Algebraic Geometry: The Minimal Model Program

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Preface

These notes are based on my course in the Summer Semester 2014 at the University of Bonn.

The notes will grow non-linearly during the course. That means two things: first, I will try and update the material weekly as the course goes on, but the material will not be in 1-1 correspondence with what is actually said in the course. Second, it is quite possible that chapters will simultaneously grow. I try to be pedagogical, and introduce new concepts only when/if needed.

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Chapter 1

The Minimal Model Program

In this chapter I will first introduce the classification procedure of algebraic varieties. I try to convince you that the classification criterion is natural and I give several motivations which lead to the same goal. From this point of view, it turns out that the classification criterion is *necessarily* the one explained in these notes – in other words, even if you try to come up with a different criterion, it will likely not be giving you anything better.

I always work over the field $\mathbb{C}$ of complex numbers; however everything in this course holds for any algebraically closed field k.

1.1 Motivation

1.1.1 Curves and surfaces

The classification of curves is classical and was done in the 19th century. The rough classification is according to the genus of a smooth projective curve.

The situation with surfaces is already more complicated. If we start with a smooth projective surface, and want our classification procedure to simplify it in tangible ways, we would therefore want some basic invariants, like the Picard number, to be as minimal as possible. To this end, recall that if $\pi\colon Y \to X$ is a blow up of a point on a smooth surface X, then the exceptional divisor $E \subseteq Y$ is a (-1)-curve, that is $E \simeq \mathbb{P}^1$ and $E^2 = -1$. The starting point of the classification of surfaces is if we start with a (-1)-curve on Y, we can invert the blowup construction:

Theorem 1.1 (Castelnuovo contraction, [Har77, Theorem V.5.7])**.** *Let Y be a nonsingular projective surface containing a (-1)-curve E. Then there exists a birational morphism $f\colon Y \to X$ to a smooth projective surface X such that E is contracted to a point, and moreover, f is a blowup of X at $f(E)$.*

Now it is easy to see how the classification works in dimension 2. Once we have resolved singularities of our surface, we ask whether the surface obtained has a (-1)-curve. If not, we have our relatively minimal model. If yes, then we use Castelnuovo contraction to contract a (-1)-curve. We repeat the process for the new surface. The process is finite since after each step, the rank of the Néron-Severi group drops, as well as the second Betti number.

Note however, that the criterion "does X have a (-1)-curve" does not have a meaningful generalisation to higher dimensions. Also, it is not clear that it gives the right notion – in other words, it is not obvious that this is an intrinsic notion of X with special implications on the geometry of X. However, note that, by the adjunction formula, E is a (-1)-curve on X if and only if $E \simeq \mathbb{P}^1$ and $K_X \cdot E < 0$. Therefore, if X has a (-1)-curve, then its canonical class cannot be nef.

There are three cases for the relatively minimal model X. First, if K_X is nef, then a further fine classification gives that it is actually semiample, hence it defines a fibration $X \to Z$, and we can further analyse X with the aid of this map. In this case, we also say that X is the (absolute) minimal model. If K_X is not nef, then one can show that either there exists a morphism $\varphi\colon X \to Z$ to a smooth projective curve Z such that X is a $\mathbb{P}^1$-bundle over Z via φ, or $X \simeq \mathbb{P}^2$. In these last two cases, one says that X is a Mori fibre space. This gives the following *hard dichotomy* for surfaces: the end product of the classification is either a minimal model (unique up to isomorphism) if $\kappa(X) \geq 0$ or a Mori fibre space if $\kappa(X) = -\infty$.

1.1.2 Higher dimensions

One of the ingenious insights of Mori was introducing a new criterion for determining whether a variety X is a minimal model:

Is K_X nef?

There are many reasons why this is a meaningful question to pose. First, it makes sense by analogy with surfaces. Second, on a random (smooth, projective) variety X it is usually very hard to find any useful divisors, especially those which carry essential information about the geometry of X – the only obvious candidate is K_X, by its very construction.

Further, in an ideal situation we would have that K_X is ample – indeed, this would mean that some multiple of K_X itself gives an embedding into a projective space, and that it enjoys many nice numerical and cohomological properties. Therefore, assume that K_X is pseudoeffective. Then, a reasonable question to pose is:

Is there a birational map $f\colon X \dashrightarrow Y$ such that the divisor f_*K_X is ample?

Here the map f should not be just any birational map, but a birational contraction – in other words, f^{-1} should not contract divisors. This is an important condition since the variety Y should be in almost every way simpler than X; in particular, some of its main invariants, such as the Picard number, should not increase. Likewise, we would like to have $K_Y = f_*K_X$, and this will almost never happen if f extracts divisors (take, for instance, an inverse of almost any blowup).

Further, we impose that f should preserve sections of all positive multiples of K_X. This is also important, since global sections are something we definitely want to keep track of, if we want the divisor $K_Y = f_*K_X$ to bear any connection with K_X. Another way to state this is as follows. Consider the *canonical ring* of X:

$$R(X, K_X) = \bigoplus_{m\in\mathbb{N}} H^0(X, mK_X).$$

Then we require that f induces an isomorphism between $R(X, K_X)$ and $R(Y, K_Y)$.

We immediately see that the answer to the question above is in general "no" – the condition would imply that K_X is a big divisor. In fact, and perhaps surprisingly, the converse is true by the following theorem of Reid [Rei80, Proposition 1.2]:

Theorem 1.2. *Let X be a smooth variety of general type, and assume that the canonical ring $R(X, K_X)$ is finitely generated. Denote $Y = \operatorname{Proj} R(X, K_X)$, and let $\varphi\colon X \dashrightarrow Y$ be the associated map. Then φ is a birational contraction and K_Y is ample.*

Proof. To start with, recall that by [Bou89, III.1.2], there exists a positive integer d such that $R(X, dK_X)$ is generated by $H^0(X, \mathcal{O}_X(dK_X))$. Let $(p, q)\colon W \to X \times Y$ be the resolution of the linear system $|dK_X|$: in other words, W is smooth, and the movable part of the linear system $p^*|dK_X|$ is basepoint free. To obtain this, we first apply [Har77, Example II.7.17.3], and then Hironaka's resolution of singularities.

$$\begin{tikzcd} W \arrow[d, "p"'] \arrow[dr, "q"] & \\ X \arrow[r, dashed, "\varphi"] & Y \end{tikzcd}$$

For each positive integer m, denote by M_m and F_m the movable part and the fixed part of $p^*(mdK_X)$, respectively. Then the fact that $R(X, dK_X)$ is generated by $H^0(X, \mathcal{O}_X(dK_X))$ implies that $M_m = mM_1$ and $F_m = mF_1$ for all m, and it is easy to see that the map q is just the semiample (Iitaka) fibration associated to M_1. Moreover, by passing to a multiple, we may assume without loss of generality that the map q is actually the morphism associated to the linear system $|M_1|$. Then $\mathcal{O}_W(M_1) = q^*\mathcal{O}_Y(1)$ for a very ample line bundle $\mathcal{O}_Y(1)$ on Y.

Denote $n = \dim X$, and let Γ be a component of F_1. To show that φ is a contraction, we need to show that Γ is contracted by q, or equivalently, that $h^0(q(\Gamma), \mathcal{O}_{q(\Gamma)}(m)) \leq O(m^{n-2})$.

Since $\mathcal{O}_W(M_m) = q^*\mathcal{O}_Y(m)$ and the natural map $\mathcal{O}_{q(\Gamma)} \to q_*\mathcal{O}_\Gamma$ is injective, we have

$$h^0(q(\Gamma), \mathcal{O}_{q(\Gamma)}(m)) \leq h^0(q(\Gamma), \mathcal{O}_Y(m) \otimes q_*\mathcal{O}_\Gamma) = h^0(\Gamma, \mathcal{O}_\Gamma(M_m)). \tag{1.1}$$

There exists effective Cartier divisors G^+ and G^- on Γ such that $\mathcal{O}_\Gamma(\Gamma) \simeq \mathcal{O}_\Gamma(G^+ - G^-)$. Consider the exact sequences

$$0 \to H^0(\Gamma, M_m|_\Gamma - G^-) \to H^0(\Gamma, M_m|_\Gamma) \to H^0(G^-, M_m|_{G^-}) \tag{1.2}$$

and

$$0 \to H^0(W, M_m) \to H^0(W, M_m + \Gamma) \to H^0(\Gamma, (M_m + \Gamma)|_\Gamma) \to H^1(W, M_m). \tag{1.3}$$

Since $F_m = mF_1$, the divisor Γ is a component of F_m, hence the first map in (1.3) is an isomorphism and the last map in (1.3) is an injection. Therefore, from (1.1), (1.2) and (1.3) we have

$$\begin{aligned} h^0(q(\Gamma), \mathcal{O}_{q(\Gamma)}(m)) &\leq h^0(\Gamma, M_m|_\Gamma) \leq h^0(\Gamma, M_m|_\Gamma - G^-) + h^0(G^-, M_m|_{G^-}) \\ &\leq h^0(\Gamma, (M_m + \Gamma)|_\Gamma) + h^0(G^-, M_m|_{G^-}) \leq h^1(W, M_m) + h^0(G^-, M_m|_{G^-}). \end{aligned}$$

As $\dim G^- = n-2$, we have $h^0(G^-, M_m|_{G^-}) \leq O(m^{n-2})$, hence it is enough to show that $h^1(W, M_m) \leq O(m^{n-2})$. To this end, from the Leray spectral sequence

$$H^p(Y, R^{1-p}q_*\mathcal{O}_W(M_m)) \Rightarrow H^1(W, \mathcal{O}_X(M_m))$$

we have

$$h^1(W, M_m) \leq h^0(Y, R^1q_*\mathcal{O}_W(M_m)) + h^1(Y, q_*\mathcal{O}_W(M_m)).$$

The terms $h^1(Y, q_*\mathcal{O}_W(M_m)) = h^1(Y, \mathcal{O}_Y(m))$ vanish for $m \gg 0$ by Serre vanishing, so we need to prove

$$h^0(Y, R^1q_*\mathcal{O}_W(M_m)) \leq O(m^{n-2}). \tag{1.4}$$

Let $U \subseteq Y$ be the maximal open subset over which q is an isomorphism. By [Har77, III.11.2], for each m the sheaf $R^1q_*\mathcal{O}_W(M_m)$ is supported on the set $Y \setminus U$ of dimension at most $n-2$, hence $\chi(Y, R^1q_*\mathcal{O}_W(M_m)) \leq O(m^{n-2})$. But by Serre vanishing again,

$$h^i(Y, R^1q_*\mathcal{O}_W(M_m)) = h^i(Y, R^1q_*\mathcal{O}_W \otimes \mathcal{O}_Y(m)) = 0$$

vanish for $m \gg 0$ and all $i > 0$, and this implies (1.4).

Finally, to see that K_Y is ample, let X_0 be an open subset of X and let Y_0 be an open subset of Y such that $\operatorname{codim}_Y(Y \setminus Y_0) \geq 2$ such that Y_0 is smooth and $\varphi|_{X_0}\colon X_0 \to Y_0$ is an isomorphism. Then it is clear that $K_{X_0} = (\varphi|_{X_0})^*(K_{Y_0})$, and since $\mathcal{O}_{X_0}(dK_{X_0}) = (\varphi|_{X_0})^*\mathcal{O}_{Y_0}(1)$, the divisor K_Y is ample by Hartogs principle. $\square$

We now return to the question we posed above, and see if we can modify it to something more probable. We can settle for something weaker, but still sufficient for our purposes: we require that the divisor K_Y is *semiample*. This then still produces an Iitaka fibration $g\colon Y \to Z$ and an ample divisor A such that $K_Y = g^*A$, and the composite map $X \dashrightarrow Z$, which is now not necessarily birational, gives an isomorphism of section rings $R(X, K_X)$ and $R(Z, A)$. In particular, this would imply that the canonical ring $R(X, K_X)$ is finitely generated. This would clearly be astonishing: we would be able to construct a projective variety $Z = \operatorname{Proj} R(X, K_X)$. In fact, the wish that the canonical ring is finitely generated predates the modern Minimal Model Program, and goes back to the seminal work of Zariski [Zar62]:

Conjecture 1.3. *Let X be a smooth projective variety. Then the canonical ring $R(X, K_X)$ is finitely generated.*

This conjecture gives another justification for the abovementioned wishful thinking. It was proved by Mumford on surfaces (in the appendix to the same paper of Zariski), and in general in [BCHM10, CL12].

Historically, by the influence of the classification of surfaces on the way we think about higher dimensional classification, this splits into two problems: finding a birational map $f\colon X \dashrightarrow Y$ such that the divisor $K_Y = f_*K_X$ is nef; and then proving that the nef divisor K_Y is semiample. This last part – the *Abundance conjecture* – is one of main open problems in higher dimensional geometry, in dimensions at least 4. We know it holds in dimensions up to 3 [Miy88b, Miy88a, Kaw92], and when the canonical divisor is big [Kaw84], but very little is known in general.

Thus, hopefully by now it is clear that the main classification criterion is whether the canonical divisor K_X is nef. If K_X is nef, we are done, at least with the first part of the programme above. Life gets much tougher, but also much more interesting when the answer is *no*.

1.1.3 The Cone and Contraction theorems

Indeed, let $\overline{\mathrm{NE}}(X) \subseteq N_1(X)_{\mathbb{R}}$ denote the closure of the cone spanned by the numerical classes of effective curves; note that the nef cone $\mathrm{Nef}(X)$ is dual to $\overline{\mathrm{NE}}(X)$ by Nakai's criterion, with respect to the intersection pairing. Since K_X is not nef, the hyperplane

$$K_X^{\perp} = \{C \in N_1(X)_{\mathbb{R}} \mid K_X \cdot C = 0\} \subseteq N_1(X)_{\mathbb{R}}$$

must cut the cone $\overline{\mathrm{NE}}(X)$ into two parts; let us denote the two pieces by $\overline{\mathrm{NE}}(X)_{K_X \geq 0}$ and $\overline{\mathrm{NE}}(X)_{K_X < 0}$. Then the celebrated Cone theorem of Mori tells that the negative part $\overline{\mathrm{NE}}(X)_{K_X < 0}$ is *locally rational polyhedral.* More precisely:

Theorem 1.4. *Let X be a smooth projective variety. Then there exist countably many extremal rays R_i of the cone $\overline{\mathrm{NE}}(X)$ such that $K_X \cdot R_i < 0$ and*

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \geq 0} + \sum R_i.$$

Moreover, for every ample $\mathbb{Q}$-divisor H on X, there exist finitely many such rays R'_i with

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X + H \geq 0} + \sum R'_i.$$

In particular, the rays R_i are discrete in the half-space $\overline{\mathrm{NE}}(X)_{K_X<0}$.

Recall that an *extremal ray* R of a closed convex cone $\mathcal{C}$, in the sense of convex geometry, is a linear subset of $\mathcal{C}$ satisfying the following condition: if $u + v \in R$ for $u, v \in \mathcal{C}$, then necessarily $u, v \in R$. Note that in the theorem, the second statement implies the first, by letting $H \to 0$, and it implies that the rays R_i can accumulate only on the hyperplane $K_X^{\perp}$. This is the standard formulation, and the proof can be found in any treatise of the subject. We will prove an analogue of this statement a bit later in the course.

There is an additional statement that we can contract any of the extremal rays R_i – this is the Contraction theorem of Kawamata and Shokurov.

Theorem 1.5. *With the notation from Theorem 1.4, fix any of the rays $R = R_i$. Then there exists a morphism with connected fibres*

$$\mathrm{cont}_R \colon X \to Y$$

to a normal projective variety Y such that a curve is contracted by cont_R if and only if its class lies in R.

The importance of the Contraction theorem is two-fold. First, it is clear that such a contraction has to be defined by a basepoint free divisor L with $L \cdot R = 0$; in general, it is very difficult to show the existence of a single non-trivial non-ample basepoint free divisor on a variety – the conclusion that there are many of them is clearly astonishing.

Second, we want to eventually end up with a variety on which the canonical divisor is nef, i.e. it has no extremal rays as above. We therefore hope that by contracting some of the rays we can make the situation better. We will see below that this is not necessarily the case, at least not immediately. However, I will argue that life indeed gets better, at least if we choose carefully *which* rays to contract.

1.1.4 Contractions in the MMP

Let us go back to the procedure in the Minimal Model Program. The Cone and Contraction theorems tell us that that if we pick a K_X-negative extremal ray R, we can contract it to obtain another normal projective variety Y, and we hope that it shares many of the properties of X that we started with, for instance $\mathbb{Q}$-factoriality. Here the situation branches into three distinct cases.

Assume first that $\dim Y < \dim X$. Then it can be shown that Y is $\mathbb{Q}$-factorial, that its singularities are manageable in a sense which I will define later, and note that the general fibre of cont_R is a Fano variety. Then we declare our procedure finished – varieties of this form can then be studied via the general fibre and the base Y, and indeed they form a well studied class called Mori fibre spaces, like in the surface case.

Assume next that the map cont_R is birational, and that the exceptional set of the map cont_R contains a prime divisor E. Then, in fact, we will prove later that we have $\mathrm{Exc}(\mathrm{cont}_R) = E$, and moreover, Y is also $\mathbb{Q}$-factorial. In this case, we say that cont_R is a *divisorial* contraction. A drawback is that Y is no longer necessarily smooth, but still it has singularities which are very close to the smooth case, and we can continue our programme on Y. However, something changed for the better: the Picard number dropped by 1 since we contracted the divisor E; our variety became simpler.

Assume next that the exceptional set of the map cont_R does not contain a prime divisor, i.e. that we have $\mathrm{codim}_X \mathrm{Exc}(\mathrm{cont}_R) \geq 2$. In this case, we say that cont_R is a *flipping* contraction. This situation is bad: not only do we have that Y is not $\mathbb{Q}$-factorial, but even $K_Y = (\mathrm{cont}_R)_* K_X$ is not a $\mathbb{Q}$-Cartier divisor. Indeed, since cont_R is an isomorphism in codimension 1, we have $K_X = \mathrm{cont}_R^* K_Y$. If C is a curve contracted by cont_R, then $K_X \cdot C < 0$, and by the projection formula this equals $K_Y \cdot (\mathrm{cont}_R)_* C = 0$, a contradiction.

The great insight of Mori, Reid and others is this. Note that the divisor K_X is anti-ample with respect to the map cont_R, and the result that we want to end up with in the end should give the canonical divisor which is nef. Thus, it is a natural thing to try to construct at least a birational map $X^+ \to Y$ which “turns the sign” of all curves contracted by cont_R; in other words, it “flips” them. Therefore, we would like to have a diagram:

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "\mathrm{cont}_R"'] & & X^+ \arrow[dl, "\mathrm{cont}_R^+"] \\ & Y & \end{tikzcd}$$

such that X^+ is $\mathbb{Q}$-factorial and K_{X^+} is ample with respect to cont_R^+.

This diagram, or just the map φ, is called *the flip* of cont_R. Since, by our requirements, the map φ should not extract divisors, the morphism cont_R^+ is also an isomorphism in codimension 1. It is then not too difficult, but crucial, to show that the existence of the diagram is equivalent to the fact that the relative canonical ring

$$R(X/Y, K_X) = \bigoplus_{n \in \mathbb{N}} (\mathrm{cont}_R)_* \mathcal{O}_X(nK_X)$$

is finitely generated as a sheaf of algebras over $(\mathrm{cont}_R)_* \mathcal{O}_X = \mathcal{O}_Y$, and moreover, then $X^+ = \mathrm{Proj}_Y R(X/Y, K_X)$; this is proved in exactly the same way as Theorem 1.2. It immediately follows from the Cone theorem that X^+ is $\mathbb{Q}$-factorial and that the Picard number of X^+ is the same as that of X.

Figure 1.1: Minimal Model Programme in higher dimensions

The flip as above is by now proved to exist in any dimension. The first proof for threefolds was given by Mori in [Mor88], and in general in [BCHM10].

Thus, the variety X^+ has all the desired features similar to X, so we continue the procedure with X^+ instead of X (again, as in the case of divisorial contractions, we lose smoothness, but we are all right if we slightly enlarge our category). Unfortunately, it is not easy to find an invariant of varieties which behaves well under flips; the only such example currently exists on threefolds. It is, therefore, a crucial problem to find a sequence of divisorial contractions and flips which terminates.

To summarise, our classification procedure – the Minimal Model Program – looks like the algorithm in Figure 1.1.

1.1.5 Pairs and their singularities

It has become clear in the last several decades that sometimes varieties are not the right objects to look at – often, it is much more convenient to look at pairs (X, Δ), where X is a normal projective variety and Δ is a Weil $\mathbb{Q}$-divisor on X such that $K_X + \Delta$ is $\mathbb{Q}$-Cartier. There are plenty of reasons for looking at these objects: they obviously generalise the concept of a ($\mathbb{Q}$-Gorenstein) variety (by taking $\Delta = 0$), they are closely related to open varieties $X \setminus \mathrm{Supp}\,\Delta$. For us, there are other, more practical reasons why it seems essential to consider this enlarged setting: it is logical that the proofs should go by induction on the dimension, and if one wants to use adjunction formula, one has to consider pairs. Finally, consider a minimal model X and a morphism $\varphi\colon X \to Z$ given as the Iitaka fibration of the semiample divisor K_X. When K_X is not big, it is in general hopeless to expect that $K_X \sim_{\mathbb{Q}} \varphi^* K_Z$ as in Theorem 1.2. However, it can be shown that there exists an effective $\mathbb{Q}$-divisor Δ on Z such that the pair has nice properties (in the sense explained a bit below) and such that $K_X \sim_{\mathbb{Q}} \varphi^*(K_Z + \Delta)$, cf. [Amb05].

Valuations. Before we see what a good notion of a pair is, we make a brief detour to define geometric valuations on the field of rational functions $k(X)$ of a normal projective variety X.

