smaller than 1, and denote $A = N - \frac{1}{k}E$. By Theorem 4.23 again, there is an effective μ-exceptional divisor F with small coefficients such that $A_Y = \mu^*A - F$ is ample, and such that $\Delta_Y = F + \frac{1}{k}\mu^*E$ has coefficients smaller than 1. Since $\mu^*B \sim_{\mathbb{Q}} A_Y + \Delta_Y$, by Step 1 we have

$$H^i(Y, \mathcal{O}_Y(K_Y + \mu^*B)) = 0 \quad \text{for } i > 0.$$

We have $\mu_*\mathcal{O}_Y(K_Y + \mu^*B) = \mathcal{O}_X(K_X + B)$ by the ramification formula and by the projection formula, hence to prove the theorem under the assumptions of this step, by Theorem 2.57 it suffices to show that

$$R^i\mu_*\mathcal{O}_Y(K_Y + \mu^*B) = 0 \quad \text{for } i > 0. \tag{4.15}$$

We will prove this using the Leray spectral sequence. First, fix an ample divisor H on X, and let r be any positive integer. Since $B + rH$ is ample, as above we show that

$$H^i(Y, \mathcal{O}_Y(K_Y + \mu^*B + \mu^*(rH)) = 0 \quad \text{for } i > 0. \tag{4.16}$$

Now, consider the Leray spectral sequence (which depends on r):

$$\begin{aligned} E_2^{pq}(r) &= H^q(X, R^p\mu_*\mathcal{O}_Y(K_Y + \mu^*B + \mu^*(rH))) \\ &\Longrightarrow H^{p+q}(Y, \mathcal{O}_Y(K_Y + \mu^*B + \mu^*(rH))). \end{aligned}$$

Since $E_2^{pq}(r) = H^q(X, R^p\mu_*\mathcal{O}_Y(K_Y + \mu^*B) \otimes \mathcal{O}_X(rH)))$ by the projection formula, we have $E_2^{pq}(r) = 0$ for $q > 0$ and for all p when $r \gg 0$ by the cohomological criterion for ampleness. In particular, the E_2-table of this spectral sequence consists of only one row, hence this implies $E_2^{p,0}(r) = E_\infty^{p,0}(r)$ for $r \gg 0$.

On the one hand, the vector space $E_\infty^{p,0}(r)$ is a direct summand of $H^p(Y, \mathcal{O}_Y(K_Y + \mu^*B + \mu^*(rH)))$, hence

$$E_\infty^{p,0}(r) = 0 \quad \text{for } p > 0$$

by (4.16). On the other hand,

$$E_2^{p,0}(r) = H^0(X, R^p\mu_*\mathcal{O}_Y(K_Y + \mu^*B) \otimes \mathcal{O}_X(rH)),$$

and the sheaf $R^p\mu_*\mathcal{O}_Y(K_Y + \mu^*B) \otimes \mathcal{O}_X(rH)$ is globally generated for $r \gg 0$ since H is ample. Hence this sheaf is the zero sheaf, and this implies (4.15).

Step 3. Now we prove the full statement of the theorem, and this is similar to the proof of Theorem 4.21 above.

The proof is by induction on the number of components of Δ. If this number is zero, the theorem follows from Step 2. Otherwise, let m be a positive integer such that $a_1 = m\delta_1$ is an integer $\leq m-1$. By Corollary 4.20 there exist a smooth variety Y, a finite surjective morphism $f\colon Y \to X$ and a line bundle $\mathcal{L}$ on Y such that $\mathcal{O}_Y(f^*\Delta_1) \simeq \mathcal{L}^m$ and $\Delta_Y = f^*\Delta$ is a divisor with simple normal crossings support. Denoting $B_Y = f^*B$ and $N_Y = f^*N$, we have $B_Y \sim_{\mathbb{Q}} N_Y + \Delta_Y$. We claim that it suffices to show that $H^i(Y, \mathcal{O}_Y(-B_Y)) = 0$ for $i < n$: indeed, there are surjective maps

$$H^i(Y, \mathcal{O}_Y(-B_Y)) \to H^q(X, \mathcal{O}_X(-B))$$

induced by $\mathrm{Tr}_{k(Y)/k(X)}$, which are dual to the maps from the proof of Lemma 4.13. Therefore, after replacing X by Y, Δ by Δ_Y, B by B_Y and N by N_Y, we may assume that there is a line bundle $\mathcal{L}$ on X such that $\mathcal{O}_X(\Delta_1) \simeq \mathcal{L}^m$.

Let $g\colon Z \to X$ be the m-cyclic covering corresponding to a section of $\mathcal{L}^m$ which defines Δ_1. Then Z is a smooth variety, and we have $g^*\Delta_1 = m\Delta_1'$ and $g^*\Delta_i = \Delta_i'$ for all $i \geq 2$, where all Δ_i' are disjoint unions of smooth prime divisors and $\Delta_Z = \sum_{i=1}^r \Delta_i'$ has simple normal crossings. Defining $B_Z = g^*B - a_1\Delta_1'$, we have $B_Z \sim_{\mathbb{Q}} g^*N + \sum_{i=2}^r b_i\Delta_i'$, hence by induction:

$$H^i(Z, \mathcal{O}_Z(-B_Z)) = 0 \quad \text{for } i < n. \tag{4.17}$$

We have $g^*\mathcal{L} \simeq \mathcal{O}_Z(\Delta_1')$ by Proposition 4.15, hence by the projection formula and by (4.5):

$$g_*\mathcal{O}_Z(-B_Z) = g_*\mathcal{O}_Z(-g^*B + a_1\Delta_1') \simeq \bigoplus_{j=0}^{m-1} \mathcal{O}_X(-B) \otimes \mathcal{L}^{a_1-j}.$$

Since g is finite, this yields

$$H^i(Z, \mathcal{O}_Z(-B_Z)) \simeq \bigoplus_{j=0}^{m-1} H^i(X, \mathcal{O}_X(-B) \otimes \mathcal{L}^{a_1-j}).$$

By assumption we have $1 \leq a_1 \leq m-1$, and the component corresponding to $j = a_1$ is $H^i(X, \mathcal{O}_X(-B))$, which together with (4.17) proves the theorem. $\square$

Note also the following easy corollary:

Corollary 4.25. *Let X be a smooth complex projective variety of dimension n, and let B be a Cartier divisor on X such that $B \sim_{\mathbb{Q}} A + \Delta$, where A is a ample $\mathbb{Q}$-divisor and $\Delta = \sum \delta_i \Delta_i$ is a $\mathbb{Q}$-divisor with simple normal crossings support such that $0 < \delta_i \leq 1$ for all i, and all Δ_i are prime divisors. Then*

$$H^i(X, \mathcal{O}_X(K_X + B)) = 0 \quad \textit{for } i > 0.$$

4.9 The canonical ring on surfaces

We have almost all the tools to prove that the canonical ring on a surface is finitely generated. The main remaining technical ingredient is the Zariski decomposition.

4.9.1 Zariski decomposition

Definition 4.26. Let X be a smooth projective surface and let $D = \sum_{i=1}^r d_i D_i$ be a $\mathbb{Q}$-divisor on X, where D_i are prime divisors and $d_i \neq 0$ for every i. Then the *intersection matrix* of D is the $(r \times r)$-matrix $(D_i \cdot D_j)$.

Theorem 4.27. *Let X be a smooth projective surface and let D be an effective $\mathbb{Q}$-divisor on X. Then there is a unique decomposition $D = P + N$, where P and N are $\mathbb{Q}$-divisors (the* positive *and* negative *parts of D) such that*

(i) P is nef,

(ii) $N \geq 0$, and if $N \neq 0$, then the intersection matrix of N is negative definite,

(iii) $P \cdot C = 0$ for every irreducible component C of N.

For an $\mathbb{R}$-divisor $D = \sum d_i D_i$ on a projective variety X, we define $\lfloor D \rfloor = \sum \lfloor d_i \rfloor D_i$. We will also use repeatedly in this subsection the fact that if C is a curve on a smooth surface X and if M is an effective divisor on X which does not contain C in its support, then $M \cdot C \geq 0$: indeed, this follows from $M \cdot C = \deg_C(M|_C)$.

The main and immediate corollary of the Zariski decomposition is the following.

Corollary 4.28. *With assumptions from Theorem 4.27, if D is an integral divisor, then the natural map*

$$H^0(X, \mathcal{O}_X(\lfloor mP \rfloor)) \to H^0(X, \mathcal{O}_X(mD))$$

is bijective for every $m \geq 1$.

Proof. Let $D' \in |mD|$. We first claim that it suffices to show that $D' \geq mN$. Indeed, the claim implies that $mN \leq \operatorname{Fix}|mD|$, and the corollary follows easily, cf. the paragraph after Example 2.29.

Hence, to prove the claim we may replace all the divisors in question by a multiple, so without loss of generality we may assume that P and N are integral and that $m = 1$. Let E_i be all the prime components of N. Then we can write $D = M + E$, where M and E are effective divisors such that $\operatorname{Supp} E \subseteq \operatorname{Supp} N$ and M does not contain any E_i in its support. Thus we want to show that $E \geq N$. Equivalently, if we write $E - N = N' - N''$, where N' and N'' are effective divisors with no common components, we need to get $N'' = 0$.

Now, since $D' - N \sim P$, we have by Theorem 4.27(iii),

$$(E - N) \cdot E_i = (D' - M - N) \cdot E_i = (P - M) \cdot E_i = -M \cdot E_i \leq 0 \quad \text{for all } i. \tag{4.18}$$

Assume for contradiction that $N'' \neq 0$. Since $\operatorname{Supp} N'' \subseteq \operatorname{Supp} N$, we have $N'' \cdot N'' < 0$ by Theorem 4.27(ii), and thus

$$(E - N) \cdot N'' = N' \cdot N'' - N'' \cdot N'' > 0,$$

which contradicts (4.18). □

Now we turn to the proof of the Zariski decomposition. We first need the following lemma.

Lemma 4.29. *Let X be a smooth projective surface, and let $N \neq 0$ be an $\mathbb{Q}$-divisor whose intersection matrix is not negative definite. Then there exists an effective nef $\mathbb{Q}$-divisor $E \neq 0$ such that* $\operatorname{Supp} E \subseteq \operatorname{Supp} N$.

Proof. Let $E_1, \dots, E_r$ be the components of N, and let $\mathcal{N}$ be the intersection matrix of N. There are two cases: either $\mathcal{N}$ is not negative semidefinite or it is negative semidefinite.