Let X be a variety. A *prime divisor over* X is any prime divisor E on a proper birational model $f\colon Y \to X$, where Y is a normal variety. If $\eta \in Y$ is the generic point of E, the local ring $\mathcal{O}_{Y,\eta} \subseteq k(X)$ is a discrete valuation ring which corresponds to the valuation mult_E given by the order of vanishing of an element $\varphi \in k(X)$. We call such a valuation on $k(X)$ a *geometric valuation*. Note that the transcendence degree of the residue field $k(\eta)$ over $\mathbb{C}$ is $\dim X - 1$. This gives a valuation on the set $\operatorname{Div}_{\mathbb{R}}(X)$ of $\mathbb{R}$-Cartier divisors on X by setting $\operatorname{mult}_E D := \operatorname{mult}_E f^*D$ for $D \in \operatorname{Div}_{\mathbb{R}}(X)$. Similarly, if we have a linear system $|D|$, then

$$\operatorname{mult}_E |D| = \inf\{\operatorname{mult}_E D' \mid D' \in |D|\}.$$

If $\mathfrak{b}_{|D|} \subseteq k(X)$ is the ideal sheaf of the base locus of $|D|$, we set $\operatorname{mult}_E \mathfrak{b}_{|D|} = \inf\{\operatorname{mult}_E f \mid f \in \mathfrak{b}_{|D|}\}$; it is clear that $\operatorname{mult}_E \mathfrak{b}_{|D|} = \operatorname{mult}_E |D|$. It is easy to see that $\operatorname{mult}_E \mathfrak{b}_{|D|} = \operatorname{mult}_E \overline{\mathfrak{b}_{|D|}}$, where the last ideal is the integral closure of the base ideal inside of $k(X)$.

Let $f'\colon Y' \to X$ be another birational morphism and let $E' \subseteq Y'$ be a prime divisor. Then we have $\operatorname{mult}_E = \operatorname{mult}_{E'}$ if and only if the induced birational map $Y \dashrightarrow Y'$ is an isomorphism at the generic points of E and E'. Therefore, the discrepancies $a(E, X, \Delta)$ (defined below) depend only on the valuation mult_E and not on the choice of the birational model f. We often do not distinguish between the valuation mult_E and a particular choice of the divisor E. And similarly for the set $c_X(E) = f(E) \subseteq X$, *the centre of the valuation* E *on* X.

Given a valuation E, it is an important question whether E can be reached from X by a sequence of blowups. The following result of Zariski shows precisely that.

Lemma 1.6. *Let X be a proper variety over a field k. Let R be a DVR of $k(X)$ with the maximal ideal $\mathfrak{m}$, and such that* $\operatorname{trdeg}(R/\mathfrak{m} : k) = \dim X - 1$. *Let $Y = \operatorname{Spec} R$, let $y \in Y$ be its unique closed point and let $f\colon Y \to X$ be the birational morphism given by the valuative criterion of properness. Define a sequence of varieties and maps as follows: set $X_0 = X$, $f_0 = f$. If $f_i\colon Y \to X_i$ is already defined, let $Z_i \subseteq X_i$ be the closure of the point $x_i = f_i(y)$, let X_{i+1} be the blowup of X_i at Z_i, and let $f_{i+1}\colon Y \to X_{i+1}$ be the birational morphism given by the valuative criterion of properness. Then f_n induces an isomorphism $\mathcal{O}_{X_n,x_n} \simeq R$ for some $n \geq 0$.*

Recall that a valuation ν on R is given by $\nu(g) = \max\{s \in \mathbb{Z} \mid g \in \mathfrak{m}^s\}$ for $g \in k(X) \setminus \{0\}$. In our case, $R = \mathcal{O}_{Y,\eta}$ and $Z_0 = c_X(E)$. Hence, the lemma says that we can reach a valuation by repeatedly blowing up its centre. The proof can be found in [KM98, Lemma 2.45].

When working with questions where finite generation of rings is involved, it is necessary to think about not only the linear system associated to a divisor D, but also to that of all of its multiples. Hence, fix a geometric valuation Γ over an algebraic variety X. If D is an *effective* $\mathbb{Q}$-Cartier divisor, then the *asymptotic order of vanishing* of D along Γ is

$$o_\Gamma(D) = \inf\{\operatorname{mult}_\Gamma D' \mid D \sim_\mathbb{Q} D' \geq 0\},$$

or equivalently,

$$o_\Gamma(D) = \inf \frac{1}{k} \operatorname{mult}_\Gamma |kD|$$

over all k sufficiently divisible. It is straightforward to see that each o_Γ is a homogeneous function of degree 1, that

$$o_\Gamma(D + D') \leq o_\Gamma(D) + o_\Gamma(D')$$

for every two effective $\mathbb{Q}$-divisors D and D', and that

$$o_\Gamma(A) = 0$$

for every semiample divisor A.

Singularities of pairs. Now assume we are given a pair (X, Δ), and let $f\colon Y \to X$ be a log resolution of the pair, i.e. f is a projective birational morphism such that Y is smooth, the set $\operatorname{Exc} f$ is a divisor, and the support of the divisor $\operatorname{Exc} f \cup f^*\Delta$ has simple normal crossings. Then it is easy to see that there exists a $\mathbb{Q}$-divisor R on Y such that

$$K_Y = f^*(K_X + \Delta) + R.$$

The divisor R is supported on the proper transform of Δ and on the exceptional divisors of f. For every prime divisor E on Y, we denote the coefficient of E in R by $a(E, X, \Delta)$, called the *discrepancy of E with respect to the pair* (X, Δ), and set $d(X, \Delta) = \inf\{a(E, X, \Delta)\}$, where the infimum is over all prime divisors lying on some birational model $Y \to X$. It is easy to see that $d(X, \Delta) \leq 1$.

We want to see how one can effectively calculate the divisor R. We claim that there is the following dichotomy: either $d(X, \Delta) \geq -1$, or $d(X, \Delta) = -\infty$. To see this, we first need a preparatory lemma, the proof is an exercise.

Lemma 1.7. *Let X be a smooth variety and let $\Delta = \sum \delta_i \Delta_i$ be a $\mathbb{Q}$-divisor on X. Let Z be a closed subvariety of X of codimension k. Let $\pi\colon Y \to X$ be the blow up of Z and let $E \subseteq Y$ be the irreducible component of the exceptional divisor which dominates Z. Then*

$$a(E, X, \Delta) = k - 1 - \sum \delta_i \operatorname{mult}_Z \Delta_i.$$

Now, to see the claim, let E be a divisor on a birational model $Y \to X$ such that $a(E, X, \Delta) = -1 - \varepsilon$ for some $\varepsilon > 0$. By taking a log resolution, we may assume that Y is smooth, and that the divisor $\Delta_Y = f^*(K_X + \Delta) - K_Y$ has simple normal crossings. Then it is easy to see that $a(F, X, \Delta) = a(F, Y, \Delta_Y)$ for every prime divisor on a birational model over X. Let $Z_0 \subseteq Y$ be a closed set of codimension 2 which is contained in E but not in any other f-exceptional divisor or in $f_*^{-1}\Delta$, and let $\pi_1 \colon Y_1 \to Y$ be the blowup of Z_0 with exceptional divisor E_1. Then $a(E_1, X, \Delta) = -\varepsilon$ by the previous lemma. Now for every $m \geq 2$, let $Z_{m-1} \subseteq Y_{m-1}$ be the intersection of E_{m-1} and the proper transform of E on Y_{m-1}, and let $\pi_m \colon Y_m \to Y_{m-1}$ be the blowup of Y_{m-1} along Z_{m-1}. Then again the discrepancy calculation shows that $a(E_m, X, \Delta) = -m\varepsilon$, hence $\lim\limits_{m\to\infty} a(E_m, X, \Delta) = -\infty$.

This shows that there is a clear cut between pairs which satisfy $d(X, \Delta) \geq -1$ and other pairs. It is possible to write down an example of a pair with $d(X, \Delta) < -1$ such that the canonical ring is not finitely generated, hence no reasonable definition of the Minimal Model Program can run for (X, Δ). Hence, we have to restrict ourselves to pairs with $d(X, \Delta) \geq -1$, in which case we say that the pair (X, Δ) has *log canonical singularities*, or just that it is log canonical. This is the largest class where the Minimal Model Program can be possibly expected to work. However, we are in good company here: we can view smooth varieties X as pairs $(X, 0)$, and they are definitely log canonical – moreover, we have $d(X, 0) > 0$ by the classical ramification formula.

However, in this course, we will restrict ourselves to a subclass of pairs with *klt singularities*: they are precisely pairs with $d(X, \Delta) > -1$. The reason is purely practical – the experience in the Minimal Model Program shows that these varieties behave much better than pairs with $d(X, \Delta) = -1$, and we simply know many more results for klt pairs than for log canonical pairs in general. It is also useful to note that it can be shown the klt condition can be shown on only one log resolution $Y \to X$ and not on all – this is an easy consequence of Lemma 1.6 and is left as an exercise.

A good way to think about klt pairs is to assume from the start that X is smooth, that $\operatorname{Supp} \Delta$ has simple normal crossings, and that all coefficients of Δ lie in the open interval $(0, 1)$. It is a fun exercise to prove that such a pair indeed has klt singularities.

Also of importance for us is that this is an open condition, in the following sense. Say you have at hand a klt pair (X, Δ) with X being $\mathbb{Q}$-factorial, and that you have an effective $\mathbb{Q}$-divisor D on X. Then for all rational $0 \leq \varepsilon \ll 1$, the pair $(X, \Delta + \varepsilon D)$ is again klt. This is easy to see from the definition.

Therefore, divisors of the form $K_X + \Delta$ are of special importance for us, and they are called *adjoint divisors*. Now we set up the Minimal Model Program in the case of pairs in exactly the same way as before, replacing K_X by $K_X + \Delta$ everywhere.

We will below construct the special version of this procedure when the pair (X, Δ) is klt and the divisor Δ is big.

Generalisations of Zariski's conjecture. The generalised Zariski's conjecture says that the *(log) canonical ring*

$$R(X, K_X + \Delta) = \bigoplus_{m \in \mathbb{N}} H^0(X, m(K_X + \Delta))$$

of a log canonical pair (X, Δ) is finitely generated. When the pair at hand is klt, this is now a theorem [BCHM10, CL12].

A note on the notation above. If X is a normal projective variety with the field of rational functions $k(X)$, and D is a $\mathbb{Q}$-divisor on X, then we define the global sections of D by

$$H^0(X, D) = \{f \in k(X) \mid \operatorname{div} f + D \geq 0\}.$$

Note that, even though D might not be an integral divisor, this makes perfect sense, and that $H^0(X, D) = H^0(X, \lfloor D \rfloor)$, where the latter H^0 is the vector space of global sections of the standard divisorial sheaf $\mathcal{O}_X(\lfloor D \rfloor)$. This is compatible with taking sums: in other words, there is a well-defined multiplication map

$$H^0(X, D_1) \otimes H^0(X, D_2) \to H^0(X, D_1 + D_2).$$

Now, if we are given a bunch of $\mathbb{Q}$-divisors $D_1, \dots, D_r$ on X, we can define the corresponding $\mathbb{N}^r$-graded *divisorial ring* as

$$\mathfrak{R} = R(X; D_1, \dots, D_r) = \bigoplus_{(n_1, \dots, n_r) \in \mathbb{N}^r} H^0(X, n_1 D_1 + \dots + n_r D_r).$$

When $r = 1$, this generalises the standard notion of the section ring $R(X, D_1)$. A special case of the divisorial ring above is when all D_i are (multiples of) adjoint divisors – we then say that the ring $\mathfrak{R}$ is an *adjoint ring*.

The following lemma summarises the main tools when operating with finite generation of divisorial rings. The proof can be found in [ADHL10].

Lemma 1.8. *Let X be a $\mathbb{Q}$-factorial projective variety, and let $D_1, \dots, D_r$ be $\mathbb{Q}$-divisors on X.*

(1) If $p_1, \dots, p_r \in \mathbb{Q}_+$, then the ring $R(X; p_1 D_1, \dots, p_r D_r)$ is finitely generated if and only if the ring $R(X; D_1, \dots, D_r)$ is finitely generated.

(2) Let $G_1, \dots, G_\ell$ be $\mathbb{Q}$-divisors such that $G_i \in \sum \mathbb{R}_+ D_i$ for all i. If the ring $R(X; D_1, \dots, D_r)$ is finitely generated, then the ring $R(X; G_1, \dots, G_\ell)$ is finitely generated.

Now we are ready to state the most important example of a finitely generated divisorial ring.

Theorem 1.9. *Let X be a $\mathbb{Q}$-factorial projective variety, and let $\Delta_1, \dots, \Delta_r$ be big $\mathbb{Q}$-divisors such that all pairs (X, Δ_i) are klt.*

Then the adjoint ring

$$R(X; K_X + \Delta_1, \dots, K_X + \Delta_r)$$

is finitely generated.

This was first proved in [BCHM10] by employing the full machinery of the classical MMP: the idea is to prove that a certain version of the Minimal Model Program works, and then to deduce the finite generation as a consequence of the generalised Zariski's conjecture above. The rough sketch is as follows. By taking a log resolution, we may assume that X is smooth and the support of the divisor $\sum \Delta_i$ has simple normal crossings. Let m be a positive integer such that $D_i = m(K_X + \Delta_i)$ is integral for every i, let $\mathcal{E} = \bigoplus_{i=1}^r \mathcal{O}_X(D_i)$, and let $Y = \mathbb{P}(\mathcal{E})$. Then it is easy to see that for every nonnegative integer k we have

$$H^0(Y, \mathcal{O}_Y(k)) = H^0(X, S^k\mathcal{E}) = \bigoplus_{n_1+\cdots+n_r=k} H^0(X, n_1D_1 + \cdots + n_rD_r),$$

hence the divisorial ring above is isomorphic to $R(Y, \mathcal{O}_Y(1))$. Now a bit more work shows that there is a divisor Δ_Y on Y such that (Y, Δ_Y) is klt, and we are done by Lemma 1.8.

However, of importance for us in this course is that Theorem 1.9 can be proved without the Minimal Model Program, and this was done in [Laz09, CL12]. We will prove it later in the course. In this chapter, we will see how Theorem 1.9 implies all the known results in the Minimal Model Program in a rather quick way.

1.2 Proof of the Cone and Contraction theorems

We will derive the Cone and Contraction theorems for klt pairs from Theorem 1.9. We first need some preparation.

1.2.1 Valuations and divisorial rings

Let X be a normal projective variety and let $D_1, \dots, D_r$ be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. Consider the divisorial ring $\mathfrak{R} = R(X; D_1, \dots, D_r)$ as above. Then we have a corresponding cone $\mathcal{C} = \sum \mathbb{R}_+ D_i$ which sits in the space of $\mathbb{R}$-divisors $\mathrm{Div}_{\mathbb{R}}(X)$.

Inside $\mathcal{C}$, there is another, much more important cone – the *support* of $\mathfrak{R}$. This cone, $\operatorname{Supp}\mathfrak{R}$, is defined as the convex hull of all integral divisors $D \in \mathcal{C}$ which have sections, i.e. $H^0(X, D) \neq 0$.

Now we have all the theory needed to state the result which gives us the main relation between finite generation and the behaviour of linear systems.

Theorem 1.10. *Let X be a normal projective variety, and let $D_1, \ldots, D_r$ be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. Assume that the ring $\mathfrak{R} = R(X; D_1, \ldots, D_r)$ is finitely generated. Then:*

(1) $\operatorname{Supp}\mathfrak{R}$ *is a rational polyhedral cone,*

(2) if $\operatorname{Supp}\mathfrak{R}$ *contains a big divisor, then all pseudo-effective divisors in $\sum \mathbb{R}_+ D_i$ are in fact effective,*

(3) there is a finite rational polyhedral subdivision $\operatorname{Supp}\mathfrak{R} = \bigcup \mathcal{C}_i$ *into cones of maximal dimension, such that o_Γ is linear on $\mathcal{C}_i$ for every geometric valuation Γ over X,*

(4) there exists a positive integer k such that $o_\Gamma(kD) = \operatorname{mult}_\Gamma |kD|$ for every integral divisor $D \in$ $\operatorname{Supp}\mathfrak{R}$ *and every geometric valuation Γ over D.*

Proof. For (1), pick generators f_i of $\mathfrak{R}$, and let $E_i \in \sum \mathbb{R}_+ D_i$ be the divisors such that $f_i \in H^0(X, E_i)$. Then clearly $\operatorname{Supp}\mathfrak{R} = \sum \mathbb{R}_+ E_i$.

For (2), fix a big divisor B in $\operatorname{Supp}\mathfrak{R}$, and let $D \in \sum \mathbb{R}_+ D_i$ be a pseudoeffective divisor. Observe that every divisor in the interval $(D, B]$ is big, hence $(D, B] \subseteq \operatorname{Supp}\mathfrak{R}$. But then $[D, B] \subseteq \operatorname{Supp}\mathfrak{R}$ since $\operatorname{Supp}\mathfrak{R}$ is closed by (1).

We extract the proofs of (3) and (4) verbatim from the proof of [ELM+06, Theorem 4.1]. Consider the system of ideals $(\mathfrak{b}_{\mathbf{n}})_{\mathbf{n}=(n_1,\ldots,n_r)\in\mathbb{N}^r}$, where $\mathfrak{b}_{\mathbf{n}}$ is the base ideal of the linear system $|n_1 D_1 + \ldots n_r D_r|$. This is a finitely generated system of ideals, so by [ELM+06, Proposition 4.7] there is a rational polyhedral subdivision $\mathbb{R}^r_+ = \bigcup \mathcal{D}_i$ and a positive integer d such that for every i, if $e^i_1, \ldots, e^i_s$ are generators of $\mathbb{N}^r \cap \mathcal{D}_i$, then

$$\overline{\mathfrak{b}_{d\sum_j p_j e^i_j}} = \overline{\prod_j \mathfrak{b}^{p_j}_{de^i_j}}$$

for every $(p_1, \ldots, p_s) \in \mathbb{N}^s$. Since a valuation of an ideal is equal to that of its integral closure, we deduce that for every geometric valuation Γ of X, o_Γ is linear on each of the cones $\mathcal{C}_i = \operatorname{Supp} R \cap \mathcal{D}_i$, and we can take $k = d$. $\square$

A simple, but as we will see important consequence is the following.

Lemma 1.11. *Let X be a normal projective variety and let D be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then D is semiample if and only if $R(X, D)$ is finitely generated and $o_\Gamma(D) = 0$ for all geometric valuations Γ over X.*

The proof is very simple: if D is semiample, then the statement is classical. Conversely, for every point $x \in X$, Theorem 1.10 implies that x does not belong to the base locus of the linear system $|mD|$ for m sufficiently divisible.

As a demonstration of the previous two results, we will see immediately how inside the cone $\operatorname{Supp} \mathfrak{R}$, all the cones that we can imagine behave nicely. As we will see, the following is effectively the proof of Mori's Cone theorem [CL13, KKL12].

Proposition 1.12. *Let X be a normal projective variety, and let $D_1, \dots, D_r$ be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X. Assume that the ring $\mathfrak{R} = R(X; D_1, \dots, D_r)$ is finitely generated, and denote by $\pi\colon \operatorname{Div}_{\mathbb{R}}(X) \to N^1(X)_{\mathbb{R}}$ the natural projection. Assume that $\operatorname{Supp} \mathfrak{R}$ contains an ample divisor. Then the cone $\operatorname{Supp} \mathfrak{R} \cap \pi^{-1}\big(\operatorname{Nef}(X)\big)$ is rational polyhedral, and every element of this cone is semiample.*

Proof. Let $\operatorname{Supp} \mathfrak{R} = \bigcup \mathcal{C}_i$ be a finite rational polyhedral subdivision as in Theorem 1.10, and let Γ be a prime divisor *over* X. If the relative interior of $\mathcal{C}_\ell$ contains an ample divisor, then $o_\Gamma|_{\mathcal{C}_\ell} \equiv 0$ for every Γ since o_Γ is a linear non-negative function on $\mathcal{C}_\ell$. Hence, every element of $\mathcal{C}_\ell$ is semiample by Lemmas 1.8 and 1.11, and so $\mathcal{C}_\ell \subseteq \operatorname{Supp} \mathfrak{R} \cap \pi^{-1}\big(\operatorname{Nef}(X)\big)$. Therefore, the cone $\operatorname{Supp} \mathfrak{R} \cap \pi^{-1}\big(\operatorname{Nef}(X)\big)$ is equal to the union of some $\mathcal{C}_i$, which suffices. $\square$

I next state the result which contains both the Cone and Contraction theorems. The new statement lives in $N^1(X)_{\mathbb{R}}$ and, by duality, involves the nef cone. This formulation has been known for a long time, and origins go back at least to [Kaw88]. However, it has only recently been realised [CL13, Theorem 4.2] that this statement is much easier to prove than Theorems 1.4 and 1.5, once we have right tools at hand.

Theorem 1.13. *Let (X, Δ) be a projective klt pair such that $K_X + \Delta$ is not nef. Let V be the* visible boundary *of $\operatorname{Nef}(X)$ from the class $\kappa = [K_X + \Delta] \in N^1(X)_{\mathbb{R}}$:*

$$V = \big\{\delta \in \partial \operatorname{Nef}(X) \mid [\kappa, \delta] \cap \operatorname{Nef}(X) = \{\delta\}\big\}.$$

Then:

(1) every compact subset F which belongs to the relative interior of V, is contained in a union of finitely many supporting rational hyperplanes of $\operatorname{Nef}(X)$,

(2) every Cartier divisor on X whose class belongs to the relative interior of V is semiample.

Proof. The proof is almost by picture. Note first that since $K_X + \Delta$ is not nef, the class κ is not in $\mathrm{Nef}(X)$. The set V is then precisely the points that κ "sees" on $\mathrm{Nef}(X)$.

Since F is compact, we can pick finitely many rational points $w_1, \ldots, w_m \in N^1(X)_{\mathbb{R}}$ very close to F, such that F is contained in the convex hull of these points. Then it is obvious that F belongs to the boundary of the cone $\mathrm{Nef}(X) \cap \sum \mathbb{R}_+ w_i$, hence it is enough to show that this cone is rational polyhedral.

Note that since each w_i is very close to F, and F belongs to the relative interior of V, the line containing κ and w_i will intersect the ample cone. Therefore, there are rational ample classes α_j and rational numbers $t_j \in (0,1)$ such that

$$w_j = t_j \kappa + (1 - t_j)\alpha_j.$$

For each j, choose an ample $\mathbb{Q}$-divisor A_j which represents the class $\frac{1-t_j}{t_j}\alpha_j$ such that the pair $(X, \Delta + A_j)$ is klt (use Bertini's theorem). Then w_j is the class of the divisor $t_j(K_X + \Delta + A_j)$. By Theorem 1.9, the adjoint ring

$$\mathfrak{R} = R(X; K_X + \Delta + A_1, \ldots, K_X + \Delta + A_m)$$

is finitely generated. Denote by $\pi \colon \mathrm{Div}_{\mathbb{R}}(X) \to N^1(X)_{\mathbb{R}}$ the natural projection. Then

$$\mathrm{Nef}(X) \cap \sum \mathbb{R}_+ w_i \subseteq \pi(\mathrm{Supp}\,\mathfrak{R})$$

by Theorem 1.10(2), and the conclusion follows by Proposition 1.12. $\square$

It is an exercise(!) to show that the statement of this result is precisely dual to the statement of the Cone theorem we saw before. It is now convenient to state the following immediate corollary of Theorem 1.9; the proof is analogous but easier than that of the Cone and Contraction theorems.

Corollary 1.14. *Let (X, Δ) be a projective klt pair where Δ is big. If $K_X + \Delta$ is pseudoeffective, then it is effective. If $K_X + \Delta$ is nef, then it is semiample.*

Proof. Let A be an ample $\mathbb{Q}$-divisor on X such that the pair $(X, \Delta + A)$ is klt. By Theorem 1.9, the ring

$$\mathfrak{R} = R(X; K_X + \Delta, K_X + \Delta + A)$$

is finitely generated, and $\mathrm{Supp}\,\mathfrak{R} = \mathbb{R}_+(K_X + \Delta) + \mathbb{R}_+(K_X + \Delta + A)$ by parts (1) and (2) of Theorem 1.10. This immediately implies the first statement.

If $K_X + \Delta$ is nef, the divisor $K_X + \Delta + \varepsilon A$ is ample for each $\varepsilon > 0$, thus $o_\Gamma(K_X + \Delta + \varepsilon A) = 0$ for every geometric valuation Γ of X. Therefore, all o_Γ are identically zero on $\mathrm{Supp}\,\mathfrak{R}$ by Theorem 1.10(3), and thus $K_X + \Delta$ is semiample by Lemmas 1.8 and 1.11. $\square$

1.3 Properties of contractions and the existence of flips

In this section, we keep the notation from the proof of Theorem 1.13. By that proof, the cone $\mathcal{N} = \operatorname{Supp}\mathfrak{R} \cap \pi^{-1}\big(\operatorname{Nef}(X)\big)$ is rational polyhedral, and every element of this cone is semiample. We pick any of its codimension 1 faces $\mathcal{F}$. Then any line bundle L in the relative interior of $\mathcal{F}$ gives a birational contraction $f = f_{\mathcal{F}} \colon X \to Y$. This map contracts the curves contained in the extremal ray $R = \pi(\mathcal{F})^{\perp} \subseteq \overline{\mathrm{NE}}(X)$, and only them. The next simple but very useful lemma tells us that the morphism f does not depend on the choice of L.

Lemma 1.15. *Let X, Y and Y' be varieties and let $\pi \colon X \to Y$ and $\pi' \colon X \to Y'$ be proper morphisms. Assume that $\pi_*\mathcal{O}_X = \mathcal{O}_Y$ and that π' contracts each fibre of π. Then there exists a morphism $\xi \colon Y \to Y'$ such that $\pi' = \xi \circ \pi$.*

In particular, if π' contracts every curve contracted by π, then π' factors through π.

Proof. Let Z be the image of the proper morphism $\psi = (\pi, \pi') \colon X \to Y \times Y'$ and let $p \colon Z \to Y$ and $p' \colon Z \to Y'$ be the two projections; note that $\pi = p \circ \psi$ and $\pi' = p' \circ \psi$, and that p is proper. For any point $y \in Y$, the fibre $\pi^{-1}(y)$ is contracted by ψ by assumption, hence $p^{-1}(y) = \psi(\pi^{-1}(y))$ is a point. We have

$$\mathcal{O}_Y \subseteq p_*\mathcal{O}_Z \subseteq p_*\psi_*\mathcal{O}_X = \pi_*\mathcal{O}_X = \mathcal{O}_Y,$$

which implies $p_*\mathcal{O}_Z = \mathcal{O}_Y$, and hence p is an isomorphism. We set $\xi = p' \circ p^{-1}$.

For the second statement, it is enough to show that every two points x and y in a fibre F of π can be connected by a curve lying in F. To see this, note that F is connected. By first blowing up x and y in F, and taking a resolution of singularities, we obtain a birational morphism $f \colon F' \to F$ from a smooth projective scheme F' and two prime divisors E and E' on F' such that $f(E) = \{x\}$ and $f(E') = \{y\}$. If H is an irreducible very ample divisor on F', then H intersects E and E', hence $f(H)$ is a connected prime divisor in F containing x and y. We finish by induction on the dimension. □

We start our analysis of the map f. First we note the following important property, which says that over Y, the numerical and the linear equivalence of divisors coincide.

Lemma 1.16. *Let M be a $\mathbb{Q}$-divisor on X such that $M \equiv_f 0$. Then $M \sim_{\mathbb{Q}} f^*M_Y$ for some $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor M_Y on Y.*

Proof. First, note that for $t = \rho(X) - 1$, we can find $\mathbb{Q}$-divisors $B_i = K_X + \Delta + A_i$, $i = 1, \ldots, t$, in the relative interior of $\mathcal{F}$ such that $M \equiv \sum \lambda_i B_i$ for some nonzero rational numbers λ_i: indeed, by assumption, the set $\pi(\mathcal{F})$ spans the hyperplane in $N^1(X)_\mathbb{R}$ which is orthogonal to R, hence the class of M belongs to this hyperplane, and we pick B_i so that their classes are suitable generators of that hyperplane. Note that all B_i are semiample by Theorem 1.13. Denote

$$B_1' = \frac{1}{\lambda_1}\big(M - \textstyle\sum_{i\geq 2}\lambda_i B_i\big).$$

Then $B_1' \equiv B_1$, hence $B_1' = K_X + \Delta + A_1'$ for some ample $\mathbb{Q}$-divisor A_1', and therefore B_1' is semiample by Corollary 1.14. Then by Lemma 1.15, the morphism $X \to \operatorname{Proj} R(X, B_1')$ is, up to isomorphism, equal to f. By the definition of f, there are ample $\mathbb{Q}$-divisors A_1' and A_i on Y such that $B_1' \sim_\mathbb{Q} f^*A_1'$ and $B_i \sim_\mathbb{Q} f^*A_i$ for all i. Therefore $M \sim_\mathbb{Q} f^*M_Y$ for $M_Y = \lambda_1 A_1' + \sum_{i\geq 2}\lambda_i A_i$. $\square$

The following is the main result of this section.

Theorem 1.17. *Let the notation and assumptions be as above. Then:*

(1) if $\dim Y < \dim X$, then Y is $\mathbb{Q}$-factorial,

(2) if f is birational and if the exceptional locus of f contains a divisor, then this locus is a single prime divisor, and Y is $\mathbb{Q}$-factorial,

(3) if f is an isomorphism in codimension 1, then there exists a diagram

$$\begin{tikzcd} X \arrow[rr, dashed, "\varphi"] \arrow[dr, "f"'] & & X^+ \arrow[dl, "f^+"] \\ & Y & \end{tikzcd}$$

such that φ is an isomorphism in codimension 1 which is not an isomorphism, and X^+ is $\mathbb{Q}$-factorial. The divisor $K_{X^+} + \varphi_*\Delta$ is f^+-ample.

We need the following important result in the proof of Theorem 1.17.

Lemma 1.18 (Negativity lemma)**.** *Let $f\colon X \to Y$ be a proper birational morphism between normal varieties. Let $-D$ be an f-nef $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X. Then D is effective if and only if f_*D is.*

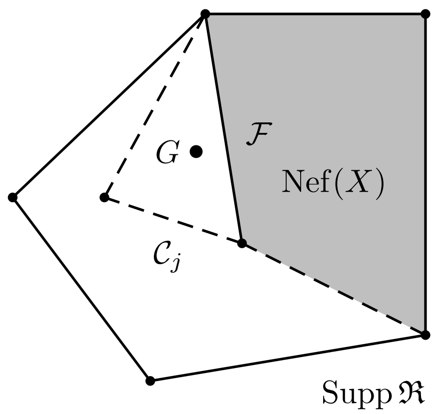
Proof. The lemma is reduced to the surface case by cutting by $\dim X - 2$ general hyperplanes, and then it follows from the Hodge index theorem. The details are in [KM98, Lemma 3.39] or [Deb01, Lemma 7.19]. $\square$

Proof of Theorem 1.17. We first show (1). Let P be a Weil divisor on Y and let $Y_0 \subseteq Y$ be the smooth locus. Let $P' \subseteq X$ be the closure of $f^{-1}(P|_{Y_0})$. Then P' is disjoint from the general fibre of f, hence $P' \cdot C = 0$ for every curve contracted by f. By Lemma 1.16, there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that $P' \sim_{\mathbb{Q}} f^*D$, and therefore $P \sim_{\mathbb{Q}} D$.

To show (2), let E be an f-exceptional prime divisor on X. If $E \cdot C \geq 0$ for some curve C contracted by f, then E is f-nef since all curves contracted by f are numerically proportional, but this contradicts Lemma 1.18. Therefore, $E \cdot C < 0$ for every curve C contracted by f, thus $C \subseteq E$, and so the exceptional locus of f equals E.

Let P be any Weil divisor on Y, and let P' be its proper transform on X. Then P' is $\mathbb{Q}$-Cartier, and since all the curves contracted by f belong to the extremal ray $R \subseteq \mathrm{NE}(X)$, there exists a rational number α such that $P' \equiv_f \alpha E$. By Lemma 1.16 there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that $P' \sim_{\mathbb{Q}} \alpha E + f^*D$. Pushing forward this relation by f, we obtain that the divisor $P \sim_{\mathbb{Q}} D$ is $\mathbb{Q}$-Cartier.

Now we show (3). Let $\operatorname{Supp}\mathfrak{R} = \bigcup \mathcal{C}_i$ be the decomposition as in the proof of Theorem 1.13. Then there is a cone $\mathcal{C}_j \nsubseteq \pi^{-1}\big(\mathrm{Nef}(X)\big)$ of dimension $\dim \operatorname{Supp}\mathfrak{R}$, such that $\mathcal{F}$ is a face of $\mathcal{C}_j$. Let G be any Cartier divisor in the interior of $\mathcal{C}_j$, and let $\varphi\colon X \dashrightarrow X^+ = \operatorname{Proj} R(X,G)$ be the birational map associated to G. We claim that φ does not depend on the choice of G, and that there exists a morphism $f^+\colon X^+ \to Y$ as in (3). Assuming the claim, let us show how it completes the proof of the theorem.



To this end, note first that φ is a birational contraction by Theorem 1.2. Since f is an isomorphism in codimension 1, then so are φ and f^+. Consider a Weil divisor P on X^+, and let P' be its proper transform on X. Since X is $\mathbb{Q}$-factorial, the divisor P' is $\mathbb{Q}$-Cartier. Since all the curves contracted by f belong to R, there exists a rational number α such that $P' \equiv_f \alpha G$. By Lemma 1.16, there exists a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor D on Y such that $P' \sim_{\mathbb{Q}} \alpha G + f^*D$. By the definition of φ, the divisor φ_*G is ample, hence $\mathbb{Q}$-Cartier. Therefore, pushing forward this relation by φ, we obtain that the divisor

$$P \sim_{\mathbb{Q}} \alpha\varphi_*G + (f^+)^*D$$

is $\mathbb{Q}$-Cartier. The fact that the divisor $K_{X^+} + \varphi_*\Delta$ is f^+-ample follows similarly, from the fact that φ_*G is an ample divisor and that $K_{X^+} + \varphi_*\Delta$ and φ_*G lie on the same side of the plane supporting $\varphi_*\mathcal{F}$ (exercise!).

Finally, we prove the claim stated above. Theorem 1.10 implies that we can find a resolution $\theta\colon \widetilde{X} \to X$ and a positive integer d such that $\operatorname{Mob}\theta^*(dD)$ is basepoint free for every Cartier divisor $D \in \operatorname{Supp}\mathfrak{R}$. Denote $M = \operatorname{Mob}\theta^*(dG)$. Then we have the induced birational morphism $\psi\colon \widetilde{X} \to X^+$, which is just the Iitaka fibration associated to M. We only show that the definition of φ does not depend on the choice of G; the existence of the diagram as in (3) follows similarly.

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\theta"'] \arrow[drr, "\psi"] & & \\ X \arrow[rr, dashed, "\varphi"] & & X^+ \end{tikzcd}$$

Pick any other Cartier divisor G' in the interior of the cone $\mathcal{C}_j$, and let $\psi'\colon \widetilde{X} \to \operatorname{Proj} R(X, G')$ be the corresponding map. There exists a Cartier divisor G'' in the interior of $\mathcal{C}_j$, together with positive integers r, r', r'' such that

$$rG = r'G' + r''G''.$$

Denoting $M' = \operatorname{Mob}\theta^*(dG')$ and $M'' = \operatorname{Mob}\theta^*(dG'')$, then we have

$$rM = r'M' + r''M'' \tag{1.5}$$

(since all functions o_Γ are linear on $\mathcal{C}_j$), and the divisors M, M', M'' are basepoint free. For any curve C on $\widetilde{X}$ contracted by ψ we have $M \cdot C = 0$, hence equation (1.5) implies $M' \cdot C = 0$, and so C is contracted by ψ'. Reversing the roles of G and G', we obtain that ψ and ψ' contract the same curves, therefore they are the same map up to isomorphism. □

Recall that in the case (1) of the theorem, the map f is a *Mori fibre space*; in the case (2), the map f is a *divisorial contraction*; and in the case (3), the map φ or the corresponding diagram is *the flip of f*.

Finally, when the map f is birational, then the resulting variety also has klt singularities – this shows that we stay in the same category of pairs in our programme:

Lemma 1.19. *Let the notation and assumptions be as in Theorem 1.17. Then:*

(1) if f is a divisorial contraction, then the pair $(Y, f_*\Delta)$ is klt,

(2) if f is an isomorphism in codimension 1, then the pair $(X^+, \varphi_*\Delta)$ is klt.

Proof. We only show (2), the proof of (1) is completely analogous. It suffices to prove that for every geometric valuation E over X we have $a(E, X, \Delta) \leq a(E, X^+, \Delta^+)$, where $\Delta^+ = \varphi_*\Delta$.

Let $(g, g^+)\colon Z \to X \times X^+$ be the resolution of the rational map φ such that E is a divisor on Z – apply [Har77, Example II.7.17.3] and Lemma 1.6. Set $h = f \circ g = f^+ \circ g^+$. From the relations

$$K_Z \sim_{\mathbb{Q}} g^*(K_X + \Delta) + \sum a(E, X, \Delta) \cdot E$$

and

$$K_Z \sim_{\mathbb{Q}} (g^+)^*(K_{X^+} + \Delta^+) + \sum a(E, X^+, \Delta^+) \cdot E$$

we obtain that the divisor $H = \sum(a(E, X, \Delta) - a(E, X^+, \Delta^+)) \cdot E$ is h-nef. Note also that H is h-exceptional, since every prime divisor in its support is g-exceptional or g^+-exceptional. Then Lemma 1.18 implies that $-H$ is effective, which is what we needed. $\square$

1.4 Termination of the MMP

The variety X^+, thus, has all the desired features similar to X, so we continue the procedure with X^+ instead of X. Unfortunately, it is not easy to find an invariant of varieties which behaves well under flips; the only such example currently exists on threefolds. It is, therefore, the crucial problem to find a sequence of divisorial contractions and flips which terminates.

We know how to do this for a klt pair (X, Δ), where Δ is a big divisor, and this was proved first in [BCHM10]. Here, I give an argument from [CL13] – I hope to convince you that it is not too difficult to deduce it as a consequence of Theorem 1.9.

Lemma 1.20. *Let X and Y be $\mathbb{Q}$-factorial projective varieties and let $f\colon X \dashrightarrow Y$ be a birational map which is an isomorphism in codimension one. Let $\mathcal{C} \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ be a cone spanned by effective divisors and fix a geometric valuation Γ of X. Then the asymptotic order of vanishing o_Γ is linear on $\mathcal{C}$ if and only if it is linear on $f_*\mathcal{C} \subseteq \mathrm{Div}_{\mathbb{R}}(Y)$.*

Proof. For every rational $D \in \mathcal{C}$, write

$$V_D = \{D_X - D \mid D \sim_{\mathbb{Q}} D_X \text{ and } D_X \geq 0\} \subseteq \mathrm{Div}_{\mathbb{R}}(X)$$

and

$$W_D = \{D_Y - f_*D \mid f_*D \sim_{\mathbb{Q}} D_Y \text{ and } D_Y \geq 0\} \subseteq \mathrm{Div}_{\mathbb{R}}(Y).$$

Note that the elements of V_D and W_D are $\mathbb{Q}$-linear combinations of principal divisors, and we have the isomorphism $f_*|_{V_D}\colon V_D \simeq W_D$: indeed, let $U \subseteq X$ and $V \subseteq Y$ be open subsets such that $f_{|U}\colon U \to V$ is an isomorphism and $\mathrm{codim}_Y(Y \setminus V) \geq 2$. Then it is enough to show the claim by restricting to U and V, where it is obvious. Similarly $\mathrm{mult}_\Gamma P_X = \mathrm{mult}_\Gamma f_*P_X$ for every $P_X \in V_D$. Therefore

$$o_\Gamma(D) - \mathrm{mult}_\Gamma D = \inf_{P_X \in V_D} \mathrm{mult}_\Gamma P_X = \inf_{P_X \in V_D} \mathrm{mult}_\Gamma f_*P_X = o_\Gamma(f_*D) - \mathrm{mult}_\Gamma f_*D,$$

hence the function $o_\Gamma(\cdot) - o_\Gamma\big(f_*(\cdot)\big)\colon \mathcal{C} \to \mathbb{R}$ is equal to the linear map $\mathrm{mult}_\Gamma(\cdot) - \mathrm{mult}_\Gamma f_*(\cdot)$. The claim now follows. $\square$

Theorem 1.21. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial klt pair with Δ big. Then:*

(1) if $K_X + \Delta$ is pseudoeffecive, there exists a sequence of $(K_X + \Delta)$-divisorial contractions and $(K_X + \Delta)$-flips which terminates with a variety on which the proper transform of $K_X + \Delta$ is semiample,

(2) if $K_X + \Delta$ is not pseudoeffective, there exists a sequence of $(K_X + \Delta)$-divisorial contractions and $(K_X + \Delta)$-flips which terminates with a Mori fibre space.

Proof. Note that we may assume to start with that any sequence of birational contractions starting from (X, Δ) is a sequence of flips, since in divisorial contractions the Picard rank drops by one.

Denote by $\pi\colon \mathrm{Div}_{\mathbb{R}}(X) \to N^1(X)_{\mathbb{R}}$ the natural projection. Similarly as in the proof of Theorem 1.13, we choose ample $\mathbb{Q}$-divisors $A_1, \dots, A_m$ such that all the pairs $(X, \Delta + A_i)$ are klt, such that the cone $\pi\big(\sum \mathbb{R}_+(K_X + \Delta + A_i)\big)$ has dimension $\rho(X)$, and that this cone contains an ample class.

By Theorem 1.9, the ring

$$\mathfrak{R} = R(X; K_X + \Delta, K_X + \Delta + A_1, \dots, K_X + \Delta + A_m)$$

is finitely generated, and denote $\mathcal{C} = \mathrm{Supp}\, \mathfrak{R}$. Let $\mathcal{C} = \bigcup_{i \in I} \mathcal{C}_i$ be the rational polyhedral decomposition as in Theorem 1.10. Fix an ample divisor A such that if the line ℓ passing through $K_X + \Delta$ and A intersects some codimension 1 face of a cone $\mathcal{C}_i$, then ℓ intersects the relative interior of that face. Set

$$\lambda = \min\{t \in \mathbb{R} \mid K_X + \Delta + tA \text{ is nef}\}.$$

Then by construction, $K_X + \Delta + \lambda A$ belongs to the relative interior of $\mathcal{F}$.

As in Theorems 1.13 and 1.17 we can construct a flip $\varphi\colon X \dashrightarrow X^+$ corresponding to a birational contraction $f\colon X \to Y$, which is an isomorphism in codimension 1, which in turn comes from a codimension 1 face $\mathcal{F}$ of the cone $\mathcal{C}\cap\pi^{-1}(\mathrm{Nef}(X))$ which intersects the line ℓ. By the proof of the cone theorem, we can assume that $\mathcal{F}$ is also a face of some cone $\mathcal{C}_j$.

The map φ is an isomorphism in codimension 1, hence it induces isomorphisms $\mathrm{Div}_{\mathbb{R}}(X) \simeq \mathrm{Div}_{\mathbb{R}}(X^+)$ and

$$\mathfrak{R} \simeq R(X^+; K_{X^+} + \varphi_*\Delta, K_{X^+} + \varphi_*\Delta + \varphi_*A_1, \dots, K_{X^+} + \varphi_*\Delta + \varphi_*A_m).$$

The cone $\mathcal{C}^+ = \varphi_*\mathcal{C} \subseteq \mathrm{Div}_{\mathbb{R}}(X^+)$ has a decomposition $\mathcal{C}^+ = \bigcup_{i\in I^+} \mathcal{C}_i^+$ as in Theorem 1.10. Lemma 1.20 shows that we can assume that $I = I^+$ and $\mathcal{C}_i^+ = \varphi_*\mathcal{C}_i$.