Assume first that $\mathcal{N}$ is not negative semidefinite. Then there exists a row $\mathbf{b} = (b_1, \dots, b_r) \in \mathbb{Q}^r$ such that $\mathbf{b}\mathcal{N}\mathbf{b}^t > 0$, and if we denote $B = \sum b_i E_i$, this is equivalent to $B^2 > 0$. Write $B = B_1 - B_2$, where B_1 and B_2 are effective divisors with no common components. Then

$$0 < B^2 = B_1^2 - 2B_1B_2 + B_2^2,$$

hence $B_1^2 > 0$ or $B_2^2 > 0$. Therefore, replacing B by B_1 or by B_2, we may assume that there exists an effective $\mathbb{Q}$-divisor B such that $B^2 > 0$.

From Theorem 3.10 we have

$$\chi(\mathcal{O}_X(mB)) = \chi(\mathcal{O}_X) + \frac{1}{2} mB \cdot (mB - K_X),$$

which clearly grows like $O(m^2)$. On the other hand, if H is a smooth very ample divisor on X, then $\deg_H(K_X - mB) = (K_X - mB) \cdot H < 0$ for $m \gg 0$, hence the divisor $K_X - mB$ cannot be linearly equivalent to an effective divisor. By the Serre duality, for $m \gg 0$ this implies

$$\begin{aligned}\chi(\mathcal{O}_X(mB)) &= h^0(X, \mathcal{O}_X(mB)) - h^1(X, \mathcal{O}_X(mB)) + h^0(X, \mathcal{O}_X(K_X - mB)) \\ &\le h^0(X, \mathcal{O}_X(mB)),\end{aligned}$$

and thus the divisor B is big. For $m \gg 0$, pick $B' \in |mB|$, and write $B' = M_m + F_m$, where M_m and F_m are the moving and fixed parts of mB respectively. Then M_m is a nonzero effective divisor. Moreover, M_m is nef: indeed, if C were a curve such that $M_m \cdot C < 0$, then C would belong to the support of M_m. Since this is then true for every element of the linear system $|M_m|$, this would imply by definition that C belongs to $\operatorname{Fix}|M_m|$, a contradiction.

Therefore, we set $E = M_m$ in this case.

Now assume that $\mathcal{N}$ is negative semidefinite. We argue by induction on r (the number of components of N).

If $r = 1$, then $N^2 = E_1^2 = 0$, hence E_1 is nef and we set $E = E_1$.

Now suppose that $r > 1$. Since $\mathcal{N}$ is negative semidefinite but not negative definite, there exists an eigenvalue of $\mathcal{N}$ which is equal to zero, and hence $\det \mathcal{N} = 0$. We claim that there exists a $\mathbb{Q}$-divisor $R = \sum r_i E_i \neq 0$ such that $R \cdot E_i = 0$ for every i: indeed, these relations give a system of linear equations whose determinant is $\det \mathcal{N}$, hence its system of solutions is a positive dimensional *rational* vector subspace of $\mathbb{R}^r$ (since the entries of $\mathcal{N}$ are integers).

If $R \ge 0$ or $-R \ge 0$, then R, respectively $-R$, is nef, and we set $E = R$, respectively $E = -R$. Otherwise, write $R = R_1 - R_2$, where R_1 and R_2 are nonzero effective divisors with no common components. Then

$$0 = R^2 = R_1^2 - 2R_1R_2 + R_2^2,$$

and since $\mathcal{N}$ is negative semidefinite, we have $R_1^2 \le 0$ and $R_2^2 \le 0$. Therefore $R_1^2 = R_2^2 = 0$. The divisor R_1 has fewer components than R, and $R_1^2 = 0$ implies that the intersection matrix of R_1 is negative semidefinite, but not negative definite. We finish by induction. □

Proof of Theorem 4.27. We first prove uniqueness. Assume that there are two decompositions $D = N + P$ and $D = N' + P'$ satisfying the assumptions of the theorem, where $N = \sum n_i E_i$ and $N' = \sum n_i' E_i$ (here we allow that some n_i and n_i' are zero). Denote $N'' = \sum \min\{n_i, n_i'\} E_i$, $N_0 = N - N'' \ge 0$ and $N_0' = N' - N'' \ge 0$. Then $P + N_0 = P' + N_0'$, and N_0 and N_0' have no common components. We have $P \cdot N_0 = 0$ by the property (iii) and $P' \cdot N_0 \ge 0$ since P' is nef. Hence, if $N_0 \neq 0$, then

$$0 > N_0^2 = (P + N_0) \cdot N_0 = (P' + N_0') \cdot N_0 \ge N_0 \cdot N_0' \ge 0,$$

a contradiction. This shows $N' \geq N$, and by symmetry $N \geq N'$, which implies uniqueness.

Now we prove existence. Write $D = \sum_{i=1}^t d_i C_i$, where C_i are prime divisors and $d_i > 0$. For every $D' = \sum_{i=1}^t x_i C_i$ with $0 \leq x_i \leq d_i$ for all i, we have that D' is nef if and only if $\sum_{i=1}^t x_i C_i \cdot C_j \geq 0$ for all $j = 1, \dots, t$. Let

$$\mathcal{K} = \bigcap_{j=1}^t \Big\{ (x_1, \dots, x_t) \in \prod_{i=1}^t [0, d_i] \mid \sum_{i=1}^t x_i C_i \cdot C_j \geq 0 \Big\}.$$

Then $\mathcal{K}$ is a rational polytope. For each $\tau \in [0,1]$, let $\mathcal{H}_\tau$ be the hyperplane $\{(x_1, \dots, x_t) \in \mathbb{R}^t \mid \sum x_i = \tau \sum d_i\}$, and let τ_0 be the maximal value of τ for which $\mathcal{H}_\tau \cap \mathcal{K} \neq \emptyset$. Then it is clear that $\tau_0 \in \mathbb{Q}$ since $\mathcal{K}$ is a rational polytope. Pick any rational t-tuple $(y_1, \dots, y_t) \in \mathcal{H}_{\tau_0} \cap \mathcal{K}$, and set $P = \sum_{i=1}^t y_i C_i$ and $N = D - P$. Then by definition P is nef and $0 \leq P \leq D$, hence $N \geq 0$, which shows (i).

For (ii), suppose that $P \cdot C_i > 0$ for some prime component C_i of N. In particular $\varepsilon C_i \leq N$ for some $0 < \varepsilon \ll 1$, hence $P + \varepsilon C_i \leq D$. Moreover, $P + \varepsilon C_i$ is nef for ε very small. But then $P + \varepsilon C_i$ belongs to $\mathcal{K}$, which contradicts the construction of P.

For (iii), suppose that $N \neq 0$ and that the intersection matrix of N is not negative definite. Then by Lemma 4.29 there exists an nonzero, nef and effective $\mathbb{Q}$-divisor E such that $\operatorname{Supp} E \subseteq \operatorname{Supp} N$. Then for $0 < \varepsilon \ll 1$ we have $N - \varepsilon E \geq 0$ and $P + \varepsilon E$ is nef. Since $P + \varepsilon E$ belongs to $\mathcal{K}$, this again contradicts the construction of P. □

4.9.2 The finite generation of the canonical ring

We can finally address the question of the finite generation of the ring

$$R(X, K_X) = \bigoplus_{m \geq 0} H^0(X, mK_X),$$

where X is a smooth surface. Recall that the Kodaira dimension of X is an element of the set $\{-\infty, 0, 1, 2\}$. If $\kappa(X) = -\infty$, then $R(X, K_X) = \mathbb{C}$. If $\kappa(X) = 0$, then some Veronese subring (cf. the proof of Theorem 3.34) of $R(X, K_X)$ is isomorphic to the polynomial ring, hence finitely generated.

If $\kappa(X) = 1$, then by the classification of surfaces, there exists a morphism $\pi\colon X \to \mathbb{P}^1$ such that the generic fibre of π is an elliptic curve. Then by Kodaira's canonical bundle formula, some multiple of K_X is the pullback of a Cartier divisor D on $\mathbb{P}^1$, and the finite generation follows since the section ring of any divisor on a curve is finitely generated (since the divisor is either ample, anti-ample or numerically trivial).

Hence, it remains to show the following.

Theorem 4.30. *Let X be a smooth projective surface of general type. Then the canonical ring of X is finitely generated.*

In the proof we need the following result which we will not prove; the proof is not too difficult but is long, and it is worth noting that it is a nice application of a more general version of Theorem 3.34.

Theorem 4.31. *Let X be a projective variety, and let D be a Cartier divisor on X such that* $\operatorname{Bs}|D|$ *is a finite set. Then D is semiample.*

Finally, we have:

Proof of Theorem 4.30. Since K_X is big, there exist an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor E such that $K_X \sim_{\mathbb{Q}} A+E$. By Theorem 4.23, there exist a smooth projective surface Y and a birational morphism $f\colon Y \to X$ such that $\operatorname{Exc}(f)+f^*E$ has simple normal crossings support, and a small effective f-exceptional divisor F on Y such that $A_Y = f^*A - F$ is ample. By the ramification formula we have

$$K_Y \sim f^*K_X + G,$$

where G is an effective f-exceptional divisor, and write $E_Y = f^*E + F + G$. Then $K_Y \sim_{\mathbb{Q}} A_Y + E_Y$, and by Bertini's theorem, there exists a $\mathbb{Q}$-divisor $A'_Y \sim_{\mathbb{Q}} A_Y$ such that the divisor $A'_Y + E_Y$ has simple normal crossings support. Furthermore, by the projection formula and by Lemma 2.24 we have $R(Y, K_Y) \simeq R(X, K_X)$. Hence replacing X by Y, A by A'_Y and E by E_Y, we may assume that the divisor $A+E$ has simple normal crossings support.

Pick a rational number $0 < \varepsilon \ll 1$ such that $\lfloor \varepsilon E \rfloor = 0$, and let

$$(1+\varepsilon)(A+E) = P + N \tag{4.19}$$

be the Zariski decomposition of $(1+\varepsilon)(A+E) \sim_{\mathbb{Q}} K_X + \varepsilon(A+E)$. Note that by the proof of Theorem 4.27, P and N are effective divisors supported on $\operatorname{Supp}(A+E)$. Let ℓ be a positive integer such that $\mathbf{B}(P) = \operatorname{Bs}|\ell P|$. If we write $\ell P = M_0 + F_0$, where $M_0 = \operatorname{Mob}(\ell P)$ and $F_0 = \operatorname{Fix}|\ell P|$, then M_0 is semiample by Theorem 4.31 and $\operatorname{Supp} F_0 \subseteq \mathbf{B}(P)$. Thus, for some integer $\ell' \gg 0$, the linear system $|\ell' M_0|$ is basepoint free, and Bertini's theorem implies that a general element M'_0 of $|\ell' M_0|$ is a smooth subvariety. Setting $M = \frac{1}{\ell\ell'}M'_0$ and $F = \frac{1}{\ell}F_0$, we have $P \sim_{\mathbb{Q}} M + F$. If we replace A by $A + \frac{1}{1+\varepsilon}(M + F - P)$, we may assume that

(a) $P = M + F$, where M and F are effective $\mathbb{Q}$-divisors,

(b) $\operatorname{Supp}(M + F + N)$ has simple normal crossings,

(c) $\operatorname{Supp} F \subseteq \mathbf{B}(P)$,

(d) the coefficients of M are much smaller than the coefficients of F.

Now, let m be a positive integer such that $m\varepsilon A$, $m\varepsilon E$, mP and mN are all integral divisors and $m(1+\varepsilon)K_X \sim m(1+\varepsilon)(A+E)$. Then the Veronese subring $R(X, K_X)^{(m(1+\varepsilon))}$ (cf. the proof of Theorem 3.34) is isomorphic to $R(X, mP)$ by Corollary 4.28. If P is semiample, then the canonical ring $R(X, K_X)$ is finitely generated by Theorem 3.34 and its proof.

Therefore, we may assume that P is not semiample, and in particular, $\mathbf{B}(P)$ contains a curve by Theorem 4.31 and thus $F \neq 0$. Our goal is to use the Kawamata-Viehweg vanishing (in the form of Corollary 4.25) to derive a contradiction. More precisely, we claim that there exist:

(i) an integral divisor R such that $0 \leq R \leq \lceil N \rceil$,

(ii) an ample $\mathbb{Q}$-divisor A' and an effective divisor B' with simple normal crossings support such that the coefficients of B' are ≤ 1 and $\lfloor B' \rfloor \subseteq \mathbf{B}(P)$, and a positive integer m with $\operatorname{Bs}|mP| = \mathbf{B}(P)$ and

$$mP + R \sim_{\mathbb{Q}} K_X + A' + B', \tag{4.20}$$

(iii) a prime divisor $S \subseteq \lfloor B' \rfloor$ such that $H^0(S, \mathcal{O}_S((mP+R)|_S)) \neq 0$.

Assuming the claim, let us see how it quickly implies the theorem. The long cohomology sequence associated to the exact sequence

$$0 \to \mathcal{O}_X(mP+R-S) \to \mathcal{O}_X(mP+R) \to \mathcal{O}_S((mP+R)|_S) \to 0$$

together with (4.20) and Corollary 4.25 imply that the map

$$H^0(X, \mathcal{O}_X(mP+R)) \to H^0(S, \mathcal{O}_S((mP+R)|_S))$$

is surjective. This and (iii) show that the map

$$H^0(X, \mathcal{O}_X(mP+R-S)) \to H^0(X, \mathcal{O}_X(mP+R))$$

is not an isomorphism, which is equivalent to $S \not\subseteq \operatorname{Bs}|mP+R|$ (exercise!). Since $0 \leq R \leq \lceil N \rceil \leq \lceil mN \rceil = mN$ by (i), the composition of injective maps

$$H^0(X, \mathcal{O}_X(mP)) \to H^0(X, \mathcal{O}_X(mP+R)) \to H^0(X, \mathcal{O}_X(mP+mN))$$

is an isomorphism by Corollary 4.28, hence so is the first map. But this implies $\operatorname{Bs}|mP+R| = \operatorname{Bs}|mP| \cup \operatorname{Supp} R$ (exercise!), and thus $S \not\subseteq \operatorname{Bs}|mP| = \mathbf{B}(P)$, a contradiction.

Now we prove the claim. Noting that $\lfloor \varepsilon E - N \rfloor \leq 0$, let

$$\lambda = \sup\{t \geq 0 \mid \lfloor \varepsilon E + tP - N \rfloor \leq 0\} > 0,$$

and denote $Q = \varepsilon E + \lambda P - N$. If $Q = \sum q_i Q_i$, where Q_i are prime divisors, then $q_i \leq 1$. Moreover, if $q_i = 1$, then $Q_i \subseteq \operatorname{Supp} F \subseteq \mathbf{B}(P)$ by (a), (c) and (d) above, and if $q_i < 0$, then $Q_i \subseteq \operatorname{Supp} N$ by construction. Set

$$R = -\sum_{q_i<0} \lfloor q_i \rfloor Q_i.$$

Then (i) is immediate. Denote

$$B' = Q + R = \sum_{q_i \geq 0} q_i Q_i + \sum_{q_i<0} (q_i - \lfloor q_i \rfloor) Q_i.$$

Then it is clear that the coefficients of B' lie in the interval $(0, 1]$ and that $\lfloor B' \rfloor \subseteq \mathbf{B}(P)$. Let $m > \lambda + 1$ be a sufficiently large positive integer such that $m\varepsilon A$, $m\varepsilon E$, mP and mN are integral divisors and $\mathbf{B}(P) = \operatorname{Bs}|mP|$. Let $A' = \varepsilon A + (m-1-\lambda)P$, and note that A' is ample since P is nef and $m - 1 - \lambda > 0$. Then by (4.19) we have

$$mP + R = (1 + \varepsilon)(A + E) - N + (m - 1)P + R \sim_{\mathbb{Q}} K_X + A' + B',$$

which shows (ii).

Finally, fix a prime divisor S in $\lfloor B' \rfloor$, and let g be the genus of S. Note that $S \nsubseteq \operatorname{Supp} R$ by construction, and therefore $\deg_S((mP + R)|_S) \geq 0$ since P is nef. In particular, this implies (iii) for $g = 0$. If $g \geq 1$, then by adjunction:

$$(mP + R)|_S \sim_{\mathbb{Q}} K_S + A'|_S + (B' - S)|_S.$$

Since $\deg_S(A'|_S + (B' - S)|_S) > 0$, by the Riemann-Roch we have

$$\begin{aligned} h^0(S, \mathcal{O}_S((mP + R)|_S)) &\geq \deg(mP + R)|_S - g + 1 \\ &= 2g - 2 + \deg(A'|_S + (B' - S)|_S) - g + 1 \\ &> g - 1 \geq 0, \end{aligned}$$

which finishes the proof of the claim. $\square$

4.10 Proof of Proposition 4.22

We now prove Proposition 4.22. Let X be a smooth projective variety and let Δ be a reduced simple normal crossings divisor on X. Recall from Section 4.5 that we have a complex of differentials with log poles $\Omega_X^\bullet(\log\Delta)$. If we denote $U = X \setminus \Delta$ and by $j\colon U \to X$ the inclusion, then we have the inclusion of complexes

$$\Omega_X^\bullet(\log\Delta) \to j_*\Omega_U^\bullet.$$

Indeed, $j_*\Omega_U^\bullet$ is the complex of sheaves of meromorphic differentials on X which are regular on U, hence the inclusion follows from the definition of $\Omega_X^\bullet(\log\Delta)$.

The first important result is the following.

Theorem 4.32. *Let X be a smooth projective variety and let Δ be a reduced simple normal crossings divisor on X. Denote $U = X \setminus \Delta$ and let $j\colon U \to X$ be the inclusion. Then the inclusion of complexes*

$$\Omega_X^\bullet(\log\Delta) \to j_*\Omega_U^\bullet$$

is a quasi-isomorphism.

Recall from Subsection 1.4.2 that to the complex $\Omega_X^\bullet(\log\Delta)$ we can associate the Hodge-to-de Rham spectral sequence

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log\Delta)) \underset{p}{\Longrightarrow} \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log\Delta)).$$

Then we have:

Theorem 4.33. *Let X be a smooth projective variety and let Δ be a reduced simple normal crossings divisor on X. Then the Hodge-to-de Rham spectral sequence*

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log\Delta)) \underset{p}{\Longrightarrow} \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log\Delta))$$

degenerates at E_1.

The first proof of Theorem 4.32 was given by Grothendieck. The first proof of Theorem 4.33 was given by Deligne, and note that when $\Delta = 0$, it follows easily from the classical Hodge theory and from (1.6). We will prove these two results algebraically in the sections below. Assuming Theorems 4.32 and 4.33, we can now prove Proposition 4.22.

Proof of Proposition 4.22. Denote $n = \dim X$ and $U = X \setminus \Delta$, and let $j\colon U \to X$ be the inclusion. Consider the Hodge-to-de Rham spectral sequences

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log\Delta)) \underset{p}{\Longrightarrow} \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log\Delta)),$$
$$\hat{E}_1^{p,q} = H^q(X, j_*\Omega_U^p) \underset{p}{\Longrightarrow} \mathbb{H}^{p+q}(X, j_*\Omega_U^\bullet).$$

Since $E_1^{p,q} = \hat{E}_1^{p,q} = 0$ for $p > n$, we have surjections $E_1^{n,q} \to E_\infty^{n,q}$ and $\hat{E}_1^{n,q} \to \hat{E}_\infty^{n,q}$ for all q, which by (1.5) induce canonical morphisms

$$\theta_q \colon E_1^{n,q} \to \mathbb{H}^{n+q}(X, \Omega_X^\bullet(\log \Delta)) \quad \text{and} \quad \hat{\theta}_q \colon \hat{E}_1^{n,q} \to \mathbb{H}^{n+q}(X, j_*\Omega_U^\bullet),$$

where θ_q are injective since $E_1^{n,q} \simeq E_\infty^{n,q}$ by Theorem 4.33. Since the constructions are functorial, we get the commutative diagram

$$\begin{tikzcd}
H^q(X, \mathcal{O}_X(K_X + \Delta)) \arrow[r, "\theta_q"] \arrow[d, "i_q"'] & \mathbb{H}^{n+q}(X, \Omega_X^\bullet(\log \Delta)) \arrow[d, "j_q"] \\
H^q(X, j_*\mathcal{O}_U(K_U)) \arrow[r, "\hat{\theta}_q"] & \mathbb{H}^{n+q}(X, j_*\Omega_U^\bullet)
\end{tikzcd}$$

The maps j_q are isomorphisms by Theorem 4.32, and hence the maps i_q are injective.