Recall that by Lemma 1.16, every $L \in \mathcal{F}$ is the pullback of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y, hence $\varphi_*L \in \mathrm{Div}_{\mathbb{R}}(X^+)$ is also the pullback of a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on Y. In particular, the divisor φ_*L is again nef, but not ample. In other words, the set $\varphi_*\mathcal{F}$ belongs to the boundary of the cone $\mathrm{Nef}(X^+)$. Note that the interiors of the cones $\varphi_*\mathrm{Nef}(X)$ and $\mathrm{Nef}(X^+)$ do not intersect, since otherwise φ would be an isomorphism. Setting

$$\lambda^+ = \min\{t \in \mathbb{R} \mid K_{X^+} + \varphi_*\Delta + t\varphi_*A \text{ is nef}\},$$

it is clear from the construction that $\lambda^+ < \lambda$ and that $K_{X^+} + \varphi_*\Delta + \lambda\varphi_*A$ belongs to the relative interior of a codimension 1 face of some cone $\mathcal{C}_j^+$. Since there are only finitely many such faces, this process must terminate.

There are two cases. When $K_X + \Delta$ is not pseudoeffecive, the process necessarily stops with a Mori fibre space. If $K_X + \Delta$ is pseudoeffective, the process stops when its proper transform becomes nef, and hence semiample by Corollary 1.14. □

We finish by noting that this allows us to finish the MMP for all pairs of log general type and for pairs which are not pseudoeffective:

Corollary 1.22. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial klt pair such that $K_X + \Delta$ is big. Then there exists a sequence of $(K_X + \Delta)$-divisorial contractions and $(K_X + \Delta)$-flips which terminates with a variety on which the proper transform of $K_X + \Delta$ is semiample.*

Proof. By Kodaira's trick, there exist an ample $\mathbb{Q}$-divisor A and an effective $\mathbb{Q}$-divisor E such that $K_X + \Delta \sim_{\mathbb{Q}} A + E$. Setting $\Delta' = \Delta + \varepsilon(A + E)$ for $0 < \varepsilon \ll 1$, we have that Δ' is a big divisor such that the pair (X, Δ') is klt, and

$$K_X + \Delta' \sim_{\mathbb{Q}} (1 + \varepsilon)(K_X + \Delta).$$

Therefore all $(K_X + \Delta)$-extremal contractions are $(K_X + \Delta')$-extremal contractions. We conclude by Theorem 1.21. □

Corollary 1.23. *Let (X, Δ) be a projective $\mathbb{Q}$-factorial klt pair such that $K_X + \Delta$ is not pseudoeffective. Then there exists a sequence of $(K_X + \Delta)$-divisorial contractions and $(K_X + \Delta)$-flips which terminates with a Mori fibre space.*

Proof. Fix an ample divisor A on X. Then there exists $0 < \mu \ll 1$ such that $K_X + \Delta + \mu A$ is also not pseudoeffective, thus all $(K_X + \Delta)$-extremal contractions are $(K_X + \Delta + \mu A)$-extremal contractions. We conclude by Theorem 1.21. $\square$

Chapter 2

Finite generation of adjoint rings

2.1 The induction scheme

This chapter is devoted to the proof of Theorem 1.9. The proof is very technical, but as before, I try to convince you that the main idea is very natural.

We say that a pair (X, Δ) is *log smooth* is X is smooth and the support of Δ has simple normal crossings. Our first observation is that in order to prove Theorem 1.9, we can freely assume that everything in sight is log smooth. More precisely, we concentrate on proving the following statement.

Theorem A. *Let X be a smooth projective variety, and let $\Delta_1, \dots, \Delta_r$ be $\mathbb{Q}$-divisors on X such that (X, Δ_i) is a log smooth pair and $\lfloor \Delta_i \rfloor = 0$ for every $i = 1, \dots, r$. If A is an ample $\mathbb{Q}$-divisor on X, then the adjoint ring*

$$R(X; K_X + \Delta_1 + A, \dots, K_X + \Delta_r + A)$$

is finitely generated.

Lemma 2.1. *Theorem 1.9 is equivalent to Theorem A.*

Proof. It is clear that Theorem 1.9 implies Theorem A. For the converse, by Kodaira's trick there exist an ample $\mathbb{Q}$-divisor $H \geq 0$ on X and effective divisors E_i such that $\Delta_i \sim_{\mathbb{Q}} E_i + H$. Pick a rational number $0 < \varepsilon \ll 1$, and set $A = \varepsilon H$ and $\Delta_i' = (1 - \varepsilon)\Delta_i + \varepsilon E_i$. Then $K_X + \Delta_i \sim_{\mathbb{Q}} K_X + \Delta_i' + A$, and the pair $(X, \Delta_i' + A)$ is klt for every i since (X, Δ_i) is klt and $\varepsilon \ll 1$. Let $f \colon Y \to X$ be a log resolution of the pair $(X, \sum \Delta_i)$. For each i, let $\Gamma_i, G_i \geq 0$ be $\mathbb{Q}$-divisors on Y without common components such that G_i is f-exceptional and $K_Y + \Gamma_i = f^*(K_X + \Delta_i) + G_i$. By Hironaka's theorem, we can find an f-exceptional $\mathbb{Q}$-divisor $F \geq 0$ on Y with arbitrarily small coefficients such that $A' = f^*A - F$ is ample, and therefore we may assume that $\lfloor \Gamma_i + F \rfloor = 0$ for all i. Then the ring

$$R(Y; K_Y + \Gamma_1 + F + A', \dots, K_X + \Gamma_r + F + A')$$

is finitely generated by Theorem A, hence the ring $R(X; K_X + \Delta_1, \dots, K_X + \Delta_r)$ is finitely generated by Lemma 1.8. $\square$

At this point, it is convenient to define several polytopes in the space $\mathrm{Div}_{\mathbb{R}}(X)$. Before we proceed, we make a small detour into stable base loci and real linear systems. During a course of the proof, we will see that we cannot avoid working with $\mathbb{R}$-divisors.

Definition 2.2. Let X be a smooth projective variety. If D is an $\mathbb{R}$-divisor on X, we denote

$$|D|_{\mathbb{R}} = \{D' \geq 0 \mid D \sim_{\mathbb{R}} D'\} \qquad \text{and} \qquad \mathbf{B}(D) = \bigcap_{D' \in |D|_{\mathbb{R}}} \operatorname{Supp} D',$$

and we call $\mathbf{B}(D)$ the *stable base locus* of D. We set $\mathbf{B}(D) = X$ if $|D|_{\mathbb{R}} = \emptyset$.

Lemma 2.3. *Let X be a smooth projective variety.*

(a) Let D be a $\mathbb{Q}$-divisor on X. Then $\mathbf{B}(D) = \bigcap_q \operatorname{Bs}|qD|$ for all q sufficiently divisible.

(b) Let $D_1, \dots, D_r$ be $\mathbb{Q}$-divisors on X such that the ring $\mathfrak{R} = R(X; D_1, \dots, D_r)$ is finitely generated and let D be an $\mathbb{R}$-divisor in the cone $\sum \mathbb{R}_+ D_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$. Then $D \in \operatorname{Supp} \mathfrak{R}$ if and only if $|D|_{\mathbb{R}} \neq \emptyset$.

Proof. To show (a), note that we have $\mathbf{B}(D) \subseteq \bigcap_q \operatorname{Bs}|qD|$. To show the reverse inclusion, fix a point $x \in X \setminus \mathbf{B}(D)$. Then there exist an $\mathbb{R}$-divisor $F \geq 0$, real numbers $r_1, \dots, r_k$ and rational functions $f_1, \dots, f_k \in k(X)$ such that $F = D + \sum_{i=1}^k r_i(f_i)$ and $x \notin \operatorname{Supp} F$. Let $W \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ be the subspace spanned by the components of D and all (f_i). Let $W_0 \subseteq W$ be the subspace of divisors $\mathbb{R}$-linearly equivalent to zero, and note that W_0 is a rational subspace of W. Consider the quotient map $\pi\colon W \to W/W_0$. Then the set $\{G \in \pi^{-1}(\pi(D)) \mid G \geq 0\}$ is not empty as it contains F, and it is cut out from W by rational hyperplanes. Thus, it contains a $\mathbb{Q}$-divisor $D' \geq 0$ such that $D \sim_{\mathbb{Q}} D'$ and $x \notin \operatorname{Supp} D'$.

For (b), as above let $F \geq 0$ be an $\mathbb{R}$-divisor such that $F \sim_{\mathbb{R}} D$. Then there exist real numbers $r_1, \dots, r_k$ and rational functions $f_1, \dots, f_k \in k(X)$ such that $F = D + \sum_{i=1}^k r_i \operatorname{div} f_i$, and let $W \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ be the finite dimensional subspace based by the components of $D, D_1, \dots, D_r$ and all $\operatorname{div} f_i$. Let $W_0 \subset W$ be the subspace of divisors $\mathbb{R}$-linearly equivalent to zero, and consider the quotient map $\pi\colon W \to W/W_0$. Then the set $\mathcal{G} = \pi^{-1}(\{G \in W \mid G \geq 0\}) \cap \sum \mathbb{R} D_i$ is nonempty as it contains D, and it is cut out in $\sum \mathbb{R} D_i$ by rational hyperplanes. If $D \notin \operatorname{Supp} \mathfrak{R}$, then since $\operatorname{Supp} \mathfrak{R}$ is closed by Theorem 1.10(1), there exists a rational divisor $D' \notin \operatorname{Supp} \mathfrak{R}$ such that $|D'|_{\mathbb{R}} \neq \emptyset$, which is a contradiction with (a). $\square$

Definition 2.4. Let $(X, S+\sum_{i=1}^p S_i)$ be a log smooth projective pair, where S and all S_i are distinct prime divisors, let $V=\sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$, and let A be a $\mathbb{Q}$-divisor on X. We define

$$\begin{aligned}
\mathcal{L}(V) &= \{B=\textstyle\sum b_iS_i \in V \mid 0\leq b_i\leq 1 \text{ for all } i\},\\
\mathcal{E}_A(V) &= \{B\in\mathcal{L}(V) \mid |K_X+A+B|_{\mathbb{R}}\neq\emptyset\},\\
\mathcal{B}_A^S(V) &= \{B\in\mathcal{L}(V) \mid S\not\subseteq \mathbf{B}(K_X+S+A+B)\}.
\end{aligned}$$

Several comments are in order. Note that the set $\mathcal{L}(V)$ is a rational polytope by definition – indeed, it is just a hypercube in V. One of the principal inputs in the proof of Theorem 1.9 will be to show that $\mathcal{E}_A(V)$ and $\mathcal{B}_A^S(V)$ are rational polytopes when A is ample. The importance of $\mathcal{B}_A^S(V)$ will be discussed shortly. First we note that Theorem A immediately implies that $\mathcal{E}_A(V)$ is a rational polytope: indeed, let Δ_i' to be the vertices of $\mathcal{L}(V)$, set $\Delta_i=\Delta_i'-\varepsilon\lfloor\Delta_i'\rfloor$ for $0<\varepsilon\ll 1$, and let $A_i=A+\varepsilon\lfloor\Delta_i'\rfloor$. Then the ring $\mathfrak{R}=R(X;K_X+\Delta_1+A_1,\dots,K_X+\Delta_r+A_r)$ is finitely generated by Theorem 1.9, and $\mathrm{Supp}\,\mathfrak{R}=\mathbb{R}_+(K_X+A+\mathcal{E}_A(V))$. Hence we have:

Theorem B. *Let $(X,\sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where $S_1,\dots,S_p$ are distinct prime divisors. Let $V=\sum_{i=1}^p\mathbb{R}S_i\subseteq\mathrm{Div}_{\mathbb{R}}(X)$, and let A be an ample $\mathbb{Q}$-divisor on X. Then $\mathcal{E}_A(V)$ is a rational polytope.*

We will, actually, in the course of the proof use Theorem B to derive Theorem A. More precisely, we use Theorem A_n to denote Theorem A in dimension n, and similarly for other theorems. Then a rough scheme of the proof looks like this:

$$\mathrm{A}_{n-1}+\mathrm{B}_n\Rightarrow\mathrm{A}_n,\qquad \mathrm{A}_{n-1}+\mathrm{B}_{n-1}\Rightarrow\mathrm{B}_n.$$

We will refine this induction scheme a little bit later. First we need the following simple, but crucial example.

Definition 2.5. Let X be a smooth projective variety, let S be a smooth prime divisor on X and let D be a $\mathbb{Q}$-divisor on X. Fix $\eta\in H^0(X,\mathcal{O}_X(S))$ such that $\mathrm{div}\,\eta=S$. From the exact sequence

$$0\to H^0(X,\mathcal{O}_X(\lfloor D\rfloor-S))\xrightarrow{\cdot\eta}H^0(X,\mathcal{O}_X(\lfloor D\rfloor))\xrightarrow{\rho_S}H^0(S,\mathcal{O}_S(\lfloor D\rfloor))$$

we define $\mathrm{res}_S\,H^0(X,\mathcal{O}_X(D))=\mathrm{im}(\rho_S)$, and for $\sigma\in H^0(X,\mathcal{O}_X(D))$, denote $\sigma_{|S}=\rho_S(\sigma)$. Note that

$$\ker(\rho_S)=H^0(X,\mathcal{O}_X(D-S))\cdot\eta,$$

and that $\mathrm{res}_S\,H^0(X,\mathcal{O}_X(D))=0$ if and only if $S\subseteq\mathrm{Bs}\,|\lfloor D\rfloor|$.

For $\mathbb{Q}$-divisors $D_1, \dots, D_\ell$, the *restriction of $R(X; D_1, \dots, D_\ell)$ to S* is the ring

$$\operatorname{res}_S R(X; D_1, \dots, D_\ell) = \bigoplus_{(n_1,\dots,n_\ell)\in\mathbb{N}^\ell} \operatorname{res}_S H^0\big(X, \mathcal{O}_X(n_1D_1 + \cdots + n_\ell D_\ell)\big).$$

Note that $\operatorname{res}_S R(X, D) = 0$ if and only if $S \subseteq \mathbf{B}(D)$.

Example 2.6. Let $(X, S + B)$ be a log smooth pair, where $\lfloor B \rfloor = 0$ and S is a prime divisor, and assume that there exists a positive rational number r such that $K_X + S + B \sim_\mathbb{Q} rS$. Then $R(X, K_X + S + B)$ is finitely generated if and only if $\operatorname{res}_S R(X/Z, K_X + S + B)$ is finitely generated.

Indeed, the harder part is sufficiency, and by Lemma 1.8 it is enough to prove that the ring $R(X, S)$ is finitely generated. Again by Lemma 1.8 we have that $\operatorname{res}_S R(X, S)$ is finitely generated, and let $\theta_1, \dots, \theta_p$ be some homogeneous generators of $\operatorname{res}_S R(X, S)$. Choose $\Theta_1, \dots, \Theta_p \in R(X, S)$ such that $\Theta_i|_S = \theta_s$. Let $\sigma_S \in H^0(X, S)$ be a section such that $\operatorname{div} \sigma_S = S$ and let $\mathcal{H} = \{\sigma_s, \Theta_1, \dots, \Theta_p\}$. Then $\mathcal{H}$ is the set of generators of $R(X, S)$: indeed, let $\varphi \in R(X, S)$ be any homogeneous section, say $\varphi \in H^0(X, dS)$ for some $d \geq 1$. Then there exists a polynomial $p \in \mathbb{C}[X_1, \dots, X_p]$ such that $\varphi|_S = p(\theta_1, \dots, \theta_p)$. From the exact sequence

$$0 \to H^0(X, (d-1)S) \xrightarrow{\cdot\sigma_S} H^0(X, dS) \to \operatorname{res}_S H^0(X, dS) \to 0$$

we get $\varphi - p(\Theta_1, \dots, \Theta_p) = \sigma_S \cdot \varphi'$ for some $\varphi' \in H^0(X, (d-1)S)$, hence we conclude by descending induction on d.

Note that in this example, $\operatorname{res}_S R(X, K_X + S + B) \subseteq R(S, K_S + B|_S)$. If we had equality instead of inclusion, we would conclude by induction that the ring $R(X.K_X + S + B)$ is finitely generated; however, this is almost never the case. The second problem that we have to deal with in our proof of Theorem 1.9 is that the above condition $K_X + S + B \sim_\mathbb{Q} rS$ also almost never happens; however, in the context of the MMP, it occurs in a special situation called *pl flips*, which was used to give the first proof of the existence of klt flips. This condition was useful for us for two reasons: (a) the section σ_S was immediately an element of $R(X, S)$, and (b) by "dividing by σ_S" we again landed in $R(X, S)$. We will have to deal with both of these issues in the next section. However, the main idea is contained in this example: in favourable circumstances, we do not have to know what the kernel of the restriction map is – rather, it is enough to know the generators of the restriction, and then we can chase the generators of the original ring by hand.

This example suggests the following result, whose role will be apparent from the proof in the following section.

Theorem C. *Let $(X, S + \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S and all S_i are distinct prime divisors. Let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \operatorname{Div}_\mathbb{R}(X)$, let*

A be an ample $\mathbb{Q}$-divisor on X, and let $B_1, \dots, B_m \in \mathcal{E}_{S+A}(V)$ be $\mathbb{Q}$-divisors. Then the ring

$$\operatorname{res}_S R(X; K_X + S + B_1 + A, \dots, K_X + S + B_m + A)$$

is finitely generated.

Now, this result implies that the set $\mathcal{B}_A^S(V)$ is a rational polytope, in exactly the same way as we showed above that Theorem A implies Theorem B. Therefore:

Theorem D. *Let $(X, S + \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S and all S_i are distinct prime divisors. Let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \operatorname{Div}_{\mathbb{R}}(X)$ and let A be an ample $\mathbb{Q}$-divisor on X. Then $\mathcal{B}_A^S(V)$ is a rational polytope.*

Now we can give a refined version of the induction procedure in the proof. In the next section we will show:

Theorem 2.7. *Theorem B_n and Theorem C_n imply Theorem A_n.*

Then we concentrate on proving Theorems C_n and D_n – indeed, we will prove a more general result which will yield both results almost at once. The induction step here is:

Theorem 2.8. *Theorem A_{n-1}, Theorem B_{n-1} and Theorem D_n imply Theorem C_n.*

Theorem 2.9. *Theorem A_{n-1} and Theorem B_{n-1} imply Theorem D_n.*

Finally, the last step is to show

Theorem 2.10. *Theorem D_n implies Theorem B_n.*

2.2 Proof of Theorem 2.7

In order to prove Theorem 2.7, we first need some additional definitions. The following generalises our previous definition of divisorial and adjoint rings.

Definition 2.11. Let X be a smooth projective variety and let $\mathcal{S} \subseteq \operatorname{Div}_{\mathbb{Q}}(X)$ be a finitely generated monoid. Then

$$R(X, \mathcal{S}) = \bigoplus_{D \in \mathcal{S}} H^0\big(X, \mathcal{O}_X(D)\big)$$

is a *divisorial $\mathcal{S}$-graded ring*. If $D_1, \dots, D_\ell$ are generators of $\mathcal{S}$, then there is a natural projection map $R(X; D_1, \dots, D_\ell) \longrightarrow R(X, \mathcal{S})$. If $D_i \sim_{\mathbb{Q}} k_i(K_X + \Delta_i)$, where $\Delta_i \geq 0$ and $k_i \in \mathbb{Q}_+$ for every i, the algebra $R(X, \mathcal{S})$ is an *adjoint ring associated to $\mathcal{S}$*.

If $\mathcal{C} \subseteq \operatorname{Div}_{\mathbb{R}}(X)$ is a rational polyhedral cone, we define the algebra $R(X, \mathcal{C})$, an *adjoint ring associated to $\mathcal{C}$*, to be $R(X, \mathcal{C} \cap \operatorname{Div}(X))$.

Note that here we used that by Gordan's lemma [Ful93, Section 1.2, Proposition 1], the monoid $\mathcal{C} \cap \mathrm{Div}(X)$ is finitely generated. The following lemma summarises the basic properties of preservation of finite generation under natural operations on the monoid $\mathcal{S}$.

Definition 2.12. Let $\mathcal{S} \subseteq \mathbb{Z}^r$ be a finitely generated monoid and let $R = \bigoplus_{s\in\mathcal{S}} R_s$ be an $\mathcal{S}$-graded algebra. If $\mathcal{S}' \subseteq \mathcal{S}$ is a finitely generated submonoid, then $R' = \bigoplus_{s\in\mathcal{S}'} R_s$ is a *Veronese subring* of R. If there exists a subgroup $\mathbb{L} \subset \mathbb{Z}^r$ of finite index such that $\mathcal{S}' = \mathcal{S} \cap \mathbb{L}$, then R' is a *Veronese subring of finite index* of R.

Lemma 2.13. *Let $\mathcal{S} \subseteq \mathbb{Z}^r$ be a finitely generated monoid and let $R = \bigoplus_{s\in\mathcal{S}} R_s$ be an $\mathcal{S}$-graded algebra. Let $\mathcal{S}' \subseteq \mathcal{S}$ be a finitely generated submonoid and let $R' = \bigoplus_{s\in\mathcal{S}'} R_s$.*

(i) If R is finitely generated over R_0, then R' is finitely generated over R_0.

(ii) If R_0 is Noetherian, R' is a Veronese subring of finite index of R, and R' is finitely generated over R_0, then R is finitely generated over R_0.

(iii) Let $s_1, \dots, s_r$ be generators of $\mathcal{S}$, and consider the free monoid $\mathcal{M} = \bigoplus_{i=1}^r \mathbb{N}s_i$ with the natural projection $\pi\colon \mathcal{M} \to \mathcal{S}$. Let M be the $\mathcal{M}$-graded ring with $M_m = R_{\pi(m)}$ for $m \in \mathcal{M}$. Then M is finitely generated if and only if R is finitely generated.

Proof. See [ADHL10, Propositions 1.2.2, 1.2.4, 1.2.6]. □

Now we can prove Theorem 2.7.

Proof of Theorem 2.7.

Step 1. We first assume that there exist $\mathbb{Q}$-divisors $F_i \geq 0$ such that

$$\big(X, \textstyle\sum_i (B_i + F_i)\big) \text{ is log smooth and } K_X + A + B_i \sim_{\mathbb{Q}} F_i \text{ for every } i. \tag{2.1}$$

We reduce the general case to this one at the end of the proof.

Let W be the subspace of $\mathrm{Div}_{\mathbb{R}}(X)$ spanned by the components of all B_i and F_i, and let $S_1, \dots, S_p$ be the prime divisors in W. Denote by

$$\mathcal{T} = \{(t_1, \dots, t_k) \mid t_i \geq 0, \sum t_i = 1\} \subseteq \mathbb{R}^k$$

the standard simplex, and for each $\tau = (t_1, \dots, t_k) \in \mathcal{T}$, set

$$B_\tau = \sum_{i=1}^k t_i B_i \qquad \text{and} \qquad F_\tau = \sum_{i=1}^k t_i F_i \sim_{\mathbb{R}} K_X + A + B_\tau. \tag{2.2}$$

Denote

$$\mathcal{B} = \{F_\tau + B \mid \tau \in \mathcal{T}, 0 \leq B \in W, B_\tau + B \in \mathcal{L}(W)\} \subseteq W,$$

and for every $j = 1, \ldots, p$, let

$$\mathcal{B}_j = \{F_\tau + B \mid \tau \in \mathcal{T}, 0 \leq B \in W, B_\tau + B \in \mathcal{L}(W), S_j \subseteq \lfloor B_\tau + B \rfloor\} \subseteq W.$$

Then $\mathcal{B}$ and $\mathcal{B}_j$ are rational polytopes, and thus $\mathcal{C} = \mathbb{R}_+\mathcal{B}$ and $\mathcal{C}_j = \mathbb{R}_+\mathcal{B}_j$ are rational polyhedral cones. Denote $\mathcal{S} = \mathcal{C} \cap \mathrm{Div}(X)$ and $\mathcal{S}_j = \mathcal{C}_j \cap \mathrm{Div}(X)$. Then it is enough to show that the ring $R(X, \mathcal{S})$ is finitely generated: indeed, let d be a positive integer such that $F_i' = dF_i$ are integral divisors for $i = 1, \ldots, k$. Pick divisors $F_{k+1}', \ldots, F_m'$ such that $F_1', \ldots, F_m'$ are generators of $\mathcal{S}$. Then $R(X; F_1', \ldots, F_m')$ is finitely generated by Lemma 2.13(iii), and so is $R(X; F_1', \ldots, F_k')$ by Lemma 2.13(i). Finally, Lemma 2.13(ii) implies that $R(X; F_1, \ldots, F_k)$ is finitely generated, and therefore so is $R(X; K_X + A + B_1, \ldots, K_X + B + A_k)$ by (2.1) and by Lemma 1.8.

We prove that the ring $R(X, \mathcal{S})$ is finitely generated in Step 3, but first we need a claim.

Step 2. We claim that:

(i) $\mathcal{C} = \bigcup_{j=1}^p \mathcal{C}_j$,

(ii) there exists $M > 0$ such that the "width" of the cones $\mathcal{C}_i$ in the half-plane $\{\sum x_i S_i \mid \sum x_i \geq M\}$ is bigger than 1; more precisely, there exists $M > 0$ such that, if $\sum \alpha_i S_i \in \mathcal{C}_j$ for some j and some $\alpha_i \in \mathbb{N}$ with $\sum \alpha_i \geq M$, then $\sum \alpha_i S_i - S_j \in \mathcal{C}$;

(iii) the ring $\mathrm{res}_{S_j} R(X, \mathcal{S}_j)$ is finitely generated for every $j = 1, \ldots, p$.

Note that (i) and (ii) are true by "looking at the picture", and (iii) follows from Theorem C_n. Note that the picture shows the situation when we only have two components S_1 and S_2, and where our ring has rank 1, but in general the picture is similar, just more complicated. We now give more details.

To see (i), fix $G \in \mathcal{C} \backslash \{0\}$. Then, by definition of $\mathcal{C}$, there exist $\tau \in \mathcal{T}, 0 \leq B \in W$ and $r > 0$ such that $B_\tau + B \in \mathcal{L}(W)$ and $G = r(F_\tau + B)$. Setting

$$\lambda = \max\{t \geq 1 \mid B_\tau + tB + (t-1)F_\tau \in \mathcal{L}(W)\}$$

and $B' = \lambda B + (\lambda - 1)F_\tau$, we have

$$\lambda G = r(F_\tau + B'),$$

and there exists j_0 such that $S_{j_0} \subseteq \lfloor B_\tau + B' \rfloor$. Therefore $G \in \mathcal{C}_{j_0}$, which proves (i).

For (ii), denote by $\|\cdot\|$ the sup-norm on V. There exists $\varepsilon > 0$ such that $\|B_i\| \leq 1 - \varepsilon$ for all i, and thus

$$\|B_\tau\| \leq 1 - \varepsilon \quad \text{for any } \tau \in \mathcal{T}. \tag{2.3}$$

Since the polytopes $\mathcal{B}_j \subseteq W$ are compact, there is a positive constant C such that $\|\Psi\| \leq C$ for any $\Psi \in \bigcup_{j=1}^p \mathcal{B}_j$, and denote $M = pC/\varepsilon$. For some $j \in \{1, \ldots, p\}$, let $G = \sum \alpha_i S_i \in \mathcal{S}_j$ be such that $\sum \alpha_i \geq M$. Since $p\|G\| \geq \sum \alpha_i$, we have

$$\|G\| \geq \frac{M}{p} = \frac{C}{\varepsilon}.$$

By definition of $\mathcal{C}_j$ and of C, we may write $G = rG'$ with $G' \in \mathcal{B}_j$, $\|G'\| \leq C$ and $r > 0$. In particular,

$$r = \frac{\|G\|}{\|G'\|} \geq \frac{1}{\varepsilon}. \tag{2.4}$$

Furthermore, $G' = F_\tau + B$ for some $\tau \in \mathcal{T}$ and $0 \leq B \in W$ such that $B_\tau + B \in \mathcal{L}(W)$ and $S_j \subseteq \lfloor B_\tau + B \rfloor$. Therefore, by (2.3) and (2.4) we have

$$\operatorname{mult}_{S_j} B = 1 - \operatorname{mult}_{S_j} B_\tau \geq \varepsilon \geq \frac{1}{r},$$

and thus

$$G - S_j = r\big(F_\tau + B - \tfrac{1}{r} S_j\big) \in \mathcal{C}.$$

Finally, to show (iii), fix $j \in \{1, \ldots, p\}$, and let $\{E_1, \ldots, E_\ell\}$ be a set of generators of $\mathcal{S}_j$. Then, by definition of $\mathcal{S}_j$ and by (2.2), for every $i = 1, \ldots, \ell$, there exist $k_i \in \mathbb{Q}_+$, $\tau_i \in \mathcal{T} \cap \mathbb{Q}^k$ and $0 \leq B_i \in W$ such that $B_{\tau_i} + B_i \in \mathcal{L}(W)$, $S_j \subseteq \lfloor B_{\tau_i} + B_i \rfloor$ and

$$E_i = k_i(F_{\tau_i} + B_i) \sim_{\mathbb{Q}} k_i(K_X + A + B_{\tau_i} + B_i).$$

Denote $E_i' = K_X + A + B_{\tau_i} + B_i$. Then the ring $\operatorname{res}_{S_j} R(X; E_1', \ldots, E_\ell')$ is finitely generated by Theorem C, and thus so is $\operatorname{res}_{S_j} R(X; E_1, \ldots, E_\ell)$ by Lemma 1.8. Since there is the natural projection $\operatorname{res}_{S_j} R(X; E_1, \ldots, E_\ell) \to \operatorname{res}_{S_j} R(X, \mathcal{S}_j)$, this proves the claim.

Step 3. Now we show how the claim shows that the ring $R(X, \mathcal{S})$ is finitely generated. The proof is similar to that from Example 2.6.

For every $i = 1, \ldots, p$, let $\sigma_i \in H^0(X, \mathcal{O}_X(S_i))$ be a section such that $\operatorname{div} \sigma_i = S_i$. Let $\mathfrak{R} \subseteq R(X; S_1, \ldots, S_p)$ be the ring spanned by $R(X, \mathcal{S})$ and $\sigma_1, \ldots, \sigma_p$, and note that $\mathfrak{R}$ is graded by $\sum_{i=1}^p \mathbb{N} S_i$. By Lemma 2.13(i), it is enough to show that $\mathfrak{R}$ is finitely generated.

For any $\alpha = (\alpha_1, \dots, \alpha_p) \in \mathbb{N}^p$, denote $D_\alpha = \sum \alpha_i S_i$ and $\deg(\alpha) = \sum \alpha_i$, and for a section $\sigma \in H^0(X, \mathcal{O}_X(D_\alpha))$, set $\deg(\sigma) = \deg(\alpha)$. By (ii), for each $j = 1, \dots, p$ there exists a finite set $\mathcal{H}_j \subseteq R(X, \mathcal{S}_j)$ such that

$$\operatorname{res}_{S_j} R(X, \mathcal{S}_j) \quad \text{is generated by the set} \quad \{\sigma_{|S_j} \mid \sigma \in \mathcal{H}_j\}. \tag{2.5}$$

Since the vector space $H^0(X, \mathcal{O}_X(D_\alpha))$ is finite-dimensional for every $\alpha \in \mathbb{N}^p$, there is a finite set $\mathcal{H} \subseteq \mathfrak{R}$ such that

$$\{\sigma_1, \dots, \sigma_p\} \cup \mathcal{H}_1 \cup \dots \cup \mathcal{H}_p \subseteq \mathcal{H}, \tag{2.6}$$

and

$$H^0(X, \mathcal{O}_X(D_\alpha)) \subseteq \mathbb{C}[\mathcal{H}] \quad \text{for every } \alpha \in \mathbb{N}^p \text{ with } D_\alpha \in \mathcal{S} \text{ and } \deg(\alpha) \leq M, \tag{2.7}$$

where $\mathbb{C}[\mathcal{H}]$ is the $\mathbb{C}$-algebra generated by the elements of $\mathcal{H}$. Observe that $\mathbb{C}[\mathcal{H}] \subseteq \mathfrak{R}$, and it suffices to show that $\mathfrak{R} \subseteq \mathbb{C}[\mathcal{H}]$.

Let $\chi \in \mathfrak{R}$. By definition of $\mathfrak{R}$, we may write $\chi = \sum_i \sigma_1^{\lambda_{1,i}} \dots \sigma_p^{\lambda_{p,i}} \chi_i$, where $\chi_i \in H^0(X, \mathcal{O}_X(D_{\alpha_i}))$ for some $D_{\alpha_i} \in \mathcal{S}$ and $\lambda_{j,i} \in \mathbb{N}$. Thus, it is enough to show that $\chi_i \in \mathbb{C}[\mathcal{H}]$, and after replacing χ by χ_i we may assume that

$$\chi \in H^0(X, \mathcal{O}_X(D_\alpha)) \quad \text{for some} \quad D_\alpha \in \mathcal{S}.$$

The proof is by induction on $\deg \chi$. If $\deg \chi \leq M$, then $\chi \in \mathbb{C}[\mathcal{H}]$ by (2.7). Now assume $\deg \chi > M$. Then there exists $1 \leq j \leq p$ such that $D_\alpha \in \mathcal{S}_j$, and so by (2.5) and (2.6) there are $\theta_1, \dots, \theta_z \in \mathcal{H}$ and a polynomial $\varphi \in \mathbb{C}[X_1, \dots, X_z]$ such that $\chi_{|S_j} = \varphi(\theta_{1|S_j}, \dots, \theta_{z|S_j})$. Therefore, from the exact sequence

$$0 \longrightarrow H^0(X, \mathcal{O}_X(D_\alpha - S_j)) \xrightarrow{\cdot \sigma_j} H^0(X, \mathcal{O}_X(D_\alpha)) \longrightarrow H^0(S_j, \mathcal{O}_{S_j}(D_\alpha))$$

we obtain

$$\chi - \varphi(\theta_1, \dots, \theta_z) = \sigma_j \cdot \chi' \quad \text{for some} \quad \chi' \in H^0(X, \mathcal{O}_X(D_\alpha - S_j)).$$

Note that $D_\alpha - S_j \in \mathcal{S}$ by (i), and since $\deg \chi' < \deg \chi$, by induction we have $\chi' \in \mathbb{C}[\mathcal{H}]$. Therefore $\chi = \sigma_j \cdot \chi' + \varphi(\theta_1, \dots, \theta_z) \in \mathbb{C}[\mathcal{H}]$, and we are done.

This completes the proof under the additional assumption that (2.1) holds.

Step 4. We finally prove the general case of the theorem – the goal is to reduce to the case covered above. This is easy, but technical; we want to use Theorem B to reduce to the case where the support of our ring is the whole cone $\sum \mathbb{R}_+(K_X + A + B_i)$, and we also need to pass to a log resolution to make everything in sight simple normal crossings. If you find this believable, I suggest you skip it in the first reading.

Let V be the subspace of $\mathrm{Div}_{\mathbb{R}}(X)$ spanned by the components of all B_i, let $\mathcal{P} \subseteq V$ be the convex hull of all B_i, and denote $\mathcal{R} = \mathbb{R}_+(K_X + A + \mathcal{P})$. Then, by Lemma 2.13(iii) it suffices to show that $R(X, \mathcal{R})$ is finitely generated. By Theorem B_n, $\mathcal{P}_{\mathcal{E}} = \mathcal{P} \cap \mathcal{E}_A(V)$ is a rational polytope, and denote $\mathcal{R}_{\mathcal{E}} = \mathbb{R}_+(K_X + A + \mathcal{P}_{\mathcal{E}})$. Since $H^0(X, \mathcal{O}_X(D)) = 0$ for every integral divisor $D \in \mathcal{R} \setminus \mathcal{R}_{\mathcal{E}}$, the ring $R(X, \mathcal{R})$ is finitely generated if and only if $R(X, \mathcal{R}_{\mathcal{E}})$ is.

By Gordan's lemma, the monoid $\mathcal{R}_{\mathcal{E}} \cap \mathrm{Div}(X)$ is finitely generated, and let R_i be its generators for $i = 1, \dots, \ell$. Then there exist $p_i \in \mathbb{Q}_+$ and $P_i \in \mathcal{P}_{\mathcal{E}} \cap \mathrm{Div}_{\mathbb{Q}}(X)$ such that $R_i = p_i(K_X + A + P_i)$. By construction, $\lfloor P_i \rfloor = 0$ and there exist $\mathbb{Q}$-divisors $G_i \geq 0$ such that

$$K_X + A + P_i \sim_{\mathbb{Q}} G_i$$

for all i. Let $f\colon Y \to X$ be a log resolution of $\big(X, \sum_i (P_i + G_i)\big)$. For every i, there are $\mathbb{Q}$-divisors $C_i, E_i \geq 0$ on Y with no common components such that E_i is f-exceptional and

$$K_Y + C_i = f^*(K_X + P_i) + E_i.$$

Note that $\lfloor C_i \rfloor = 0$, and denote $F_i^\circ = p_i(f^*G_i + E_i) \geq 0$. Let $H \geq 0$ be an f-exceptional $\mathbb{Q}$-divisor on Y such that A° is ample and $\lfloor C_i^\circ \rfloor = 0$ for all i, where $A^\circ = f^*A - H$ is ample and $C_i^\circ = C_i + H$, and denote $D_i^\circ = K_Y + A^\circ + C_i^\circ$. Then

$$p_i D_i^\circ \sim_{\mathbb{Q}} f^*R_i + p_i E_i \sim_{\mathbb{Q}} F_i^\circ.$$

This last relation implies two things: first, it follows from Steps 1–3 and by Lemma 2.13 that the adjoint ring $R(Y; D_1^\circ, \dots, D_\ell^\circ)$ is finitely generated. Second, the ring $R(X; R_1, \dots, R_\ell)$ is then finitely generated by Lemma 1.8. Since there is the natural projection map $R(X; R_1, \dots, R_\ell) \to R(X, \mathcal{R}_{\mathcal{E}})$, the ring $R(X, \mathcal{R}_{\mathcal{E}})$ is finitely generated, and we are done. □

2.3 Nakayama functions

We need several definitions and results from [Nak04]. We would like to find a meaningful extension of the asymptotic valuations o_Γ that we defined before, to the case of pseudo-effective divisors for which we do not necessarily know that they are effective. The starting point is the following simple lemma.

Lemma 2.14. *Let X be a $\mathbb{Q}$-factorial projective variety, let A be an ample $\mathbb{Q}$-divisor on X, and let D and D' be two big $\mathbb{Q}$-divisors on X such that $D \equiv D'$. Let Γ be a prime divisor on X. Then $o_\Gamma(D) = o_\Gamma(D')$ and*

$$o_\Gamma(D) = \lim_{\varepsilon \downarrow 0} o_\Gamma(D + \varepsilon A).$$

Proof. We first prove the second statement. Note that by Kodaira's trick we can write $D \sim_{\mathbb{Q}} \delta A + E$ for some rational $\delta > 0$ and an effective $\mathbb{Q}$-divisor E. Therefore

$$(1+\varepsilon)o_\Gamma(D) = o_\Gamma(D + \varepsilon\delta A + \varepsilon E) \le o_\Gamma(D + \varepsilon\delta A) + \varepsilon o_\Gamma(E) \le o_\Gamma(D) + \varepsilon o_\Gamma(E),$$

and we obtain the claim by letting $\varepsilon \downarrow 0$.

Now, fix an ample divisor A and a rational number $\varepsilon > 0$. Since the divisor $D - D' + \varepsilon A$ is numerically equivalent to εA, and thus ample, we have

$$o_\Gamma(D + \varepsilon A) = o_\Gamma\big(D' + (D - D' + \varepsilon A)\big) \le o_\Gamma(D').$$

Letting $\varepsilon \downarrow 0$ and applying the claim, we get $o_\Gamma(D) \le o_\Gamma(D')$. The reverse inequality is analogous. □

This motivates the following definition.

Definition 2.15. Let X be a smooth projective variety, let A be an ample $\mathbb{Q}$-divisor, and let Γ be a prime divisor. If $D \in \mathrm{Div}_{\mathbb{R}}(X)$ is pseudo-effective, set

$$\sigma_\Gamma(D) = \lim_{\varepsilon\downarrow 0} o_\Gamma(D + \varepsilon A) \qquad \text{and} \qquad N_\sigma(D) = \textstyle\sum_\Gamma \sigma_\Gamma(D)\cdot\Gamma,$$

where the sum runs over all prime divisors Γ on X.

Lemma 2.16. *Let X be a smooth projective variety, let A be an ample $\mathbb{R}$-divisor, let D be a pseudo-effective $\mathbb{R}$-divisor, and let Γ be a prime divisor. Then $\sigma_\Gamma(D)$ exists as a limit, it is independent of the choice of A, it depends only on the numerical equivalence class of D, and $\sigma_\Gamma(D) = o_\Gamma(D)$ if D is big. The function σ_Γ is homogeneous of degree one, convex and lower semi-continuous on the cone of pseudo-effective divisors on X, and it is continuous on the cone of big divisors. For every pseudo-effective $\mathbb{R}$-divisor E we have $\sigma_\Gamma(D) = \lim_{\varepsilon\downarrow 0}\sigma_\Gamma(D + \varepsilon E)$.*

Furthermore, $N_\sigma(D)$ is an $\mathbb{R}$-divisor on X, $D - N_\sigma(D)$ is pseudo-effective, and for any $\mathbb{R}$-divisor $0 \le F \le N_\sigma(D)$ we have $N_\sigma(D - F) = N_\sigma(D) - F$.

Proof. See [Nak04, §III.1]. □

Remark 2.17. Let X be a smooth projective variety, let D_m be a sequence of pseudo-effective $\mathbb{R}$-divisors which converge to an $\mathbb{R}$-divisor D, and let Γ be a prime divisor on X. Then the sequence $\sigma_\Gamma(D_m)$ is bounded. Indeed, pick $k \gg 0$ such that $D - k\Gamma$ is not pseudo-effective, and assume that $\sigma_\Gamma(D_m) > k$ for infinitely many m. Then $D_m - k\Gamma$ is pseudo-effective for infinitely many m by Lemma 2.16, a contradiction.

Remark 2.18. Let X be a smooth projective variety, let D be a pseudo-effective $\mathbb{R}$-divisor, let A be an ample $\mathbb{R}$-divisor, and let $x \in X \setminus \bigcup_{\varepsilon>0} \operatorname{Bs}(D+\varepsilon A)$. Let $f \colon Y \to X$ be the blowup of X along x with the exceptional divisor E. Then $\sigma_E(f^*D) = 0$. To see this, observe that $E \not\subseteq \operatorname{Bs}(f^*D + \varepsilon f^*A)$, and thus $o_E(f^*D + \varepsilon f^*A) = 0$. Letting $\varepsilon \to 0$, we conclude by Lemma 2.16.

Lemma 2.19. *Let X be a smooth projective variety, let D be a pseudo-effective $\mathbb{R}$-divisor, and let A be an ample $\mathbb{Q}$-divisor. If $D \not\equiv N_\sigma(D)$, then there exist a positive integer k and a positive rational number β such that kA is integral and*

$$h^0(X, \mathcal{O}_X(mD + kA)) > \beta m \quad \text{for all} \quad m \gg 0.$$

Proof. Replacing D by $D - N_\sigma(D)$, we may assume that $N_\sigma(D) = 0$. Now apply [Nak04, Theorem V.1.11]. □

Lemma 2.20. *Let X be a smooth projective variety, let D be a pseudo-effective $\mathbb{R}$-divisor on X, and let $\Gamma_1, \dots, \Gamma_\ell$ be distinct prime divisors such that $\sigma_{\Gamma_i}(D) > 0$ for all i. Then for any $\gamma_j \in \mathbb{R}_+$ we have $\sigma_{\Gamma_i}(\sum_{j=1}^{\ell} \gamma_j \Gamma_j) = \gamma_i$ for every i. In particular, if $D \geq 0$ and if $\sigma_\Gamma(D) > 0$ for every component Γ of D, then $D = N_\sigma(D)$.*

Proof. This is [Nak04, Proposition III.1.10]. □

Lemma 2.21. *Let X be a smooth projective variety and let Γ be a prime divisor. Let D be a pseudo-effective $\mathbb{R}$-divisor and let A be an ample $\mathbb{R}$-divisor.*

(i) If $\sigma_\Gamma(D) = 0$, then $\Gamma \not\subseteq \operatorname{Bs}(D + A)$.