Now, consider the inclusions

$$\mathcal{O}_X \to \mathcal{O}_X(D) \to j_*\mathcal{O}_U.$$

Tensoring this sequence with $\mathcal{O}_X(K_X + \Delta)$ and passing to the long exact sequence in cohomology, for each $q \geq 0$ we obtain the diagram

$$\begin{tikzcd}
H^q(X, \mathcal{O}_X(K_X + \Delta)) \arrow[r, "\delta_q"] \arrow[dr, "i_q"'] & H^q(X, \mathcal{O}_X(K_X + \Delta + D)) \arrow[d] \\
 & H^q(X, j_*\mathcal{O}_U(K_U))
\end{tikzcd}$$

The injectivity of i_q implies that δ_q is injective, which is what we needed. $\square$

4.11 Residues*

In this section we prove Theorem 4.32. To this end, we need to extend the setup already used in the second proof of the Kodaira vanishing.

Definition 4.34. Let X be a smooth variety and let D be a reduced simple normal crossings divisor on X. Let $(x_1, \dots, x_r)$ be a local coordinate system around a point $p \in X$ such that D_j is locally given by x_j for $j = 1, \dots, s$. Denote

$$\delta_j = \begin{cases} \frac{dx_j}{x_j} & \text{if } j \leq s, \\ dx_j & \text{if } j > s, \end{cases}$$

and for $I = \{i_1, \dots, i_j\} \subseteq \{1, \dots, s\}$ with $i_1 < j_2 < \cdots < i_j$, set $\delta_I = \delta_{i_1} \wedge \cdots \wedge \delta_{i_j}$. For $a \geq 1$ and for each local section $\varphi \in \Omega_X^a(\log D)$, we can write

$$\varphi = \varphi_1 + \varphi_2 \wedge \delta_1,$$

where φ_1 is an a-form which lies in the span of δ_I with $1 \notin I$, and φ_2 is an $(a-1)$-form which lies in the span of δ_I with $1 \notin I$. Then we define the map

$$\begin{aligned}\beta_a \colon \Omega^a_X(\log D) &\to \Omega^{a-1}_{D_1}(\log(D - D_1)|_{D_1})\\ \varphi &\mapsto \varphi_2|_{D_1}.\end{aligned}$$

For $a \geq 0$, let

$$\gamma_a \colon \Omega^a_X(\log(D - D_1)) \to \Omega^a_{D_1}(\log(D - D_1)|_{D_1})$$

be the restriction of differential forms.

Then one can easily prove the following:

Proposition 4.35. *With notation from Definition 4.34, there are exact sequences*

$$\begin{aligned}0 \to \Omega^a_X(\log(D - D_1)) &\xrightarrow{\imath_a} \Omega^a_X(\log D)\\ &\xrightarrow{\beta_a} \Omega^{a-1}_{D_1}(\log(D - D_1)|_{D_1}) \to 0 \end{aligned} \tag{4.21}$$

and

$$\begin{aligned}0 \to \Omega^a_X(\log D) \otimes \mathcal{O}_X(-D_1) &\xrightarrow{\jmath_a} \Omega^a_X(\log(D - D_1))\\ &\xrightarrow{\gamma_a} \Omega^a_{D_1}(\log(D - D_1)|_{D_1}) \to 0, \end{aligned} \tag{4.22}$$

where $\imath_a$ and $\jmath_a$ are obvious inclusions of forms.

Proof. The maps β_a and γ_a are clearly surjective, so we only need to find the kernels of these maps.

With the conventions from Definition 4.34, the kernel of β_a consists of locally of forms $\varphi = \varphi_1 + \varphi_2 \wedge \delta_1$, where φ_2 is divisible by x_1. But these forms are clearly log forms which are regular along D_1, hence (4.21) follows.

For (4.22), each local section $\varphi \in \Omega^a_X(\log(D - D_1))$ can be written as $\varphi = \varphi_1 + \varphi_2 \wedge dx_1$, where φ_1 is an a-form which lies in the span of δ_I with $1 \notin I$, and φ_2 is an $(a-1)$-form which lies in the span of δ_I with $1 \notin I$. Then $\varphi \in \ker \gamma_a$ if and only if $\varphi_1 = x_1\varphi_1'$ for some a-form which lies in the span of δ_I with $1 \notin I$. This implies

$$\varphi = x_1\varphi_1' + \varphi_2 \wedge (x_1\delta_1) = x_1(\varphi_1' + \varphi_2 \wedge \delta_1),$$

which finishes the proof. □

Definition 4.36. With notation from Definition 4.34, let $\mathcal{E}$ be a locally free coherent sheaf on X. Let

$$\nabla \colon \mathcal{E} \to \Omega^1_X(\log D) \otimes \mathcal{E}$$

be a $\mathbb{C}$-linear map such that for local sections $f \in \mathcal{O}_X$ and $e \in \mathcal{E}$ we have

$$\nabla(fe) = f\nabla(e) + df \otimes e.$$

For local sections $\omega \in \Omega^a_X(\log D)$ and $e \in \mathcal{E}$ we define

$$\begin{aligned}\nabla_a \colon \Omega^a_X(\log D) \otimes \mathcal{E} &\to \Omega^{a+1}_X(\log D) \otimes E \\ \omega \otimes e &\mapsto d\omega \otimes e + (-1)^a \omega \wedge \nabla(e).\end{aligned}$$

If $\nabla_{a+1} \circ \nabla_a = 0$ for all a, then ∇ is an *(integrable logarithmic) connection along D*, and the complex $(\Omega^\bullet_X(\log D) \otimes \mathcal{E}, \nabla_\bullet)$ is the *logarithmic de Rham complex* of $(\mathcal{E}, \nabla)$. The *residue map* along D_1 is the map

$$\operatorname{Res}_{D_1}(\nabla) \colon \mathcal{E} \xrightarrow{\nabla} \Omega^1_X(\log D) \otimes \mathcal{E} \xrightarrow{\beta_1 \otimes \mathrm{id}_{\mathcal{E}}} \mathcal{O}_{D_1} \otimes \mathcal{E} = \mathcal{E}|_{D_1}.$$

We will only use connections in the case when $\mathcal{E}$ is a line bundle.

Lemma 4.37. *With notation from Definition 4.34, let $(\mathcal{E}, \Delta)$ be a locally free coherent sheaf on X with a connection along D. Then:*

(a) $\operatorname{Res}_{D_1}(\nabla)$ *is $\mathcal{O}_X$-linear, and there exists a map $\operatorname{Res}^0_{D_1}(\nabla) \colon \mathcal{E}|_{D_1} \to \mathcal{E}|_{D_1}$ such that we have the commutative diagram*

$$\begin{tikzcd} \mathcal{E} \arrow[r, "r"] \arrow[dr, "\operatorname{Res}_{D_1}(\nabla)"'] & \mathcal{E}|_{D_1} \arrow[d, "\operatorname{Res}^0_{D_1}(\nabla)"] \\ & \mathcal{E}|_{D_1} \end{tikzcd}$$

where r is the restriction to D_1,

(b) if $\imath_a \colon \Omega^a_X(\log(D - D_1)) \to \Omega^a_X(\log D)$ is the obvious inclusion, then for each $a \geq 0$ we have the commutative diagram

$$\begin{tikzcd} \Omega^a_X(\log(D-D_1)) \otimes \mathcal{E} \arrow[r, "\nabla_a \circ (\imath_a \otimes \mathrm{id}_{\mathcal{E}})"] \arrow[d, "\gamma_a \otimes \mathrm{id}_{\mathcal{E}}"'] & \Omega^a_X(\log D) \otimes \mathcal{E} \arrow[d, "\beta_a \otimes \mathrm{id}_{\mathcal{E}}"] \\ \Omega^a_{D_1}(\log(D-D_1)|_{D_1}) \otimes \mathcal{E} \arrow[d, equal] & \Omega^a_{D_1}(\log(D-D_1)|_{D_1}) \otimes \mathcal{E} \arrow[d, equal] \\ \Omega^a_{D_1}(\log(D-D_1)|_{D_1}) \otimes \mathcal{E}|_{D_1} \arrow[r, "((-1)^a \mathrm{id}) \otimes \operatorname{Res}^0_{D_1}(\nabla)"] & \Omega^a_{D_1}(\log(D-D_1)|_{D_1}) \otimes \mathcal{E}|_{D_1} \end{tikzcd}$$

Proof. Denote $\beta'_a = \beta_a \otimes \mathrm{id}_{\mathcal{E}}$. For local sections $f \in \mathcal{O}_X$ and $e \in \mathcal{E}$ we have

$$\operatorname{Res}_{D_1}(\nabla)(fe) = \beta'_1(f\nabla(e) + df \otimes e) = \beta'_1(f\nabla(e)) = f|_{D_1} \operatorname{Res}_{D_1}(\nabla)(e).$$

Since $r(fe) = f|_{D_1} \otimes e$, we set $\mathrm{Res}^0_{D_1}(\nabla)(f|_{D_1} \otimes e) = f|_{D_1} \mathrm{Res}_{D_1}(\nabla)(e)$, which proves (a).

For (b), let $\omega \in \Omega^a_X(\log(D - D_1))$ and $e \in \mathcal{E}$ be local sections. The form ω can be written as $\omega = \omega_1 + \omega_2 \wedge dx_1$, where ω_1 is an a-form which lies in the span of δ_I with $1 \notin I$, and ω_2 is an $(a-1)$-form which lies in the span of δ_I with $1 \notin I$. Then

$$\begin{aligned}\beta'_a(\nabla_a(\omega \otimes e)) &= \beta'_a(d\omega \otimes e + (-1)^a \omega \wedge \nabla(e)) = \beta'_a((-1)^a \omega \wedge \nabla(e)) \\ &= \beta'_a((-1)^a \omega_1 \wedge \nabla(e)) + \beta'_a((-1)^a x_1 \omega_2 \wedge \delta_1 \wedge \nabla(e)) \\ &= \beta'_a((-1)^a \omega_1 \wedge \nabla(e)) = (-1)^a \omega_1|_{D_1} \mathrm{Res}^0_{D_1}(\nabla)(e|_{D_1}).\end{aligned}$$

On the other hand, $\gamma_a(\omega) \otimes e = \omega_1|_{D_1} \otimes e$, and the conclusion follows. $\square$

Notation 4.38. Let X be a variety, let D be a Cartier divisor on X and let $\mathcal{F}$ be a sheaf on X. Then we use the notation $\mathcal{F}(D) = \mathcal{F} \otimes \mathcal{O}_X(D)$.