(ii) If $\sigma_\Gamma(D) > 0$, then $\Gamma \subseteq \operatorname{Bs}(D + \varepsilon A)$ for $0 < \varepsilon \ll 1$.

Proof. For (i), note that $\sigma_\Gamma(D + \frac{1}{2}A) \leq \sigma_\Gamma(D) = 0$. By Lemma 2.16 there exists $0 \leq D' \sim_\mathbb{R} D + \frac{1}{2}A$ such that $\gamma = \operatorname{mult}_\Gamma D' \ll 1$, and in particular $\frac{1}{2}A + \gamma\Gamma$ is ample. Pick $A' \sim_\mathbb{R} \frac{1}{2}A + \gamma\Gamma$ such that $A' \geq 0$ and $\operatorname{mult}_\Gamma A' = 0$. Then

$$D + A \sim_\mathbb{R} D' - \gamma\Gamma + A' \geq 0 \qquad \text{and} \qquad \operatorname{mult}_\Gamma(D' - \gamma\Gamma + A') = 0.$$

This proves the first claim. The second claim follows from $0 < \sigma_\Gamma(D) = \lim_{\varepsilon \downarrow 0} o_\Gamma(D + \varepsilon A)$, since then $o_\Gamma(D + \varepsilon A) > 0$ for $0 < \varepsilon \ll 1$. □

Lemma 2.22. *Assume Theorem D_n. Let $(X, S + \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S and S_i are distinct prime divisors, let A be an ample $\mathbb{Q}$-divisor on X, and let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \operatorname{Div}_\mathbb{R}(X)$. Then*

$$\mathcal{B}_A^S(V) = \{B \in \mathcal{L}(V) \mid \sigma_S(K_X + S + A + B) = 0\}.$$

Proof. Let $V \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ be the vector space spanned by the components of V. Denoting $\mathcal{Q} = \{B \in \mathcal{L}(V) \mid \sigma_S(K_X + S + A + B) = 0\}$, then clearly $\mathcal{Q} \supseteq \mathcal{B}_A^S(V)$.

For the reverse inclusion, fix $B \in \mathcal{Q}$, and let H be a very ample divisor such that $(X, S + \sum_{i=1}^p S_i + H)$ is log smooth and $H \not\subseteq \mathrm{Supp}(S + \sum_{i=1}^p S_i)$. Let $V_H = \mathbb{R}H + V \subseteq \mathrm{Div}_{\mathbb{R}}(X)$, and note that

$$\sigma_S(K_X + S + A + B + tH) \leq \sigma_S(K_X + S + A + B) = 0 \quad \text{for } t > 0.$$

Then $B + tH \in \mathcal{B}_A^S(V_H)$ for any $0 < t < 1$ by Lemma 2.21(i), hence $B \in \mathcal{B}_A^S(V_H)$ since $\mathcal{B}_A^S(V_H)$ is closed. Therefore $B \in \mathcal{B}_A^S(V)$. □

2.4 Proof of Theorem 2.10

In this section we prove that Theorem D_n implies Theorem B_n. To this end, let $(X, \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S_i are distinct prime divisors, let A be an ample $\mathbb{Q}$-divisor on X, and let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$. Consider the set

$$\mathcal{P}_A(V) = \{B \in \mathcal{L}(V) \mid K_X + A + B \equiv D \text{ for some } \mathbb{R}\text{-divisor } D \geq 0\}.$$

The strategy is to show that this set is a rational polytope, and that it equals $\mathcal{E}_A(V)$. The moral of the story is that for divisors of the form $K_X + B + A$, the effectivity is the numerical property.

2.4.1 Numerical effectivity

We start with the following lemma which makes this more precise.

Lemma 2.23. *Let (X, B) be a log smooth pair, where B is a $\mathbb{Q}$-divisor such that $\lfloor B \rfloor = 0$. Let A be a nef and big $\mathbb{Q}$-divisor, and assume that there exists an $\mathbb{R}$-divisor $D \geq 0$ such that $K_X + A + B \equiv D$. Then there exists a $\mathbb{Q}$-divisor $D' \geq 0$ such that $K_X + A + B \sim_{\mathbb{Q}} D'$.*

Proof. Let $V \subseteq \mathrm{Div}(X)_{\mathbb{R}}$ be the vector space spanned by the components of K_X, A, B and D, and let $\phi\colon V \longrightarrow N^1(X)_{\mathbb{R}}$ be the linear map sending an $\mathbb{R}$-divisor to its numerical class. Since $\phi^{-1}(\phi(K_X + A + B))$ is a rational affine subspace of V, we can assume that $D \geq 0$ is a $\mathbb{Q}$-divisor.

First assume that $(X, B + D)$ is log smooth. Let m be a positive integer such that $m(A + B)$ and mD are integral. Denoting $F = (m-1)D + B$, $L = m(K_X + A + B) - \lfloor F \rfloor$ and $L' = mD - \lfloor F \rfloor$, we have

$$L \equiv L' = D - B + \{F\} \equiv K_X + A + \{F\}.$$

Thus, Kawamata-Viehweg vanishing implies that $h^i(X, \mathcal{O}_X(L)) = h^i(X, \mathcal{O}_X(L')) = 0$ for all $i > 0$, and since the Euler characteristic is a numerical invariant, this yields $h^0(X, \mathcal{O}_X(L)) = h^0(X, \mathcal{O}_X(L'))$. As mD is integral and $\lfloor B \rfloor = 0$, it follows that

$$L' = mD - \lfloor (m-1)D + B \rfloor = \lceil D - B \rceil \geq 0,$$

and thus $h^0\big(X, \mathcal{O}_X(m(K_X + A + B))\big) = h^0(X, \mathcal{O}_X(L + \lfloor F \rfloor)) \geq h^0(X, \mathcal{O}_X(L)) = h^0(X, \mathcal{O}_X(L')) > 0$.

In the general case, let $f\colon Y \to X$ be a log resolution of $(X, B + D)$. Then there exist $\mathbb{Q}$-divisors $B', E \geq 0$ with no common components such that E is f-exceptional and $K_Y + B' = f^*(K_X + B) + E$. Therefore $K_Y + f^*A + B' \equiv f^*D + E \geq 0$, so by above there exists a $\mathbb{Q}$-divisor $D^\circ \geq 0$ such that $K_Y + f^*A + B' \sim_{\mathbb{Q}} D^\circ$. Hence $K_X + A + B \sim_{\mathbb{Q}} f_*D^\circ \geq 0$. □

Corollary 2.24. *Let $(X, \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S_i are distinct prime divisors, let A be an ample $\mathbb{Q}$-divisor on X, and let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$. If $\mathcal{P}_A(V)$ is a rational polytope, then $\mathcal{E}_A(V) = \mathcal{P}_A(V)$.*

Proof. Let $B_1, \dots, B_q$ be the extreme points of $\mathcal{P}_A(V)$, and choose $\varepsilon > 0$ such that $A + \varepsilon B_i$ is ample for every i. Since $K_X + A + B_i = K_X + (A + \varepsilon B_i) + (1 - \varepsilon)B_i$ and $\lfloor (1-\varepsilon)B_i \rfloor = 0$, Lemma 2.23 implies that there exist $\mathbb{Q}$-divisors $D_i \geq 0$ such that $K_X + A + B_i \sim_{\mathbb{Q}} D_i$. Thus $B_i \in \mathcal{E}_A(V)$ for every i, and therefore $\mathcal{P}_A(V) \subseteq \mathcal{E}_A(V)$ as $\mathcal{E}_A(V)$ is convex. Since obviously $\mathcal{E}_A(V) \subseteq \mathcal{P}_A(V)$, the corollary follows. □

Lemma 2.25. *Let $(X, \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S_i are distinct prime divisors, let A be an ample $\mathbb{Q}$-divisor on X, and let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$. If $\mathcal{P}_A(V)$ is a polytope, then it is a rational polytope.*

Proof. Let $B_1, \dots, B_q$ be the extreme points of $\mathcal{P}_A(V)$. Then there exist $\mathbb{R}$-divisors $D_i \geq 0$ such that $K_X + A + B_i \equiv D_i$ for all i. Let $W \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ be the vector space spanned by V and by the components of $K_X + A$ and $\sum_{i=1}^q D_i$. Note that for every $\tau = (t_1, \dots, t_q) \in \mathbb{R}_+^q$ such that $\sum t_i = 1$, we have $B_\tau = \sum t_i B_i \in \mathcal{P}_A(V)$ and $K_X + A + B_\tau \equiv \sum t_i D_i \in W$. Let $\phi\colon W \longrightarrow N^1(X)_{\mathbb{R}}$ be the linear map sending an $\mathbb{R}$-divisor to its numerical class. Then $W_0 = \phi^{-1}(0)$ is a rational subspace of W and

$$\mathcal{P}_A(V) = \{B \in \mathcal{L}(V) \mid B = -K_X - A + D + R, \text{ where } 0 \leq D \in W, R \in W_0\}.$$

Therefore, $\mathcal{P}_A(V)$ is cut out from $\mathcal{L}(V) \subseteq W$ by finitely many rational half-spaces, and thus is a rational polytope. □

2.4.2 Compactness

Hence, until the end of the section we prove that $\mathcal{P}_A(V)$ is a polytope, which suffices by Corollary 2.24 and by Lemma 2.25. We start with a few lemmas, which will first enable us to conclude that $\mathcal{P}_A(V)$ is a closed set. As we will see, this is essentially equivalent to the statement that if an adjoint divisor K_X+A+B is pseudo-effective, then it is numerically equivalent to an effective divisor. This statement is usually referred to as "non-vanishing."

Lemma 2.26. *Let X be a smooth projective variety of dimension n and let $x \in X$. Let $D \in \operatorname{Div}(X)$ and assume that s is a positive integer such that $h^0(X, \mathcal{O}_X(D)) > \binom{s+n}{n}$. Then there exists $D' \in |D|$ such that $\operatorname{mult}_x D' > s$.*

Proof. Let $\mathfrak{m} \subseteq \mathcal{O}_X$ be the ideal sheaf of x. Then we have

$$h^0(X, \mathcal{O}_X/\mathfrak{m}^{s+1}) = \dim_{\mathbb{C}} \mathbb{C}[x_1, \ldots, x_n]/(x_1, \ldots, x_n)^{s+1} = \binom{s+n}{n},$$

hence $h^0(X, \mathcal{O}_X(D)) > h^0(X, \mathcal{O}_X/\mathfrak{m}^{s+1})$. Therefore the exact sequence

$$0 \to \mathfrak{m}^{s+1} \otimes \mathcal{O}_X(D) \to \mathcal{O}_X(D) \to (\mathcal{O}_X/\mathfrak{m}^{s+1}) \otimes \mathcal{O}_X(D) \simeq \mathcal{O}_X/\mathfrak{m}^{s+1} \to 0$$

yields $h^0\big(X, \mathfrak{m}^{s+1} \otimes \mathcal{O}_X(D)\big) > 0$, so there exists a divisor $D' \in |D|$ with multiplicity at least $s+1$ at x. □

Lemma 2.27. *Assume Theorem D_n. Let (X, B) be a log smooth pair of dimension n, where B is an $\mathbb{R}$-divisor such that $\lfloor B \rfloor = 0$. Let A be an ample $\mathbb{Q}$-divisor on X, and assume that K_X+A+B is a pseudo-effective $\mathbb{R}$-divisor such that $K_X+A+B \not\equiv N_\sigma(K_X+A+B)$. Then there exists an $\mathbb{R}$-divisor $F \geq 0$ such that $K_X+A+B \sim_{\mathbb{R}} F$.*

Proof. By Lemma 2.19, we have $h^0(X, \mathcal{O}_X(mk(K_X+A+B)+kA)) > \binom{nk+n}{n}$ for any sufficiently divisible positive integers m and k. Fix a point

$$x \in X \setminus \bigcup_{\varepsilon > 0} \operatorname{Bs}(K_X + A + B + \varepsilon A).$$

Then, by Lemma 2.26 there exists an $\mathbb{R}$-divisor $G \geq 0$ such that $G \sim_{\mathbb{R}} mk(K_X + A + B) + kA$ and $\operatorname{mult}_x G > nk$, so setting $D = \frac{1}{mk} G$, we have

$$D \sim_{\mathbb{R}} K_X + A + B + \frac{1}{m} A \qquad \text{and} \qquad \operatorname{mult}_x D > \frac{n}{m}. \tag{2.8}$$

For any $t \in [0, m]$, define $A_t = \frac{m-t}{m} A$ and $\Psi_t = B + tD$, so that

$$(1+t)(K_X + A + B) \sim_{\mathbb{R}} K_X + A + B + t\big(D - \tfrac{1}{m} A\big) = K_X + A_t + \Psi_t. \tag{2.9}$$

Let $f \colon Y \to X$ be a log resolution of $(X, B+D)$ constructed by first blowing up X at x. Then for every $t \in [0, m]$, there exist $\mathbb{R}$-divisors $C_t, E_t \geq 0$ with no common components such that E_t is f-exceptional and

$$K_Y + C_t = f^*(K_X + \Psi_t) + E_t. \tag{2.10}$$

The exceptional divisor of the initial blowup gives a prime divisor $P \subseteq Y$ such that $\operatorname{mult}_P(K_Y - f^*K_X) = n - 1$, $\operatorname{mult}_P f^*\Psi_t = \operatorname{mult}_x \Psi_t$, and $P \notin \operatorname{Supp} N_\sigma(f^*(K_X + A + B))$ by Remark 2.18. Since $\operatorname{mult}_x \Psi_m > n$ by (2.8), it follows from (2.10) that

$$\operatorname{mult}_P E_m = 0 \quad \text{and} \quad \operatorname{mult}_P C_m > 1. \tag{2.11}$$

Note that $\lfloor C_0 \rfloor = 0$, and denote

$$B_t = C_t - C_t \wedge N_\sigma(K_Y + f^*A_t + C_t).$$

Observe that by (2.9) and (2.10) we have

$$\begin{aligned} N_\sigma(K_Y + f^*A_t + C_t) &= N_\sigma\big(f^*(K_X + A_t + \Psi_t)\big) + E_t \\ &= (1+t)N_\sigma\big(f^*(K_X + A + B)\big) + E_t, \end{aligned}$$

hence B_t is a continuous function in t. Moreover $P \not\subseteq \operatorname{Supp} N_\sigma(K_Y + f^*A_m + B_m)$ by the choice of x and by (2.11), and in particular $\operatorname{mult}_P B_m > 1$. Pick $0 < \varepsilon \ll 1$ such that $\operatorname{mult}_P B_{m-\varepsilon} > 1$, and let $H \geq 0$ be an f-exceptional $\mathbb{Q}$-divisor on Y such that $\lfloor B_0 + H \rfloor = 0$ and $f^*A_{m-\varepsilon} - H$ is ample. Then there exists a minimal $\lambda < m - \varepsilon$ such that $\lfloor B_\lambda + H \rfloor \neq 0$, and let $S \subseteq \lfloor B_\lambda + H \rfloor$ be a prime divisor. Since $\lfloor H \rfloor = 0$, we have $S \subseteq \operatorname{Supp} B_\lambda$. As $B_\lambda \wedge N_\sigma(K_Y + f^*A_\lambda + B_\lambda) = 0$ by Lemma 2.16, we deduce that $S \not\subseteq \operatorname{Supp} N_\sigma(K_Y + f^*A_\lambda + B_\lambda)$.

Let $A' = f^*A_\lambda - H = f^*(\frac{m-\varepsilon-\lambda}{m}A) + (f^*A_{m-\varepsilon} - H)$. Then A' is ample, and since $\sigma_S(K_Y + A' + B_\lambda + H) = \sigma_S(K_Y + f^*A_\lambda + B_\lambda) = 0$ by what we proved above, Lemma 2.22 implies that $S \not\subseteq \operatorname{Bs}(K_Y + A' + B_\lambda + H) = \operatorname{Bs}(K_Y + f^*A_\lambda + B_\lambda)$. In particular, there exists an $\mathbb{R}$-divisor $F' \geq 0$ such that $K_Y + f^*A_\lambda + B_\lambda \sim_\mathbb{R} F'$, and thus, by (2.9) and (2.10),

$$K_X + \Delta \sim_\mathbb{R} \frac{1}{1+\lambda} f_*(K_Y + f^*A_\lambda + C_\lambda) \sim_\mathbb{R} \frac{1}{1+\lambda} f_*(F' + C_\lambda - B_\lambda) \geq 0.$$

This finishes the proof. $\square$

Corollary 2.28. *Let $(X, \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S_i are distinct prime divisors, let A be an ample $\mathbb{Q}$-divisor on X, and let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \operatorname{Div}_\mathbb{R}(X)$. Then $\mathcal{P}_A(V)$ is a closed set and*

$$\mathcal{P}_A(V) = \{B \in \mathcal{L}(V) \mid K_X + A + B \text{ is pseudo-effective}\}.$$

Proof. The last claim follows immediately from Lemma 2.27. For compactness, fix $B \in \overline{\mathcal{P}_A(V)}$ and denote $\Delta = A + B$. In particular, $K_X + \Delta$ is pseudo-effective. If $K_X + \Delta \equiv N_\sigma(K_X + \Delta)$, then it follows immediately that $B \in \mathcal{P}_A(V)$. If $K_X + \Delta \not\equiv N_\sigma(K_X + \Delta)$, assume first that $\lfloor B \rfloor = 0$. Then by Lemma 2.27 there exists an $\mathbb{R}$-divisor $F \geq 0$ such that $K_X + \Delta \sim_\mathbb{R} F$, and in particular $B \in \mathcal{P}_A(V)$. If $\lfloor B \rfloor \neq 0$, pick a $\mathbb{Q}$-divisor $0 \leq G \in V$ such that $A + G$ is ample and $\lfloor B - G \rfloor = 0$. Then $B - G \in \mathcal{P}_{A+G}(V)$ by above, and hence $B \in \mathcal{P}_A(V)$. This implies that $\mathcal{P}_A(V)$ is compact. $\square$

2.4.3 Finitely many extremal points

The technique applied in Lemma 2.27 is often called *tie-breaking*: the idea is to "scale-up" the an adjoint divisor until some of divisor contains a component with coefficient one; the additionally we demand some other properties – in the case of Lemma 2.27, we demanded that the Nakayama function of the adjoint divisor along the component is zero.

Tie-breaking in Lemma 2.27 was a bit involved, since we did not have an effective representative in the (linear equivalence) class of the divisor to start with. Once we have such an effective representative, tie-breaking produces some additional properties. That is the content of the following lemma.

Lemma 2.29. *Assume Theorem D_n. Let (X, B) be a log smooth pair of dimension n, where B is an $\mathbb{R}$-divisor such that $\lfloor B \rfloor = 0$. Let A be an ample $\mathbb{Q}$-divisor on X, assume that $K_X + A + B \not\equiv N_\sigma(K_X + A + B)$, and let $F \geq 0$ be an $\mathbb{R}$-divisor such that $K_X + A + B \sim_{\mathbb{R}} F$, cf. Lemma 2.27.*

Then there exist a positive real number μ such that, if we denote

$$\Phi_\mu = B + \mu F, \quad \Lambda = \Phi_\mu \wedge N_\sigma\big((1+\mu)F\big), \quad \Upsilon_\mu = \Phi_\mu - \Lambda, \quad \Sigma = (1+\mu)F - \Lambda,$$

then the coefficients of Φ_μ are between 0 and 1, we have

$$\Sigma \geq 0 \quad \textit{and} \quad K_X + A + \Upsilon_\mu \sim_{\mathbb{R}} \Sigma, \tag{2.12}$$

and there exists a prime divisor $S \subseteq \lfloor \Phi_\mu \rfloor$ such that

$$\sigma_S(K_X + A + \Upsilon_\mu) = 0 \quad \textit{and} \quad \operatorname{mult}_S \Sigma > 0.$$

Proof. For any $t \geq 0$, define

$$\Phi_t = B + tF, \tag{2.13}$$

so that

$$(1+t)(K_X + A + B) \sim_{\mathbb{R}} K_X + A + B + tF = K_X + A + \Phi_t.$$

Note that $\lfloor \Phi_0 \rfloor = 0$ and

$$N_\sigma(K_X + A + \Phi_t) = (1+t)N_\sigma(K_X + A + B) = (1+t)N_\sigma(F). \tag{2.14}$$

Thus, if we denote

$$\Upsilon_t = \Phi_t - \Phi_t \wedge N_\sigma(K_X + A + \Phi_t), \tag{2.15}$$

then Υ_t is a continuous function in t.

Write $F = \sum_{j=1}^{\ell} f_j F_j$, where F_j are prime divisors and $f_j > 0$ for all j. Since $F \not\equiv N_\sigma(F)$, Lemma 2.20 implies that there exists $j \in \{1, \dots, \ell\}$ such that $\sigma_{F_j}(F) = 0$. Thus, by (2.13), (2.14) and (2.15),

$$\operatorname{mult}_{F_j} \Upsilon_t = \operatorname{mult}_{F_j} B + t f_j,$$

so there exists a minimal $\mu > 0$ such that $\lfloor \Upsilon_\mu \rfloor \neq 0$. Note that $\lfloor \Upsilon_\mu \rfloor \subseteq \operatorname{Supp} F$, but F_j is not necessarily a component of $\lfloor \Upsilon_\mu \rfloor$. Fixing a prime divisor $S \subseteq \lfloor \Upsilon_\mu \rfloor$, we immediately have

$$\sigma_S(K_X + A + \Upsilon_\mu) = 0$$

by (2.15). Moreover,

$$\begin{aligned} \sigma_S\big((1+\mu)F\big) &= \sigma_S(K_X + A + \Phi_\mu) = \operatorname{mult}_S \Phi_\mu - \operatorname{mult}_S \Upsilon_\mu \\ &= \operatorname{mult}_S B + \mu \operatorname{mult}_S F - 1 < \mu \operatorname{mult}_S F \end{aligned}$$

by (2.13), (2.14) and (2.15), hence

$$\operatorname{mult}_S \Sigma \geq (1+\mu) \operatorname{mult}_S F - \sigma_S\big((1+\mu)F\big) > \operatorname{mult}_S F \geq 0.$$

The relations in (2.12) are clear from the construction. $\square$

Now we have all the tools to show that $\mathcal{P}_A(V)$ is a polytope. We do it in the following way: Assume for contradiction that $\mathcal{P}_A(V)$ is not a polytope. Then there exists an infinite sequence of distinct extreme points $B_m \in \mathcal{P}_A(V)$. By compactness and by passing to a subsequence we can assume that there is a point $B \in \mathcal{P}_A(V)$ such that $\lim\limits_{m \to \infty} B_m = B$. We will show that for infinitely many m there exist $B'_m \in \mathcal{P}_A(V)$ such that $B_m \in (B, B'_m)$, so that in particular, no such B_m can be an extreme point of $\mathcal{P}_A(V)$. We do it in Lemmas 2.32 and 2.33, depending on the properties of the point B.

But first we need a simple lemma from convex geometry which characterises polytopes.

Lemma 2.30. *Let $\mathcal{P}$ be a compact convex set in $\mathbb{R}^N$, and fix any norm $\|\cdot\|$ on $\mathbb{R}^N$. Then $\mathcal{P}$ is a polytope if and only if for every point $x \in \mathcal{P}$ there exists a real number $\delta = \delta(x, \mathcal{P}) > 0$, such that for every $y \in \mathbb{R}^N$ with $0 < \|x - y\| < \delta$, if $(x, y) \cap \mathcal{P} \neq \emptyset$, then $y \in \mathcal{P}$.*

Proof. Suppose that $\mathcal{P}$ is a polytope and let $x \in \mathcal{P}$. Let $F_1, \dots, F_k$ be the set of all the faces of $\mathcal{P}$ which do not contain x. Then it is enough to define

$$\delta(x, \mathcal{P}) = \min\{\|x - y\| \mid y \in F_i \text{ for some } i = 1, \dots, k\}.$$

Conversely, assume that $\mathcal{P}$ is not a polytope, and let x_n be an infinite sequence of distinct extreme points of $\mathcal{P}$. Since $\mathcal{P}$ is compact, by passing to a subsequence we may assume that there exists $x = \lim_{n\to\infty} x_n \in \mathcal{P}$. For any real number $\delta > 0$ pick $k \in \mathbb{N}$ such that $0 < \|x - x_k\| < \delta$, and set $x' = x + t(x_k - x)$ for some $1 < t < \delta/\|x - x_k\|$. Then $0 < \|x - x'\| < \delta$ and $\emptyset \neq (x, x_k) \subseteq (x, x') \cap \mathcal{P}$, but $x' \notin \mathcal{P}$ since x_k is an extreme point of $\mathcal{P}$. This proves the lemma. $\square$

Remark 2.31. With assumptions from Lemma 2.30, assume additionally that $\mathcal{P}$ does not contain the origin, and let $\mathcal{C} = \mathbb{R}_+\mathcal{P}$. Then the same proof shows that $\mathcal{C}$ is a polyhedral cone if and only if for every point $x \in \mathcal{C}$ there exists a real number $\delta = \delta(x, \mathcal{C}) > 0$, such that for every $y \in \mathbb{R}^N$ with $0 < \|x - y\| < \delta$, if $(x, y) \cap \mathcal{C} \neq \emptyset$, then $y \in \mathcal{C}$.

Lemma 2.32. *Assume Theorem D_n. Let $(X, \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S_i are distinct prime divisors, let A be an ample $\mathbb{Q}$-divisor on X, and let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \operatorname{Div}_{\mathbb{R}}(X)$. Fix $B \in \mathcal{P}_A(V)$, and let $B_m \in \mathcal{P}_A(V)$ be a sequence of distinct points such that $\lim_{m\to\infty} B_m = B$. Assume that $\lfloor B \rfloor = 0$ and*

$$K_X + A + B \not\equiv N_\sigma(K_X + A + B).$$

Then for infinitely many m there exist $B'_m \in \mathcal{P}_A(V)$ such that $B_m \in (B, B'_m)$.

Proof. By Lemma 2.27, there exists an $\mathbb{R}$-divisor $F \geq 0$ such that $K_X + A + B \sim_{\mathbb{R}} F$. We first prove the lemma under an additional assumption that $F \in V$, and treat the general case at the end of the proof.

Step 1. We use notation from Lemma 2.29. For every $m \in \mathbb{N}$, define $\Phi_{\mu,m} = B_m + \mu(F + B_m - B)$. Then

$$\lim_{m\to\infty} \Phi_{\mu,m} = \Phi_\mu \qquad \text{and} \qquad (1+\mu)(K_X + A + B_m) \sim_{\mathbb{R}} K_X + A + \Phi_{\mu,m} \quad (2.16)$$

by assumption, and let

$$\Lambda_m = \Phi_{\mu,m} \wedge \sum_{Z \subseteq \operatorname{Supp} \Lambda} \sigma_Z(K_X + A + \Phi_{\mu,m}) \cdot Z.$$

Note that $0 \leq \Lambda_m \leq N_\sigma(K_X + A + \Phi_{\mu,m})$. By Lemma 2.16, we have $\Lambda \leq \liminf_{m\to\infty} \Lambda_m$, and in particular, $\operatorname{Supp} \Lambda_m = \operatorname{Supp} \Lambda$ for $m \gg 0$. Thus, there exists an increasing sequence of rational numbers $\varepsilon_m > 0$ such that $\lim_{m\to\infty} \varepsilon_m = 1$ and $\Lambda_m \geq \varepsilon_m \Lambda$, and define $\Upsilon_{\mu,m} = \Phi_{\mu,m} - \varepsilon_m \Lambda$.

Note that $K_X + A + \Upsilon_{\mu,m}$ is pseudo-effective by Lemma 2.16, and

$$\lim_{m\to\infty} \Upsilon_{\mu,m} = \Upsilon_\mu \qquad (2.17)$$

by (2.16). We claim that by passing to a subsequence, for every m there exist $\Upsilon'_m \in V$ and $0 < \alpha_m \ll 1$ such that

$$K_X + A + \Upsilon'_m \quad \text{is pseudo-effective} \qquad \text{and} \qquad \Upsilon_{\mu,m} = \alpha_m \Upsilon_\mu + (1 - \alpha_m)\Upsilon'_m.$$

This immediately implies the lemma under our additional assumption that $F \in V$: indeed, setting $B'_m = \frac{1}{1-\alpha_m}(B_m - \alpha_m B)$, we have $B_m = \alpha_m B + (1 - \alpha_m)B'_m$, and an easy calculation involving (2.13), (2.16) and (2.17) shows that

$$K_X + A + B'_m \sim_{\mathbb{R}} \frac{1}{1+\mu}\Big(K_X + A + \Upsilon'_m + \frac{\varepsilon_m - \alpha_m}{1 - \alpha_m}\Lambda\Big).$$

In particular, $K_X + A + B'_m$ is pseudo-effective for $m \gg 0$. Since $\mathcal{L}(V)$ is a rational polytope, Lemma 2.30 yields $B'_m \in \mathcal{L}(V)$ for $m \gg 0$, hence $B'_m \in \mathcal{P}_A(V)$ by Corollary 2.28.

Step 2. In this step we prove the claim from Step 1. By relabelling if necessary, we may assume that $S = S_1$ and denote $W = \sum_{i=2}^p \mathbb{R}S_i \subseteq \operatorname{Div}_{\mathbb{R}}(X)$. Let

$$\Sigma_m = \Sigma + \Upsilon_m - \Upsilon \sim_{\mathbb{R}} K_X + A + \Upsilon_m \quad \text{and} \quad \Gamma_m = \Sigma_m - \sigma_S(\Sigma_m)S.$$

Then Γ_m is pseudo-effective by Lemma 2.16. Let

$$Z = \sum_{\operatorname{mult}_{S_i}\Upsilon = 1} S_i - \sum_{\operatorname{mult}_{S_j}\Upsilon = 0} S_j,$$

and pick a rational number $0 < \varepsilon \ll 1$ such that the $\mathbb{Q}$-divisor $A' = A + \varepsilon Z$ is ample. Setting $\Upsilon' = \Upsilon - S + \varepsilon Z$, we have

$$\Upsilon' \in \sum_{i=1}^p [\varepsilon S_i, (1-\varepsilon)S_i] \quad \text{and} \quad K_X + S + A' + \Upsilon' \sim_{\mathbb{R}} \Sigma. \tag{2.18}$$

By Theorem D_n, $\mathcal{B}^S_{A'}(V)$ is a rational polytope, and denote

$$\mathcal{P} = \Sigma - \Upsilon' + \mathcal{B}^S_{A'}(W) \quad \text{and} \quad \mathcal{D} = \mathbb{R}_+\mathcal{P} \subseteq V.$$

Then $\mathcal{P}$ is a rational polytope and $\mathcal{D}$ is a rational polyhedral cone. Since

$$\sigma_S(K_X + S + A' + \Upsilon') = \sigma_S(\Sigma) = \sigma_S(K_X + A + \Upsilon) = 0$$

by assumption, Lemma 2.22 implies that $\Upsilon' \in \mathcal{B}^S_{A'}(W)$, and therefore $\Sigma \in \mathcal{P}$. By the definition of $\mathcal{P}$ and by (2.18), for every $D \in \mathcal{P}$ there exists $B \in \mathcal{B}^S_{A'}(W)$ such that

$$D = \Sigma - \Upsilon' + B \sim_{\mathbb{R}} K_X + S + A' + B.$$

Since $\operatorname{mult}_S \Upsilon' = \operatorname{mult}_S B = 0$, this implies $\operatorname{mult}_S D = \operatorname{mult}_S \Sigma > 0$ and, in particular, $\mathcal{P}$ does not contain the origin. Moreover, by the definition of $\mathcal{B}^S_{A'}(V)$, every such D is pseudo-effective, hence every element of $\mathcal{D}$ is pseudo-effective.

We will show that, after passing to a subsequence, we have

$$\Gamma_m \in \mathcal{D} \quad \text{for all } m > 0, \text{ and } \quad \lim_{m\to\infty} \Gamma_m = \Sigma. \tag{2.19}$$

This immediately implies the claim from Step 1: indeed, Remark 2.31 applied to $\mathcal{D}$ and to the point $\Sigma \in \mathcal{D}$ shows that for any $m \gg 0$ there exist $\Psi_m \in \mathcal{D}$ and $0 < \mu_m < 1$ such that $\Gamma_m = \mu_m \Sigma + (1 - \mu_m)\Psi_m$. Then Ψ_m is pseudo-effective, and thus so is the $\mathbb{R}$-divisor

$$\Sigma'_m = \Psi_m + \frac{1}{1-\mu_m}(\Sigma_m - \Gamma_m) = \Psi_m + \frac{\sigma_S(\Sigma_m)}{1-\mu_m} S.$$

Let $\Upsilon'_m = \frac{1}{1-\mu_m}(\Upsilon_m - \mu_m \Upsilon) \in V$. Then it is easy to check that $\Upsilon_m \in (\Upsilon, \Upsilon'_m)$ and $K_X + A + \Upsilon'_m \sim_{\mathbb{R}} \Sigma'_m$ is pseudo-effective as desired.

It remains to prove (2.19). Note that

$$\{\Sigma + \Theta \mid \Theta \in \mathcal{L}(V), \|\Theta\| \le \varepsilon\} \subseteq \mathcal{D},$$

and therefore $\dim \mathcal{D} = \dim V$. If Σ belongs to the interior of $\mathcal{D}$, then $\Sigma_m \in \mathcal{D}$ for $m \gg 0$ and, in particular, $\sigma_S(\Sigma_m) = 0$. Therefore, $\Gamma_m = \Sigma_m$ and the claim follows.

Otherwise, Σ belongs to the boundary of $\mathcal{D}$. Let $\mathcal{H}_i$ be the supporting hyperplanes of maximal faces of $\mathcal{D}$ containing Σ, for $i = 1, \dots, \ell \le \dim V - 1$. Let $\mathcal{W}_i$ be the half-spaces bounded by $\mathcal{H}_i$ containing $\mathcal{D}$, and denote $\mathcal{Q} = \bigcap_{i=1}^{\ell} \mathcal{W}_i$. Note that $\mathcal{Q}$ is an unbounded polygon which contains $\mathcal{D}$. If $\Sigma_m \in \mathcal{Q}$ for infinitely many m, then $\Sigma_m \in \mathcal{D}$, and again $\Gamma_m = \Sigma_m$.

Thus, after taking a subsequence, we may assume that $\Sigma_m \notin \mathcal{Q}$ for all m. Since $\operatorname{mult}_S \Sigma > 0$, let $\lambda_m = \operatorname{mult}_S \Gamma_m / \operatorname{mult}_S \Sigma \in \mathbb{R}$, and for every m choose $0 < \beta_m \ll 1$ such that $\delta_m = \beta_m \lambda_m < 1$ and $\beta_m \|\Gamma_m - \lambda_m \Sigma\| < \varepsilon$. Denote $R_m = \Upsilon' + \beta_m \Gamma_m - \delta_m \Sigma$, and note that by the choice of β_m and δ_m we have $\operatorname{mult}_S R_m = 0$. Furthermore, since $\|\beta_m \Gamma_m - \delta_m \Sigma\| < \varepsilon$, by (2.18) we have $R_m \in \mathcal{L}(V)$, and note that

$$(1 - \delta_m)\Sigma + \beta_m \Gamma_m \sim_{\mathbb{R}} K_X + A + R_m = K_X + S + A' + R_m. \tag{2.20}$$

By assumption and by definition of Γ_m, we have

$$\sigma_S\big((1-\delta_m)\Sigma + \beta_m \Gamma_m\big) \le (1-\delta_m)\sigma_S(\Sigma) + \beta_m \sigma_S(\Gamma_m) = 0, \tag{2.21}$$

hence Lemma 2.22 implies that $R_m \in \mathcal{B}^S_{A'}(V)$, and in particular

$$(1-\delta_m)\Sigma + \beta_m \Gamma_m \in \mathcal{D}. \tag{2.22}$$

As $\Sigma \in \mathcal{H}_i$ for every i, the convex cone $\mathbb{R}_{>0}\Sigma + \mathbb{R}_{>0}\Gamma_m$ intersects $\mathcal{W}_i$ for every i. This implies that $\Gamma_m \in \mathcal{W}_i$, and thus $\Gamma_m \in \mathcal{Q}$. Therefore, after passing to a subsequence we may assume that there is $i_0 \in \{1, \dots, \ell\}$, such that for all m there exists $P_m \in [\Sigma_m, \Gamma_m] \cap \mathcal{H}_{i_0}$. In particular $\lim\limits_{m\to\infty} P_m = \Sigma$, and thus $P_m \in \mathcal{D}$ for $m \gg 0$. This implies $\sigma_S(P_m) = 0$, and finally $\Gamma_m = P_m \in \mathcal{D}$ and $\lim\limits_{m\to\infty} \Gamma_m = \Sigma$.

Step 3. To show the general case of the lemma when F is not necessarily an element of V, let $f\colon Y \to X$ be a log resolution of $(X, B + F)$. Then there are $\mathbb{R}$-divisors $C, E \geq 0$ on Y with no common components and $C_m, E_m \geq 0$ on Y with no common components such that E and E_m are f-exceptional and

$$K_Y + C = f^*(K_X + B) + E \quad \text{and} \quad K_Y + C_m = f^*(K_X + B_m) + E_m.$$

Note that $\lim\limits_{m\to\infty} C_m = C$. Let $G \geq 0$ be an f-exceptional $\mathbb{Q}$-divisor on Y such that A° is ample, $\lfloor C^\circ \rfloor = 0$, and $\lfloor C^\circ_m \rfloor = 0$ for all $m \gg 0$, where

$$A^\circ = f^*A - G, \quad C^\circ = C + G \quad \text{and} \quad C^\circ_m = C_m + G.$$

Denoting $F^\circ = f^*F + E \geq 0$, we have

$$f_*C^\circ = B, \quad f_*C^\circ_m = B_m, \quad \text{and} \quad K_Y + A^\circ + C^\circ \sim_{\mathbb{R}} F^\circ.$$

Let $V^\circ \subseteq \mathrm{Div}_{\mathbb{R}}(Y)$ be the vector space spanned by the components of $\sum_{i=1}^p f_*^{-1}S_i + f_*^{-1}F$ plus all exceptional prime divisors, and note that $F^\circ \in V^\circ$. By what we proved above, for infinitely many m there exist $C'_m \in \mathcal{P}_{A^\circ}(V^\circ)$ such that $C^\circ_m \in (C^\circ, C'_m)$. Note that $\operatorname{Supp} C'_m$ is a subset of $\sum_{i=1}^p f_*^{-1}S_i$ plus all exceptional prime divisors, and denote $B'_m = f_*C'_m \in \mathcal{L}(V)$. Then $B_m \in (B, B'_m)$, and the divisor

$$K_X + A + B'_m = f_*(K_Y + A^\circ + C'_m)$$

is numerically equivalent to an effective divisor, hence $B'_m \in \mathcal{P}_A(V)$. $\square$

Lemma 2.33. *Let $(X, \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S_i are distinct prime divisors, let A be an ample $\mathbb{Q}$-divisor on X, and let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$. Fix $B \in \mathcal{P}_A(V)$, and let $B_m \in \mathcal{P}_A(V)$ be a sequence of distinct points such that $\lim\limits_{m\to\infty} B_m = B$. Assume that $\lfloor B \rfloor = 0$ and*

$$K_X + A + B \equiv N_\sigma(K_X + A + B).$$

Then for infinitely many m there exist $B'_m \in \mathcal{P}_A(V)$ such that $B_m \in (B, B'_m)$.

Proof. Let $D_m \geq 0$ be $\mathbb{R}$-divisors such that $K_X + A + B_m \equiv D_m$. By Lemma 2.21(ii), there exists an ample $\mathbb{R}$-divisor H such that

$$\operatorname{Supp} N_\sigma(K_X + A + B) \subseteq \mathrm{Bs}(K_X + A + B + H),$$

and as $H+(K_X+A+B-D_m)\equiv H+(B-B_m)$ is ample for all $m\gg 0$, by passing to a subsequence we may assume that

$$\begin{aligned}\operatorname{Supp} N_\sigma(K_X+A+B) &\subseteq \operatorname{Bs}\big(D_m+H+(K_X+A+B-D_m)\big) \\ &\subseteq \operatorname{Bs}(D_m)\subseteq \operatorname{Supp} D_m\end{aligned} \tag{2.23}$$

for all m. For $m\in\mathbb{N}$ and $t>1$, denote $C_{m,t}=B+t(B_m-B)$, and observe that

$$B_m=\frac{1}{t}C_{m,t}+\frac{t-1}{t}B \tag{2.24}$$

and

$$K_X+A+C_{m,t}\equiv tD_m-(t-1)(K_X+A+B)\equiv tD_m-(t-1)N_\sigma(K_X+A+B). \tag{2.25}$$

Since $\mathcal{L}(V)$ is a polytope and $B\in\mathcal{L}(V)$, pick $\delta=\delta(B,\mathcal{L}(V))>0$ as in Lemma 2.30. By passing to a subsequence we may assume that $\|B_m-B\|\leq\delta/2$ for every m, and as $\|C_{m,t}-B\|=t\|B_m-B\|$, Lemma 2.30 gives $C_{m,t}\in\mathcal{L}(V)$ for all m and $1<t<2$.

Fix m. By (2.23) there exists $1<t_m<2$ such that $t_mD_m-(t_m-1)N_\sigma(K_X+A+B)\geq 0$, and denote $B'_m=C_{m,t_m}$. Then (2.25) implies $B'_m\in\mathcal{P}_A(V)$, and thus (2.24) proves the lemma. □

Corollary 2.34. *Assume Theorem D_n. Let $(X,\sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S_i are distinct prime divisors, let A be an ample $\mathbb{Q}$-divisor on X, and let $V=\sum_{i=1}^p\mathbb{R}S_i\subseteq\operatorname{Div}_\mathbb{R}(X)$. Fix $B\in\mathcal{P}_A(V)$, and let $B_m\in\mathcal{P}_A(V)$ be a sequence of distinct points such that $\lim\limits_{m\to\infty}B_m=B$. Then for infinitely many m there exist $B'_m\in\mathcal{P}_A(V)$ such that $B_m\in(B,B'_m)$. In particular, $\mathcal{P}_A(V)$ is a polytope.*

Proof. Pick $\delta=\delta(B,\mathcal{L}(V))$ as in Lemma 2.30. By passing to a subsequence, we may choose a $\mathbb{Q}$-divisor $0\leq G\in V$ such that A° is ample, $\lfloor B^\circ\rfloor=0$ and all $\lfloor B_m^\circ\rfloor=0$, where

$$A^\circ=A+G,\quad B^\circ=B-G\quad\text{and}\quad B_m^\circ=B_m-G.$$

By Lemmas 2.32 and 2.33, for infinitely many m there exist $F_m\in\mathcal{P}_{A^\circ}(V)$ such that $B_m^\circ\in(B^\circ,F_m)$. In particular, setting $B'_m=F_m+G$, we have $B_m\in(B,B'_m)$. Since $B-B'_m=B^\circ-F_m$, we may assume that $\|B-B'_m\|\leq\delta$ for $m\gg 0$ by choosing F_m closer to B° if necessary. Therefore, by Lemma 2.30 applied to the polytope $\mathcal{L}(V)$ and the point $B\in\mathcal{L}(V)$, we have $B'_m\in\mathcal{L}(V)$ for $m\gg 0$, and thus $B'_m\in\mathcal{P}_A(V)$ since $K_X+A+B'_m=K_X+A^\circ+F_m$ is numerically equivalent to an effective divisor. □

This finishes the proof of Theorem 2.10.

2.5 Proofs of Theorems 2.8 and 2.9

In this section we finally finish the circle of induction, by proving that Theorems A_{n-1} and B_{n-1} imply Theorems C_n and D_n. This is the only step which really involves induction on the dimension, and hence we have to relate global sections of pluricanonical bundles with the corresponding bundles in dimension one lower. This is done via so called extension theorems.