Lemma 4.39. *With notation from Definition 4.34, let $(\mathcal{E}, \Delta)$ be a locally free coherent sheaf on X with a connection along D. Let $B = \sum_{j=1}^r \mu_j D_j$ be any divisor supported on D. Then ∇ induces a connection with logarithmic poles ∇^B on $\mathcal{E}(B)$, and we have*

$$\mathrm{Res}^0_{D_1}(\nabla^B) = (\mathrm{Res}^0_{D_1}(\nabla) - \mu_1 \mathrm{id}_{D_1}) \otimes \mathrm{id}_{\mathcal{O}_X(B)}.$$

Proof. Let $\sigma = eb$ be a local section of $\mathcal{E}(B)$, for local sections $e \in \mathcal{E}$ and $b = \prod_{j=1}^s x_j^{-\mu_j} \in \mathcal{O}_X(B)$. We set

$$\nabla^B(\sigma) = b\nabla(e) + db \otimes e = b\nabla(e) + \sum_{k=1}^s (-\mu_k \delta_k b) \otimes e \in \Omega^1_X(\log D) \otimes \mathcal{E}(B),$$

and hence

$$\mathrm{Res}_{D_1}(\nabla^B(\sigma)) = b \otimes (\mathrm{Res}_{D_1}(\nabla(e)) - \mu_1 \otimes e|_{D_1}),$$

which finishes the proof. $\square$

Lemma 4.40. *With notation from Definition 4.34, let $(\mathcal{E}, \Delta)$ be a locally free coherent sheaf on X with a connection along D. Assume that the map $\mathrm{Res}^0_{D_1}(\nabla) \colon \mathcal{E}|_{D_1} \to \mathcal{E}|_{D_1}$ is an isomorphism. Then the inclusion of complexes*

$$(\Omega^\bullet_X(\log D) \otimes \mathcal{E}(-D_1), \nabla^{-D_1}_\bullet) \to (\Omega^\bullet_X(\log D) \otimes \mathcal{E}, \nabla_\bullet)$$

is a quasi-isomorphism.

In particular, let $B = \sum_{j=1}^r \mu_j D_j$ be any divisor supported on D. Then the complexes $(\Omega^\bullet_X(\log D), d)$ and $(\Omega^\bullet_X(\log D) \otimes \mathcal{O}_X(B), d)$ are quasi-isomorphic.

Proof. For each $a = 0, \dots, n$, let $\imath_a$ and $\jmath_a$ be the maps from Proposition 4.35, and consider the complex $\mathcal{E}^{(a)}$:

$$\begin{aligned}\mathcal{E}(-D_1) \xrightarrow{\nabla_1^{-D_1}} \dots \xrightarrow{\nabla_{a-2}^{-D_1}} & \ \Omega_X^{a-1}(\log D) \otimes \mathcal{E}(-D_1) \\ \xrightarrow{(\jmath_a \otimes \mathrm{id}_{\mathcal{E}}) \circ \nabla_{a-1}^{-D_1}} & \ \Omega_X^{a}(\log(D - D_1)) \otimes \mathcal{E} \\ \xrightarrow{\nabla_a \circ (\imath_a \otimes \mathrm{id}_{\mathcal{E}})} & \ \Omega_X^{a+1}(\log D) \otimes \mathcal{E} \xrightarrow{\nabla_{a+1}} \dots \xrightarrow{\nabla_{n-1}} \Omega_X^n(\log D) \otimes \mathcal{E}.\end{aligned}$$

We need to show that $\mathcal{E}^{(0)}$ and $\mathcal{E}^{(n)}$ are quasi-isomorphic. For $a \leq n-1$, there are inclusions $\mathcal{E}^{(a+1)} \to \mathcal{E}^{(a)}$ induced by $\imath_{a+1}$ and $\jmath_a$. Therefore, combining (4.21) and (4.22) with Lemma 4.37(b), we obtain that the complex $\mathcal{E}^{(a)}/\mathcal{E}^{(a+1)}$ is

$$0 \to \Omega_{D_1}^a(\log(D - D_1)|_{D_1}) \otimes \mathcal{E} \xrightarrow{((-1)^a \mathrm{id}) \otimes \mathrm{Res}_{D_1}^0(\nabla)} \Omega_{D_1}^a(\log(D - D_1)|_{D_1}) \otimes \mathcal{E} \to 0.$$

Since $\mathrm{Res}_{D_1}^0(\nabla)$ is an isomorphism, the complex $\mathcal{E}^{(a)}/\mathcal{E}^{(a+1)}$ has no cohomology, hence $\mathcal{E}^{(a)}$ and $\mathcal{E}^{(a+1)}$ are quasi-isomorphic.

For the second statement, observe that the residues $\mathrm{Res}_{D_i}^0(d)\colon \mathcal{O}_{D_i} \to \mathcal{O}_{D_i}$ associated to the standard differential on $\mathcal{O}_X$ are zero maps for all i. Then by Lemma 4.39, for every divisor $B = \sum_{j=1}^r \mu_j D_j$ with $\mu_i \geq 1$ the residue

$$\mathrm{Res}_{D_i}^0(d)\colon \mathcal{O}(B)|_{D_i} \to \mathcal{O}(B)|_{D_i}$$

is an isomorphism, hence the complexes

$$(\Omega_X^\bullet(\log D) \otimes \mathcal{O}_X(B - D_i), d) \quad \text{and} \quad (\Omega_X^\bullet(\log D) \otimes \mathcal{O}_X(B), d)$$

are quasi-isomorphic by the first statement of the lemma. Now the proof follows by induction on the sum $\sum_{j=1}^r \mu_j$. □

Finally, we have:

Proof of Theorem 4.32. Consider the complexes $\mathcal{C}_m = \Omega_X^\bullet(\log \Delta) \otimes \mathcal{O}_X(m\Delta)$ for $m \geq 0$. It is clear that $\mathcal{C}_m \subseteq \mathcal{C}_{m+1}$ for all m, and that these are subcomplexes of $j_* \Omega_U^\bullet$ such that $j_* \Omega_U^\bullet = \bigcup_{m \geq 0} \mathcal{C}_m$. By Lemma 4.40, the complexes $\mathcal{C}_0$ and $\mathcal{C}_m$ are quasi-isomorphic, hence $\mathcal{H}^i(\mathcal{C}_m/\mathcal{C}_0) = 0$ for every $m \geq 0$ and every i. Therefore,

$$\mathcal{H}^i(j_* \Omega_U^\bullet / \Omega_X^\bullet(\log \Delta)) = \varinjlim \mathcal{H}^i(\mathcal{C}_{m+1}/\mathcal{C}_0) = 0 \quad \text{for all } i,$$

which finishes the proof. □

4.12 Degeneration of Hodge-to-de Rham*

4.12.1 Good reduction modulo p

Let X be a complex projective variety and let Δ be a reduced divisor on X. We can assume that $X \subseteq \mathbb{P}^N_{\mathbb{C}}$ for some positive integer N, and hence X and Δ are defined in $\mathbb{P}^N_{\mathbb{C}}$ by a finite set of equations. Let $a_1, \dots, a_\ell$ be the coefficients of these equations, set $R = \mathbb{Z}[a_1, \dots, a_\ell]$, and let $\mathcal{X}$ and $\mathcal{D}$ be subvarieties of $\mathbb{P}^N_R$ cut out by the same equations. Then we have the Cartesian diagram

$$\begin{tikzcd} (X,\Delta) \arrow[r] \arrow[d] & (\mathcal{X},\mathcal{D}) \arrow[d, "f"] \\ \operatorname{Spec}\mathbb{C} \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

In other words, the fibre over the generic point $0 \in \operatorname{Spec} R$ of the map $f\colon \mathcal{X} \to \operatorname{Spec} R$ is precisely X. We want to investigate nearby fibres. First we have the following useful lemma.

Lemma 4.41. *The finitely generated ring R satisfies:*

(a) for each maximal ideal $\mathfrak{m} \subseteq R$, the residue field $R/\mathfrak{m}$ is finite,

(b) the set of maximal ideals of $\operatorname{Spec} R$ *is dense in* $\operatorname{Spec} R$.

Proof. For (a), denote $k = R/\mathfrak{m}$ and $S = \mathbb{Z}/\mathbb{Z}\cap\mathfrak{m}$. Then k is a finitely generated S-algebra, hence k is a finite extension of the quotient field of S by [Mat89, Theorem 5.2]. Assume that k is infinite. This implies that S is infinite, hence $\mathbb{Z} \cap \mathfrak{m} = \{0\}$ and k is a finite dimensional $\mathbb{Q}$-vector space with basis $e_1, \dots, e_m$. Denoting $\bar{a}_i = a_i \mod \mathfrak{m} \in k$, there exists an integer λ such that $\lambda\bar{a}_i \in \bigoplus_{j=1}^m \mathbb{Z}e_i$, hence $k \subseteq \bigoplus_{j=1}^m \mathbb{Z}[1/\lambda]e_i$, which is clearly impossible.

The claim (b) is equivalent to the claim that the intersection of all maximal ideals of R is $\{0\}$. Assume that this intersection contains a nonzero $\varphi \in R$, and pick a maximal ideal $\mathfrak{n} \in R_\varphi$. Since the residue field $R_\varphi/\mathfrak{n}$ is finite by (a), its subring $R/R \cap \mathfrak{n}$ is a finite integral domain, hence a field. This implies that $R \cap \mathfrak{n}$ is a maximal ideal of R which does not contain φ, a contradiction. □

Now, by shrinking $\operatorname{Spec} R$ (in other words, by replacing R by the localisation R_φ for some nonzero $\varphi \in R$) we may assume that the map f is smooth. The sheaves $R^j f_*\Omega^i_{\mathcal{X}/R}$ are coherent by [Har77, Theorem III.8.8], and hence by the Hodge-to-de Rham spectral sequence, so are the sheaves $R^j f_*\Omega^\bullet_{\mathcal{X}/R}$. By shrinking $\operatorname{Spec} R$ again, we may assume that all these sheaves are locally free of finite rank: indeed, the stalk of $R^j f_*\Omega^i_{\mathcal{X}/R}$ at the generic point of $\operatorname{Spec} R$ is a finite dimensional vector space over the field of fractions of R, hence we conclude by [Har77, Exercise II.5.7]. Therefore, by [Har77, Theorem III.12.11], the functions $\dim_{k(s)} H^j(X_s, \Omega^i_{X_s}(\log \Delta_s))$ and $\dim_{k(s)} \mathbb{H}^j(X_s, \Omega^\bullet_{X_s}(\log \Delta_s))$ are locally constant on $\operatorname{Spec} R$, where for $s \in \operatorname{Spec} R$ we denote by X_s the fibre of f over s and by $k(s)$ the residue field at s. Additionally, by shrinking $\operatorname{Spec} R$ yet again, we may assume that the map $\operatorname{Spec} R \to \operatorname{Spec} \mathbb{Z}$ is smooth.

Now we return to the question of the degeneration of the Hodge-to-de Rham spectral sequence at E_1, Theorem 4.33, in characteristic zero. The goal of the construction is to prove the corresponding statement in characteristic p for sufficiently large p and under certain additional restrictions, and to lift the result to characteristic zero via the preceding reduction argument. Recall that by (1.6) we have to show that

$$\dim_{\mathbb{C}} \mathbb{H}^k(X, \Omega^\bullet_X(\log \Delta)) = \sum_{p+q=k} \dim_{\mathbb{C}} H^q(X, \Omega^p_X(\log \Delta)) \quad \text{for all } k.$$

By shrinking $\operatorname{Spec} R$, we may assume that the number $d = \dim X$ is invertible in R, and hence the residue field $k(s)$ for every $s \in \operatorname{Spec} R$ is of characteristic $p_s > d$.

Fix a closed point $\mathfrak{m} \in \operatorname{Spec} R$ mapping to a point $(p) \in \operatorname{Spec} \mathbb{Z}$ and with a residue field k of characteristic p. Since $\operatorname{Spec} R$ is smooth over $\operatorname{Spec} \mathbb{Z}$, we have $\mathfrak{m} = (p)\mathcal{O}_{\operatorname{Spec} R}$, and hence $\mathcal{O}_{\operatorname{Spec} R}/\mathfrak{m}^2 \times_{\mathbb{Z}/p^2} \mathbb{Z}/p = k$. It can be shown that this defines $\mathcal{O}_{\operatorname{Spec} R}/\mathfrak{m}^2$ uniquely up to isomorphism: it is isomorphic to the *Witt ring* $W_2(k) = k \times k$ *of vectors of length* 2, where the operations are given by

$$\begin{aligned} (a_1, a_2) + (b_1, b_2) &= \left(a_1 + b_1, a_2 + b_2 - \tfrac{1}{p} \textstyle\sum_{i=1}^{p-1} \binom{p}{i} a_1^i b_1^{p-i}\right), \\ (a_1, a_2) \cdot (b_1, b_2) &= (a_1 b_1, b_1^p a_2 + a_1^p b_2). \end{aligned}$$

Note that $W_2(\mathbb{Z}/p) = \mathbb{Z}/p^2$. Denote as above by $X_{\mathfrak{m}}$ the fibre of f over $\mathfrak{m}$ and $\tilde{X}_{\mathfrak{m}} = \mathcal{X} \times_{\operatorname{Spec} R} \operatorname{Spec} W_2(k)$, and similarly for $\Delta_{\mathfrak{m}}$ and $\tilde{\Delta}_{\mathfrak{m}}$. Then we have the following diagram of Cartesian squares:

$$\begin{tikzcd} (X_{\mathfrak{m}}, \Delta_{\mathfrak{m}}) \arrow[r] \arrow[d] & (\tilde{X}_{\mathfrak{m}}, \tilde{\Delta}_{\mathfrak{m}}) \arrow[r] \arrow[d] & (\mathcal{X}, \mathcal{D}) \arrow[d, "f"] \\ \operatorname{Spec} k \arrow[r] & \operatorname{Spec} W_2(k) \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

Note that the importance of the map $\tilde{X}_{\mathfrak{m}} \to \operatorname{Spec} W_2(k)$ lies in the fact that unlike in k, in $W_2(k)$ the element p is invertible. This inspires the following definition.

Definition 4.42. Let X be variety defined over a perfect field k of characteristic p, and let $D = \sum_{j=1}^r D_j$ be a divisor on X. We say that the pair (X, D) can be lifted to $W_2(k)$ if there exist a variety $\tilde{X}$ and divisors $\tilde{D}_j$ over $\operatorname{Spec} W_2(k)$ such that $X = \tilde{X} \times_{\operatorname{Spec} W_2(k)} \operatorname{Spec} k$ and $D_j = \tilde{D}_j \times_{\operatorname{Spec} W_2(k)} \operatorname{Spec} k$.

Therefore, by construction $X_{\mathfrak{m}}$ is a smooth projective scheme of dimension smaller than p which has a lifting $\tilde{X}_{\mathfrak{m}}$ over $W_2(k)$. Hence, in order to prove Theorem 4.33, it suffices to show the following.

Theorem 4.43. *Let X be a smooth projective scheme defined over a perfect field of characteristic p such that $\dim X < p$, and let Δ be a reduced simple normal crossings divisor on X. Assume that the pair (X, Δ) can be lifted to $W_2(k)$. Then the Hodge-to-de Rham spectral sequence*

$$E_1^{p,q} = H^q(X, \Omega_X^p(\log \Delta)) \underset{p}{\Longrightarrow} \mathbb{H}^{p+q}(X, \Omega_X^\bullet(\log \Delta))$$

degenerates at E_1.

Remark 4.44. With assumptions and notation from Definition 4.42, assume that $k = \mathbb{Z}/p$. Then there is an inclusion $X \subseteq \tilde{X}$ such that the ideal sheaf I_X of X in $\tilde{X}$ satisfies $I_X^2 = 0$, hence the inclusion is the identity on the underlying topological spaces. Since $\tilde{X}$ is flat over $\operatorname{Spec} \mathbb{Z}/p^2$, tensoring the exact sequence of $\mathbb{Z}/p^2$-modules

$$0 \to p \cdot \mathbb{Z}/p^2 \to \mathbb{Z}/p^2 \to \mathbb{Z}/p \to 0$$

by $\mathcal{O}_{\tilde{X}}$ we get the exact sequence of $\mathcal{O}_{\tilde{X}}$-modules

$$0 \to p \cdot \mathcal{O}_{\tilde{X}} \to \mathcal{O}_{\tilde{X}} \to \mathcal{O}_X \to 0.$$

Furthermore, from the isomorphism of $\mathbb{Z}/p^2$-modules $\mathbb{Z}/p \simeq p \cdot \mathbb{Z}/p^2$ we have the isomorphism of $\mathcal{O}_{\tilde{X}}$-modules

$$\mathbf{p}\colon \mathcal{O}_X \to p \cdot \mathcal{O}_{\tilde{X}}.$$

4.12.2 Frobenius

Let X be an n-dimensional scheme of characteristic p, i.e. assume that there exists a morphism $X \to \operatorname{Spec} \mathbb{Z}/p$ which factors the morphism $X \to \operatorname{Spec} \mathbb{Z}$. Then the *absolute Frobenius morphism* of X is the endomorphism $F_X\colon X \to X$ which is the identity on the underlying topological space of X, and the map $F_X^*\colon \mathcal{O}_X \to \mathcal{O}_X$ is locally given by $F_X^*(x) = x^p$. If $f\colon X \to Y$ is a morphism of schemes, then there is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "F_X"] \arrow[d, "f"'] & X \arrow[d, "f"] \\ Y \arrow[r, "F_Y"] & Y \end{tikzcd}$$

Denote $X^{(p)} = Y \times_Y X$ induced by the map F_Y. Then the above diagram factors as

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[dr, "f"'] & X^{(p)} \arrow[r, "\mathrm{pr}_1"] \arrow[d] & X \arrow[d, "f"] \\ & Y \arrow[r, "F_Y"] & Y \end{tikzcd}$$

where the map $F = F_{X/Y} \colon X \to X^{(p)}$ is the *relative Frobenius of* X *over* Y. Here, for local sections $x \in \mathcal{O}_X$ and $y \in \mathcal{O}_Y$ we have $F^*(x \otimes y) = x^p y$ and $\mathrm{pr}_1^*(x) = x \otimes 1$. Explicitly, locally we have $X = \mathcal{O}_Y[t_1, \ldots, t_r]/(f_1, \ldots, f_s)$, and $F^*(t_i) = t_i^p$ and $\mathrm{pr}_1^*(yt_i) = y_i^p t_i$. Note that F and pr_1 are isomorphisms on topological spaces, but note that, in general, X and $X^{(p)}$ are not isomorphic as schemes over Y.

Furthermore, if the morphism f is smooth, then F is a finite flat morphism of degree p. Indeed, we have locally the following diagram:

$$\begin{tikzcd} X \arrow[r, "F"] \arrow[d, "\pi"'] & X^{(p)} \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\pi^{(p)}"] & X \arrow[d, "\pi"] \\ \mathbb{A}^n_Y \arrow[r, "F_{\mathbb{A}^n_Y/Y}"] \arrow[dr] & (\mathbb{A}^n_Y)^{(p)} \arrow[r, "\mathrm{pr}_1"] \arrow[d] & \mathbb{A}^n_Y \arrow[d] \\ & Y \arrow[r, "F_Y"] & Y \end{tikzcd} \tag{4.23}$$

Since $\mathbb{A}^n_Y = (\mathbb{A}^n_Y)^{(p)} = \mathcal{O}_Y[t_1, \ldots, t_n]$ and $F^{-1}\mathcal{O}_{(\mathbb{A}^n_Y)^{(p)}} = \mathcal{O}_Y[t_1^p, \ldots, t_n^p]$, the sheaf $F_*\mathcal{O}_{\mathbb{A}^n_Y}$ is a free $\mathcal{O}_{(\mathbb{A}^n_Y)^{(p)}}$-module generated by monomials $t_1^{a_1} \cdots t_n^{a_n}$ for $0 \le a_i \le p-1$. Therefore, the morphism $F_{\mathbb{A}^n_Y/Y}$ is étale and the upper left square in the diagram is Cartesian, hence F is also étale. More generally, for any locally free sheaf $\mathcal{F}$ on X, the sheaf $F_*\mathcal{F}$ is locally free on $X^{(p)}$.

Definition 4.45. Assume that X is a scheme over $\operatorname{Spec} \mathbb{Z}/p$ which has a lifting $\tilde{X}$ to $\mathbb{Z}/p^2$. Then a lifting $\tilde{F}_{\tilde{X}}$ of F_X is a finite morphism $\tilde{F}_X \colon \tilde{X} \to \tilde{X}$ such that $\tilde{F}_{\tilde{X}}|_X = F_X$.

Let $S = \operatorname{Spec} k$, where k is a perfect field of characteristic p, and let X be a smooth S-scheme, where $S = \operatorname{Spec} k$. Assume that X and $X^{(p)}$ have liftings $\tilde{X}$ and $\tilde{X}^{(p)}$, respectively, to $\tilde{S} = \operatorname{Spec} W_2(k)$. Then a lifting $\tilde{F} = \tilde{F}_{X/\tilde{S}}$ of the relative Frobenius $F = F_{X/S}$ is a finite morphism $\tilde{F} \colon \tilde{X} \to \tilde{X}^{(p)}$ such that $\tilde{F}|_X = F$.

Remark 4.46. With notation from Definition 4.45, note that since the absolute Frobenius F_S is an isomorphism, so is the relative Frobenius $F \colon X \to X^{(p)}$. In particular, X has a lifting to $W_2(k)$ if and only if $X^{(p)}$ does.

From Remark 4.44 we obtain the exact sequence of $\mathcal{O}_{\tilde{X}^{(p)}}$-modules

$$0 \to p \cdot \tilde{F}_*\mathcal{O}_{\tilde{X}} \to \tilde{F}_*\mathcal{O}_{\tilde{X}} \to F_*\mathcal{O}_X \to 0$$

and an $\mathcal{O}_{\tilde{X}^{(p)}}$-isomorphism

$$\mathbf{p} \colon F_*\mathcal{O}_X \to p \cdot \tilde{F}_*\mathcal{O}_{\tilde{X}}.$$

Similarly, for every $i \ge 0$ one has an exact sequence of $\mathcal{O}_{\tilde{X}}$-modules

$$0 \to p \cdot \Omega^i_{\tilde{X}/\tilde{S}}(\log \tilde{\Delta}) \to \Omega^i_{\tilde{X}/\tilde{S}}(\log \tilde{\Delta}) \to \Omega^i_{X/S}(\log \Delta) \to 0$$

and an $\mathcal{O}_{\tilde{X}}$-isomorphism

$$\mathbf{p}\colon \Omega^i_{X/S}(\log\Delta) \to p\cdot\Omega^i_{\tilde{X}/\tilde{S}}(\log\tilde{\Delta}).$$

Lemma 4.47. *Let $S = \operatorname{Spec} k$, where k is a perfect field of characteristic p, let X be a smooth S-scheme, and let Δ be a simple normal crossing divisor relative to S. Let $(\tilde{X}^{(p)}, \tilde{\Delta}^{(p)})$ be a lifting of $(X^{(p)}, \Delta^{(p)})$. Then locally in the Zariski topology, (X, Δ) has a lifting $(\tilde{X}, \tilde{\Delta})$ such that F lifts to $\tilde{F}\colon \tilde{X} \to \tilde{X}^{(p)}$ with $\tilde{F}^*\mathcal{O}_{\tilde{X}^{(p)}}(-\tilde{\Delta}^{(p)}) = \mathcal{O}_{\tilde{X}}(-p\tilde{\Delta})$.*

Proof. By Remark 4.46, we only need to show the existence of the lifting of F locally. Locally we have an étale morphism

$$\pi\colon X \to \mathbb{A}^n_S = \operatorname{Spec}\mathcal{O}_S[t_1, \ldots, t_n]$$

such that Δ_j is defined by $\varphi_j = \pi^*(t_j)$. It is clear that we can choose liftings $\tilde{\varphi}_j \in \mathcal{O}_{\tilde{X}}$ of φ_j to which are local equations of $\tilde{\Delta}_j$, and we can choose liftings $\tilde{\varphi}_j^{(p)}$ of $\varphi_j^{(p)} = F_S^*(\varphi_j) = \varphi_j \otimes 1$ which are local equations of $\tilde{\Delta}_j^{(p)}$. Now if we define $\tilde{F}$ by $\tilde{F}^*(\tilde{\varphi}_j^{(p)}) = \tilde{\varphi}_j^p$, it is clear that $\tilde{F}$ lifts F and satisfies the conclusion of the lemma. $\square$

4.12.3 Cartier operator

We start with the following important result.

Theorem 4.48. *Let $S = \operatorname{Spec} k$, where k is a perfect field of characteristic p, and let X be a smooth S-scheme. Let Δ be a simple normal crossings divisor on X over S. Consider the map of $\mathcal{O}_{X^{(p)}}$-algebras*

$$\gamma = \bigoplus \gamma^i\colon \bigoplus \Omega^i_{X^{(p)}/S}(\log\Delta^{(p)}) \to \bigoplus \mathcal{H}^i(F_*\Omega^\bullet_{X/S}(\log\Delta))$$

which satisfies:

(a) γ^0 is the morphism $\mathcal{O}_{X^{(p)}} \to F_*\mathcal{O}_X$,

(b) for a local section $x \in \mathcal{O}_X$ and a local generator t of Δ we have

$$\gamma^1(d(x\otimes 1)) = [x^{p-1}dx] \quad \textit{and} \quad \gamma^1(d(t\otimes 1)/t\otimes 1) = [dt/t],$$

(c) $\gamma^i = \bigwedge^i \gamma^1$.

Then γ is an isomorphism.

Proof. The map γ^0 is an isomorphism since X is smooth over S by the discussion before Definition 4.45. We show first that all γ^i are well defined, and observe that it suffices to prove the claim for γ^1, since $\Omega^1_{X^{(p)}/S}(\log \Delta^{(p)})$ is generated by elements of the form $d(x \otimes 1)$ and $d(t \otimes 1)/t \otimes 1$.

To this end, the identity

$$(x+y)^{p-1}d(x+y) - x^{p-1}dx - y^{p-1}dy = d\left(\frac{1}{p}\sum_{i=1}^{p-1}\binom{p}{i}x^i y^{p-i}\right)$$

implies that $\gamma^1\big(d((x+y)\otimes 1)\big) = \gamma^1\big(d(x\otimes 1)\big) + \gamma^1\big(d(y \otimes 1)\big)$, and the identity $(xy)^{p-1}d(xy) = x^p y^{p-1}dy + y^p x^{p-1}dx$ shows that γ^1 is compatible with the derivation rule on $X^{(p)}$. Furthermore,

$$\gamma^1(d(t\otimes 1)) = \gamma^1\big((t\otimes 1)d(t\otimes 1)/t\otimes 1\big) = F^*(t\otimes 1)\gamma^1\big(d(t\otimes 1)/t \otimes 1\big) = [t^p dt/t],$$

hence the relations in (b) are compatible.

Now we show that the maps γ^i are isomorphisms. From the local Cartesian square (4.23), since the morphisms π and $\pi^{(p)}$ are étale, we deduce that it is enough to prove that γ^i are isomorphisms in the case when $X = \mathbb{A}^n_S = \operatorname{Spec} \mathcal{O}_S[t_1, \ldots, t_n]$ and D is the zero set of $t_1 \cdots t_r$. Further, by extension of scalars, it is enough to prove the claim when $S = \operatorname{Spec} \mathbb{Z}/p$.

In this case, for each a, the sheaf $F_*\Omega^a_{\mathbb{A}^n_S}(\log \Delta)$ is associated to the $\mathbb{Z}/p$-vector space freely generated by $t_1^{a_1}\cdots t_n^{a_n}\omega_{i_1}\wedge\cdots\wedge\omega_{i_a}$ for $0 \le a_i \le p-1$ and $1 \le i_1 < \cdots < i_a \le n$, where $\omega_i = dt_i/t_i$ if $i \le r$, and $\omega_i = dt_i$ otherwise. We are thus reduced to proving the following:

(i) $\mathcal{H}^0(\mathcal{F}^\bullet) = \mathbb{Z}/p$,

(ii) $\mathcal{H}^1(\mathcal{F}^\bullet) = \bigoplus_{i=1}^r (\mathbb{Z}/p)\frac{dt_i}{t_i} \oplus \bigoplus_{i=r+1}^n (\mathbb{Z}/p)t_i^{p-1}dt_i$,

(iii) $\mathcal{H}^a(\mathcal{F}^\bullet) = \bigwedge^a \mathcal{H}^1(\mathcal{F}^\bullet)$ for $a \ge 1$.

To show this, consider the complexes $\mathcal{A}^\bullet = (\mathcal{A}^0 \to \mathcal{A}^1)$ and $\mathcal{B}^\bullet = (\mathcal{B}^0 \to \mathcal{B}^1)$ with the standard differentiation, where

$$\mathcal{A}^0 = \mathcal{B}^0 = \bigoplus_{i=0}^{p-1}(\mathbb{Z}/p)t^i, \quad \mathcal{A}^1 = \bigoplus_{i=0}^{p-1}(\mathbb{Z}/p)t^i dt, \quad \mathcal{B}^1 = \bigoplus_{i=0}^{p-1}(\mathbb{Z}/p)t^i\frac{dt}{t}.$$

It is easy to see that $\mathcal{H}^0(\mathcal{A}^\bullet) = \mathcal{H}^0(\mathcal{B}^\bullet) = \mathbb{Z}/p$, $\mathcal{H}^1(\mathcal{A}^\bullet) = (\mathbb{Z}/p)t^{p-1}dt$ and $\mathcal{H}^1(\mathcal{B}^\bullet) = (\mathbb{Z}/p)dt/t$. Since $\mathcal{F}^\bullet \simeq \bigotimes_{i=1}^r \mathcal{B}^\bullet \otimes_{\mathbb{Z}/p} \bigotimes_{i=r+1}^n \mathcal{A}^\bullet$, the conclusion follows from the Künneth formula, cf. Example 1.43. □

Let us try to reinterpret this result. The algebra $\bigoplus \Omega^i_{X^{(p)}/S}(\log \Delta^{(p)})$ can be considered as a complex

$$\Omega^\bullet = \bigoplus \Omega^i_{X^{(p)}/S}(\log \Delta^{(p)})[-i],$$

where $\Omega^i = \Omega^i_{X^{(p)}/S}(\log \Delta^{(p)})$ and the differentials are $d = 0$. Then Theorem 4.48 says that there is an isomorphism

$$\mathcal{H}^i(\Omega^\bullet) \simeq \mathcal{H}^i(F_*\Omega^\bullet_{X/S}(\log \Delta)).$$

This suggests (or is at least wishful thinking) that there should exist a morphism of complexes $\Omega^\bullet \to F_*\Omega^\bullet_{X/S}(\log \Delta)$ which is a quasi-isomorphism. In practice, this is too much to ask unless we are in very special circumstances; however, under some additional hypotheses, something similar holds.

Theorem 4.49. *Let k be a perfect field of characteristic p, and let X be a smooth scheme over $S = \operatorname{Spec} k$ such that $\dim X < p$. Assume that there exists a lifting of X to $W_2(k)$. Then there exists a complex of $\mathcal{O}_{X^{(p)}}$-modules $\mathcal{C}^\bullet$ and a diagram of morphisms of complexes*

$$\Omega^\bullet \xrightarrow{\theta} \mathcal{C}^\bullet \xleftarrow{\xi} F_*\Omega^\bullet_{X/S}(\log \Delta)$$

such that θ and ξ are quasi-isomorphisms.

Let us show how this immediately implies Theorem 4.43, and hence Theorem 4.33. First note that F_S is a flat morphism, therefore by base change we have $H^j(X^{(p)}, \Omega^i_{X^{(p)}/k}(\log \Delta^{(p)})) \simeq H^j(X, \Omega^i_{X/k}(\log \Delta)) \otimes_{F_S} \operatorname{Spec} k$ for all i, j, and in particular

$$\dim_k H^j(X^{(p)}, \Omega^i_{X^{(p)}/k}(\log \Delta^{(p)})) = \dim_k H^j(X, \Omega^i_{X/k}(\log \Delta)).$$

Since the relative Frobenius F is an isomorphism (since F_S is), we have

$$\begin{aligned}\mathbb{H}^a(X, \Omega^\bullet_{X/k}(\log \Delta)) &= \mathbb{H}^a(X^{(p)}, F_*\Omega^\bullet_{X/k}(\log \Delta)) \simeq \mathbb{H}(X^{(p)}, \mathcal{C}^\bullet) \\ &\simeq \mathbb{H}^a(X^{(p)}, \Omega^\bullet) = \bigoplus_i H^{a-i}(X^{(p)}, \Omega^i_{X^{(p)}/S}(\log \Delta^{(p)}),\end{aligned}$$

and hence

$$\dim_k \mathbb{H}^a(X, \Omega^\bullet_{X/k}(\log \Delta)) = \sum_i \dim_k H^{a-i}(X, \Omega^i_{X/k}(\log \Delta)),$$

which suffices by (1.6).

4.12.4 Proof of Theorem 4.49

Fix an affine cover $\mathcal{U}$ of $X^{(p)}$, and set $\mathcal{C}^\bullet = \check{\mathcal{C}}(\mathcal{U}, F_*\Omega^\bullet_{X/S}(\log \Delta))$; we will actually choose precisely the covering during the course of the proof. Let

$$\xi\colon F_*\Omega^\bullet_{X/S}(\log \Delta) \to \mathcal{C}^\bullet$$

be the natural map defined in Example 1.44. Then ξ is automatically a quasi-isomorphism, and we need to construct morphisms $\theta^i\colon \Omega^i \to \mathcal{C}^i$; note that we can view θ^i as a morphism of complexes $\theta^i\colon \Omega^i[-i] \to \mathcal{C}$. Assume first that θ^1 is defined, and let us show how to construct θ^i for $i \geq 1$.

Indeed, if S_i is the symmetric group of the set with i elements, then we define a morphism

$$\delta_i\colon \Omega^i[-i] = \bigwedge^i \Omega^1_{X^{(p)}/S}(\log D^{(p)})[-1] \to \left(\Omega^1_{X^{(p)}/S}(\log D^{(p)})[-1]\right)^{\otimes i} = (\Omega^1[-1])^{\otimes i}$$

by

$$\delta_i(\omega_1 \otimes \cdots \otimes \omega_i) = \frac{1}{i!}\sum_{\sigma \in S_i} \operatorname{sgn}(\sigma) \cdot \omega_{\sigma(1)} \wedge \cdots \wedge \omega_{\sigma(i)}.$$

Note that this is well defined since $i \leq \dim X < p$. Then the map θ^i is the composite

$$\Omega^i[-i] \xrightarrow{\delta_i} (\Omega^1[-1])^{\otimes i} \xrightarrow{(\theta^1)^{\otimes i}} (\mathcal{C})^{\otimes i} \to \mathcal{C},$$

where the last map is the evaluation map. Since γ is a graded algebra homomorphism and θ^i are multiplicative, if θ^1 induces an isomorphism on cohomology, then so do all θ^i.

It remains to construct θ^1, and we do it in two steps. Assume first that (X, Δ) and $(X^{(p)}, \Delta^{(p)})$ lift to $(\tilde{X}, \tilde{\Delta})$ and $(\tilde{X}^{(p)}, \tilde{\Delta}^{(p)})$ respectively, and that F lifts to $\tilde{F}\colon \tilde{X} \to \tilde{X}^{(p)}$ with $\tilde{F}^*\mathcal{O}_{\tilde{X}^{(p)}}(-\tilde{\Delta}^{(p)}) = \mathcal{O}_{\tilde{X}}(-p\tilde{\Delta})$. Then we set $\mathcal{U} = \{X^{(p)}\}$, and we will construct a map $\theta^1\colon \Omega^1_{X^{(p)}/S}(\log \Delta^{(p)}) \to F_*\Omega^\bullet_{X/S}(\log \Delta)$ such that θ^1 induces isomorphism on $\mathcal{H}^1$.

The morphism

$$F^*\colon \Omega^1_{X^{(p)}/S}(\log \Delta^{(p)}) \to F_*\Omega^1_{X/S}(\log \Delta)$$

is the zero map since $F^*(d(t \otimes 1)) = d(t^p) = 0$. By Remark 4.46, we have a commutative diagram

$$\begin{tikzcd}
\Omega^1_{X^{(p)}/S}(\log \Delta^{(p)}) \arrow[r, "F^*"] \arrow[d, "\mathbf{p}"', "\simeq"] & F_*\Omega^1_{X/S}(\log \Delta) \arrow[d, "\mathbf{p}", "\simeq"'] \\
p\Omega^1_{\tilde{X}^{(p)}/\tilde{S}}(\log \tilde{\Delta}^{(p)}) \arrow[r, "\tilde{F}^*"] & pF_*\Omega^1_{\tilde{X}/\tilde{S}}(\log \tilde{\Delta})
\end{tikzcd}$$

and thus we obtain the induced map

$$\mathbf{p}^{-1} \circ \tilde{F}^* \colon \Omega^1_{X^{(p)}/S}(\log \Delta^{(p)}) \to F_* \Omega^1_{X/S}(\log \Delta).$$

To spell this out in local coordinates, let $\tilde{x} \in \mathcal{O}_{\tilde{X}}$ be a lifting of $x \in \mathcal{O}_X$, and let $\tilde{x}^{(p)} \in \mathcal{O}_{\tilde{X}^{(p)}}$ be a lifting of $x^{(p)} = x \otimes 1 \in \mathcal{O}_{X^{(p)}}$. Then

$$\tilde{F}^*(\tilde{x}^{(p)}) = \tilde{x}^p + \tilde{u}$$

for some local section $\tilde{u} \in \mathcal{O}_{\tilde{X}}$. Restricting this relation to X, and remembering that $\tilde{F}|_X = F$ and $F^*(x^{(p)}) = x^p$, we get $\tilde{u}|_X = 0$, hence $\tilde{u} = p \cdot u$ for some local section $u \in \mathcal{O}_{\tilde{X}}$ by Remark 4.44. Then

$$\mathbf{p}^{-1}\big(\tilde{F}^*(d\tilde{x}^{(p)})\big) = x^{p-1}dx + d\bar{u} \in F_*\Omega^1_{X/S}(\log \Delta),$$

and we define θ^1 as a composition of $\mathbf{p}^{-1} \circ \tilde{F}^*$ and the inclusion $F_*\Omega^1_{X/S}(\log \Delta)[-1] \to F_*\Omega^\bullet_{X/S}(\log \Delta)$.

Now we return to the general situation. By Lemma 4.47, there is a covering $\mathcal{U} = \{U_i\}$ such that $(U_i, \Delta|_{U_i})$ and $(U_i^{(p)}, \Delta^{(p)}|_{U_i^{(p)}})$ lift to $(\tilde{U}_i, \tilde{\Delta}|_{\tilde{U}_i})$ and $(\tilde{U}_i^{(p)}, \tilde{\Delta}^{(p)}|_{\tilde{U}_i^{(p)}})$ respectively, and that $F_i = F|_{U_i}$ lifts to $\tilde{F}_i \colon \tilde{U}_i \to \tilde{U}_i^{(p)}$ with $\tilde{F}_i^* \mathcal{O}_{\tilde{U}_i^{(p)}}(-\tilde{\Delta}^{(p)}) = \mathcal{O}_{\tilde{U}_i}(-p\tilde{\Delta})$. We have to define a map

$$\theta^1 \colon \Omega^1_{X^{(p)}/S}(\log \Delta^{(p)}) \to \check{\mathcal{C}}^1(\mathcal{U}, F_*\mathcal{O}_{X/S}) \oplus \check{\mathcal{C}}^0(\mathcal{U}, F_*\Omega^1_{X/S}(\log \Delta)).$$

Fix a local section of $x \in \mathcal{O}_X$ and its any lift $\tilde{x} \in \mathcal{O}_{\tilde{X}}$. Denote $\omega = dx \otimes 1 \in \Omega^1_{X^{(p)}/S}(\log \Delta^{(p)})$. Using the notation as above, we have

$$\tilde{F}_i^*(\tilde{x}^{(p)}|_{\tilde{U}_i}) = \tilde{x}^p + p \cdot u_i(\tilde{x})$$

for some section $u_i(\tilde{x}) \in \mathcal{O}_{\tilde{U}_i}$, and set

$$h_{ij}(\omega) = u_j(\tilde{x})|_{\tilde{U}_i \cap \tilde{U}_j} - u_i(\tilde{x})|_{\tilde{U}_i \cap \tilde{U}_j}.$$

Then we define θ^1 as the map

$$\omega \mapsto ((h_{ij}(\omega))_{ij}, (\theta^1_i(\omega|_{U_i}))_i)$$

It is easy to check that θ^1 is independent of choices of lifts $\tilde{x}$ up to coboundary, and it satisfies $d\theta^1 = 0$. Further, it is also easy to see that the morphism does not change upon passing to a refinement of $\mathcal{U}$, hence is independent of the choice of the cover $\mathcal{U}$. Finally, θ^1 induces isomorphism on $\mathcal{H}^1$ since this question is local, hence it follows from the first step. This concludes the proof of Theorem 4.49, and thus that of Theorem 4.33.

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