2.5.1 Extension theorem

As always, let $(X, \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S_i are distinct prime divisors, let A be an ample $\mathbb{Q}$-divisor on X, and let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$. Our goal is to analyse what precisely is the vector space $\mathrm{res}_S H^0(X, m(K_X + S + A + B))$ for $B \in \mathcal{E}_A(V)$, at least when $m \gg 0$. We know from before that this space is only interesting when $B \in \mathcal{B}_A^S(V)$, and in any case, we know that $\mathrm{res}_S H^0(X, m(K_X+S+A+B)) \subseteq H^0(S, m(K_S+A_{|S}+B_{|S}))$ by definition. In practice, this inclusion is almost never an equality. Our goal is to show that the vector space we are looking for is actually a complete linear system on S, and not just any linear system – it is a linear system associated to an adjoint line bundle on S. This is precisely the content of Theorem 2.37, or more precisely, of Corollary 2.39 below. We will prove these results later in the course. Their formulations look (and are) terrifying, but let us first see what they mean.

First we need a few definitions.

Definition 2.35. Let X be a smooth projective variety and let S be a smooth prime divisor. Let C and D be $\mathbb{Q}$-divisors on X such that $|C|_{\mathbb{Q}} \neq \emptyset$, $|D|_{\mathbb{Q}} \neq \emptyset$ and $S \nsubseteq \mathrm{Bs}(D)$. Then $|D|_S$ denotes the image of the linear system $|D|$ under restriction to S, and we define

$$\mathbf{Fix}(C) = \liminf \frac{1}{k} \operatorname{Fix} |kC| \qquad \text{and} \qquad \mathbf{Fix}_S(D) = \liminf \frac{1}{k} \operatorname{Fix} |kD|_S$$

for all k sufficiently divisible.

If V is any linear system on X, then $\mathrm{Fix}(V)$ denotes the fixed divisor of V, i.e. the maximal divisor smaller than any divisor in V. Then $\mathrm{Mov}(V) = \{D - \mathrm{Fix}(V) \mid D \in V\}$ is the movable part of V.

Definition 2.36. Let (X, Δ) be a log pair with $\lfloor \Delta \rfloor = 0$. Then (X, Δ) has *canonical*, respectively *terminal*, singularities if for every log resolution $f\colon Y \to X$, if we write

$$K_Y + f_*^{-1}\Delta = f^*(K_X + \Delta) + E,$$

we have $E \geq 0$, respectively $E \geq 0$ and $\operatorname{Supp} E = \operatorname{Exc} f$. Note that if (X, Δ) is terminal, then for every $\mathbb{R}$-divisor G, the pair $(X, \Delta + \varepsilon G)$ is also terminal for every $0 \leq \varepsilon \ll 1$.

A typical example of a terminal pair is a log smooth pair (X, Δ), where the components of Δ are *disjoint* (exercise!). Starting from a klt pair we can always reach a terminal pair on a log resolution; we will see a slight generalisation of this in Lemma below.

Now we can state the extension theorem.

Theorem 2.37. *Let $(X, S+B)$ be a log smooth projective pair of dimension n, where S is a prime divisor, and B is a $\mathbb{Q}$-divisor such that $S \not\subseteq \operatorname{Supp} B$ and $\lfloor B \rfloor = 0$. Let A be an ample $\mathbb{Q}$-divisor on X and denote $\Delta = S + A + B$. Let $C \geq 0$ be a $\mathbb{Q}$-divisor on S such that (S, C) is canonical, and let m be a positive integer such that mA, mB and mC are integral.*

Assume that for some rational number $0 \leq \varepsilon < \frac{1}{m}$ we have $S \not\subseteq \mathbf{B}(K_X + \Delta + \varepsilon A)$ and

$$C \leq B_{|S} - B_{|S} \wedge \mathbf{Fix}_S(K_X + \Delta + \varepsilon A).$$

Then

$$|m(K_S + A_{|S} + C)| + m(B_{|S} - C) \subseteq |m(K_X + \Delta)|_S.$$

In particular, if $|m(K_S + A_{|S} + C)| \neq \emptyset$, then $|m(K_X + \Delta)|_S \neq \emptyset$, and

$$\operatorname{Fix} |m(K_S + A_{|S} + C)| + m(B_{|S} - C) \geq \operatorname{Fix} |m(K_X + \Delta)|_S \geq m \, \mathbf{Fix}_S(K_X + \Delta).$$

Furthermore, if we assume Theorem A_{n-1}, then

$$\mathbf{Fix}(K_S + A_{|S} + C) + (B_{|S} - C) \geq \mathbf{Fix}_S(K_X + \Delta).$$

The presence of the divisor C may seem very strange, however we will see that this precise form of the theorem will be crucial in our proofs below. The following lemma shows how we can, and will, achieve the condition that the pair (S, C) is canonical (even terminal).

Lemma 2.38. *Let $(X, S + B)$ be a log smooth projective pair, where S is a prime divisor and B is a $\mathbb{Q}$-divisor such that $\lfloor B \rfloor = 0$ and $S \not\subseteq \operatorname{Supp} B$. Then there exist a log resolution $f \colon Y \to X$ of $(X, S + B)$ and $\mathbb{Q}$-divisors $C, E \geq 0$ on Y with no common components, such that the components of C are disjoint, E is f-exceptional, and if $T = f_*^{-1} S$, then*

$$K_Y + T + C = f^*(K_X + S + B) + E.$$

In particular, the pair $(T, C_{|T})$ is terminal.

Proof. By [KM98, Proposition 2.36], there exist a log resolution $f \colon Y \to X$ which is a sequence of blow-ups along intersections of components of B, and $\mathbb{Q}$-divisors

$C, E \geq 0$ on Y with no common components, such that the components of C are disjoint, E is f-exceptional, and

$$K_Y + C = f^*(K_X + B) + E.$$

Since $(X, S + B)$ is log smooth, it follows that if some components of B intersect, then no irreducible component of their intersection is contained in S. Thus $T = f^*S$, and the lemma follows. $\square$

Corollary 2.39. *Let $(X, S+B)$ be a log smooth projective pair, where S is a prime divisor, and B is a $\mathbb{Q}$-divisor such that $S \not\subseteq \operatorname{Supp} B$, $\lfloor B \rfloor = 0$ and $(S, B_{|S})$ is canonical. Let A be an ample $\mathbb{Q}$-divisor on X and denote $\Delta = S + A + B$. Let m be a positive integer such that mA and mB are integral, and such that $S \not\subseteq \operatorname{Bs}|m(K_X + \Delta)|$. Denote $\Phi_m = B_{|S} - B_{|S} \wedge \frac{1}{m} \operatorname{Fix}|m(K_X + \Delta)|_S$.*

Then

$$|m(K_S + A_{|S} + \Phi_m)| + m(B_{|S} - \Phi_m) = |m(K_X + \Delta)|_S.$$

In other words, if we consider linear systems on S as subsets of $k(S)$, then

$$\operatorname{res}_S H^0(X, m(K_X + \Delta)) \simeq H^0(S, m(K_S + A_{|S} + \Phi_m)).$$

Proof. Since $\Phi_m \leq B_{|S} - B_{|S} \wedge \frac{1}{qm} \operatorname{Fix}|qm(K_X + \Delta + \frac{1}{m}A)|_S$ for any positive integer q, the inclusion $|m(K_S + A_{|S} + \Phi_m)| + m(B_{|S} - \Phi_m) \subseteq |m(K_X + \Delta)|_S$ follows from Theorem 2.37.

For the reverse inclusion, it suffices to note that $m(B_{|S} - \Phi_m) \leq \operatorname{Fix}|m(K_X + \Delta)|_S$, and hence $\operatorname{Mov}|m(K_X + \Delta)|_S \subseteq |m(K_S + A_{|S} + \Phi_m)|$. $\square$

2.5.2 Proof of Theorem D

The following result contains the heart of the proof.

Proposition 2.40. *Assume Theorem A_{n-1} and Theorem B_{n-1}. Let $(X, S+\sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S and all S_i are distinct prime divisors. Let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \operatorname{Div}_{\mathbb{R}}(X)$, let A be an ample $\mathbb{Q}$-divisor on X, and let $W \subseteq \operatorname{Div}_{\mathbb{R}}(S)$ be the subspace spanned by the components of $\sum S_{i|S}$.*

(i) Define the set

$$\mathcal{F} = \{E \in \mathcal{E}_{A_{|S}}(W) \mid E \wedge \mathbf{Fix}(K_S + A_{|S} + E) = 0\}.$$

Then there are finitely many rational polytopes $\mathcal{F}_i$ such that $\mathcal{F} = \bigcup_i \mathcal{F}_i$.

(ii) Let $\mathcal{G}$ be a rational polytope contained in the interior of $\mathcal{L}(V)$, and assume that $(S, B_{|S})$ is terminal for every $B \in \mathcal{G}$. For each i, define

$$\mathcal{Q}'_i = \{(B,C) \in \mathrm{Div}_{\mathbb{Q}}(X) \times \mathrm{Div}_{\mathbb{Q}}(S) \mid B \in \mathcal{G} \cap \mathcal{B}_A^S(V), C \in \mathcal{F}_i,$$
$$C \leq B_{|S} - B_{|S} \wedge \mathbf{Fix}_S(K_X + S + A + B)\}.$$

Then the convex hull of $\mathcal{Q}'_i$ is a rational polytope.

(iii) The set $\mathcal{G} \cap \mathcal{B}_A^S(V)$ is a rational polytope.

This result immediately implies Theorem 2.9:

Proof of Theorem 2.9. Fix $B \in \overline{\mathcal{B}_A^S(V)}$, and let $B_m \in \overline{\mathcal{B}_A^S(V)}$ be a sequence of distinct points such that $\lim\limits_{m\to\infty} B_m = B$. It is enough to find a rational polytope $\mathcal{G} \subseteq \mathcal{B}_A^S(V)$ such that the points B and B_m belong to $\mathcal{G}$: indeed, since B is arbitrary, this implies that $\mathcal{B}_A^S(V)$ is closed, and that around every point there are only finitely many extreme points of $\mathcal{B}_A^S(V)$, hence $\mathcal{B}_A^S(V)$ is a polytope. If, in particular, B is an extremal point of $\mathcal{B}_A^S(V)$, this further shows that B is rational.

Let $G \in V$ be a $\mathbb{Q}$-divisor such that $B - G$ is contained in the interior of $\mathcal{L}(V)$, and that $A + G$ is ample. Denote

$$B^G = B - G, \quad B_m^G = B_m - G \quad \text{and} \quad A^G = A + G,$$

and observe that B^G and B_m^G belong to $\overline{\mathcal{B}_{A^G}^S(V)}$ for $m \gg 0$. By Lemma 2.38, there exist a log resolution $f\colon Y \to X$ of $(X, S + B^G)$ and $\mathbb{Q}$-divisors $C, E \geq 0$ on Y with no common components, such that the components of C are disjoint, $\lfloor C \rfloor = 0$, $T = f_*^{-1}S \nsubseteq \operatorname{Supp} C$, and

$$K_Y + T + C = f^*(K_X + S + B^G) + E.$$

We may then write

$$K_Y + T + C_m = f^*(K_X + S + B_m^G) + E_m,$$

where $C_m, E_m \geq 0$ are $\mathbb{Q}$-divisors on Y with no common components, $\lfloor C_m \rfloor = 0$, $T \nsubseteq \operatorname{Supp} C_m$, and note that $\lim\limits_{m\to\infty} C_m = C$. Let $V^\circ \subseteq \mathrm{Div}_{\mathbb{R}}(Y)$ be the subspace spanned by the components of C and by all f-exceptional prime divisors. Then there exists an f-exceptional $\mathbb{Q}$-divisor $F \geq 0$ such that $f^*A^G - F$ is ample, $C + F$ lies in the interior of $\mathcal{L}(V^\circ)$ and $(T, (C+F)_{|T})$ is terminal. Denote

$$A^\circ = f^*A^G - F, \quad C^\circ = C + F \quad \text{and} \quad C_m^\circ = C_m + F,$$

and observe that C° and C_m° belong to $\overline{\mathcal{B}_{A^\circ}^T(V^\circ)}$ for $m \gg 0$.

Let $\mathcal{P}$ be a rational polytope of dimension $\dim V^\circ$ contained in the interior of $\mathcal{L}(V^\circ)$ and containing C° in its interior, such that $(T, \Theta_{|T})$ is terminal for every $\Theta \in \mathcal{P}$. Then $\mathcal{P}' = \mathcal{P} \cap \mathcal{B}^T_{A^\circ}(V^\circ)$ is a rational polytope by Proposition 2.40. In particular, it is closed, so C° and C°_m belong to $\mathcal{B}^T_{A^\circ}(V^\circ)$ for $m \gg 0$. Therefore, $B^G = f_* C^\circ$ and $B^G_m = f_* C^\circ_m$ belong to $\mathcal{B}^S_{A^G}(V)$ for $m \gg 0$, and hence $B, B_m \in \mathcal{B}^S_A(V)$.

The set $f_* \mathcal{P}' \subseteq \mathcal{B}^S_{A^G}(V)$ is a polytope, and thus the set

$$\mathcal{G} = \mathcal{L}(V) \cap (G + f_* \mathcal{P}') \subseteq \mathcal{B}^S_A(V)$$

is also a polytope which contains the points B and B_m for $m \gg 0$, which concludes the proof. $\square$

Proof of Proposition 2.40(i)

The set $\mathcal{E}_{A_{|S}}(W)$ is a rational polytope by Theorem B_{n-1}, and if $E_1, \dots, E_d$ are its extreme points, the ring

$$\mathfrak{R} = R(S; K_S + A_{|S} + E_1, \dots, K_S + A_{|S} + E_d)$$

is finitely generated by Theorem A_{n-1}. Therefore, the function

$$\mathbf{Fix}\colon \operatorname{Supp} \mathfrak{R} \cap \operatorname{Div}_{\mathbb{Q}}(X) \to \mathbb{R}$$

extends to a rational piecewise linear function on $\operatorname{Supp} \mathfrak{R} = \mathbb{R}_+(K_S + A_{|S} + \mathcal{E}_{A_{|S}}(W))$ by Theorem 1.10. Then $\mathcal{F}$ is a subset of $\mathcal{E}_{A_{|S}}(W)$ defined by finitely many linear equalities and inequalities. Thus, there are finitely many rational polytopes $\mathcal{F}_i$ such that $\mathcal{F} = \bigcup_i \mathcal{F}_i$.

Proof of Proposition 2.40(ii)

We proceed in several steps.

Step 0. We fix some notation until the end of the proof. By abuse of notation, $\|\cdot\|$ denotes the sup-norm on $\operatorname{Div}_{\mathbb{R}}(X)$, $\operatorname{Div}_{\mathbb{R}}(S)$ and on $\operatorname{Div}_{\mathbb{R}}(X) \times \operatorname{Div}_{\mathbb{R}}(S)$. Denote by $\mathcal{Q}_i$ the convex hull of $\mathcal{Q}'_i$, and set

$$\mathbf{\Phi}(B) = B_{|S} - B_{|S} \wedge \mathbf{Fix}_S(K_X + S + A + B)$$

for a $\mathbb{Q}$-divisor $B \in \mathcal{B}^S_A(V)$. By Theorem 1.10 there exists a positive integer k with the property that

$$\mathbf{Fix}(K_S + A_{|S} + E) = \frac{1}{m} \operatorname{Fix} |m(K_S + A_{|S} + E)| \tag{2.26}$$

for every rational $E \in \mathcal{E}_{A_{|S}}(W)$ and every $m \in \mathbb{N}$ such that mA/k and mE/k are integral; note that, in particular, $|m(K_S + A_{|S} + E)| \neq \emptyset$ for every such m.

Fix a rational number $0 < \varepsilon \ll 1$ such that $D + \frac{1}{4}A$ is ample for any $D \in V$ with $\|D\| < \varepsilon$, and $\varepsilon(K_X + S + A + B) + \frac{1}{4}A$ is ample for any $B \in \mathcal{L}(V)$.

Step 1. In this step we prove that $\mathcal{Q}_i'$ is dense in $\mathcal{Q}_i$.

To this end, fix $(B_0, C_0), (B_1, C_1) \in \mathcal{Q}_i'$, and for a rational number $0 \le t \le 1$ set

$$(B_t, C_t) = \big((1-t)B_0 + tB_1, (1-t)C_0 + tC_1\big) \in \mathcal{P} \times \mathcal{F}_i.$$

It suffices to show that $(B_t, C_t) \in \mathcal{Q}_i'$, i.e. that $C_t \le \mathbf{\Phi}(B_t)$ for every t.

Let T be a prime divisor in W. If $\operatorname{mult}_T C_t = 0$ for some $0 < t < 1$, then since $\operatorname{mult}_T C_0 \ge 0$ and $\operatorname{mult}_T C_1 \ge 0$ we must have $\operatorname{mult}_T C_t = 0$ for all rational $t \in [0,1]$, and in particular $\operatorname{mult}_T C_t \le \operatorname{mult}_T \mathbf{\Phi}(B_t)$.

Otherwise, we have $\operatorname{mult}_T C_t > 0$ for all $0 < t < 1$, and it follows from the definition of $\mathcal{F}_i$ and by continuity of the function $\mathbf{Fix}$, cf. the proof of part (i), that

$$\operatorname{mult}_T \mathbf{Fix}(K_S + A_{|S} + C_t) = 0 \quad \text{for all} \quad t \in [0,1]. \tag{2.27}$$

By Theorem 2.37 we have

$$\mathbf{Fix}_S(K_X + S + A + B_j) \le \mathbf{Fix}(K_S + A_{|S} + C_j) + (B_{j|S} - C_j),$$

and therefore $\operatorname{mult}_T \big(B_{j|S} - \mathbf{Fix}_S(K_X + S + A + B_j)\big) \ge \operatorname{mult}_T C_j$ by (2.27). Hence,

$$\operatorname{mult}_T C_t \le \operatorname{mult}_T \big(B_{t|S} - \mathbf{Fix}_S(K_X + S + A + B_t)\big) \le \operatorname{mult}_T \mathbf{\Phi}(B_t)$$

for all t by convexity of the function $\mathbf{Fix}_S$.

Step 2. Let

$$\mathcal{C}_i = \{(G, F) \in \mathcal{G} \times \mathcal{F}_i \mid F \le G_{|S}\}.$$

Note that $\mathcal{C}_i$ is a rational polytope and $\overline{\mathcal{Q}_i} \subseteq \mathcal{C}_i$. Recall the definition of ε from Step 0. We claim:

Claim 2.41. Suppose we are given $(B, C) \in \overline{\mathcal{Q}_i}$ and $(\Gamma, \Psi) \in \operatorname{face}\big(\mathcal{C}_i, (B, C)\big)$. Assume that there exist a positive integer m and a rational number $0 < \phi \le 1$ such that mA/k, $m\Gamma/k$ and $m\Psi/k$ are integral, that $\|(B, C) - (\Gamma, \Psi)\| < \frac{\phi\varepsilon}{2m}$, and that for any prime divisor T on S we have

$$\operatorname{mult}_T(B_{|S} - C) > \phi \qquad \text{or} \qquad \operatorname{mult}_T(B_{|S} - C) \le \operatorname{mult}_T(\Gamma_{|S} - \Psi).$$

Then $(\Gamma, \Psi) \in \mathcal{Q}_i'$.

Assuming the claim, let us see how it implies Proposition 2.40(ii). Fix a point $(B, C) \in \overline{\mathcal{Q}_i}$, and let Π be the set of prime divisors T on S such that $\operatorname{mult}_T(B_{|S} - C) > 0$. If $\Pi \ne \emptyset$, pick a positive rational number

$$\phi < \min\{\operatorname{mult}_T(B_{|S} - C) \mid T \in \Pi\} \le 1,$$

and set $\phi = 1$ if $\Pi = \emptyset$. By Lemma 2.42, there exist finitely many points $(\Gamma_j, \Psi_j) \in \operatorname{face}\big(\mathcal{C}_i, (B, C)\big)$ and positive integers m_j divisible by k, such that $m_j A/k$, $m_j \Gamma_j/k$ and $m_j \Psi_j/k$ are integral, (B, C) is a convex linear combination of all (Γ_j, Ψ_j), and

$$\|(B, C) - (\Gamma_j, \Psi_j)\| < \frac{\phi\varepsilon}{2m_j}.$$

Now Claim 2.41 implies $(\Gamma_j, \Psi_j) \in \mathcal{Q}'_i$ for all j, hence $(B, C) \in \mathcal{Q}_i$. This shows that $\mathcal{Q}_i$ is closed and that all of its extreme points are rational.

Next we show that $\mathcal{Q}_i$ is a rational polytope. Assume for a contradiction that $\mathcal{Q}_i$ is not a polytope. Then there exist infinitely many distinct rational extreme points $v_n = (B_n, C_n)$ of $\mathcal{Q}_i$, with $n \in \mathbb{N}$. Since $\mathcal{Q}_i$ is compact and $\mathcal{C}_i$ is a rational polytope, by passing to a subsequence there exist $v_\infty = (B_\infty, C_\infty) \in \mathcal{Q}_i$ and a positive dimensional face $\mathcal{V}$ of $\mathcal{C}_i$ such that

$$v_\infty = \lim_{n\to\infty} v_n \qquad \text{and} \qquad \operatorname{face}(\mathcal{C}_i, v_n) = \mathcal{V} \quad \text{for all } n \in \mathbb{N}. \tag{2.28}$$

In particular, $v_\infty \in \mathcal{V}$. Let Π_∞ be the set of all prime divisors T on S such that $\operatorname{mult}_T(B_{\infty|S} - C_\infty) > 0$. If $\Pi_\infty \neq \emptyset$, pick a positive rational number

$$\phi < \min\{\operatorname{mult}_T(B_{\infty|S} - C_\infty) \mid T \in \Pi_\infty\} \leq 1,$$

and set $\phi = 1$ if $\Pi_\infty = \emptyset$. Then, if k is the positive integer from Step 0, then by Lemma 2.42 there exist $v'_\infty \in \operatorname{face}(\mathcal{C}_i, v_\infty)$, and a positive integer m divisible by k, such that $\frac{m}{k} v'_\infty$ is integral and $\|v_\infty - v'_\infty\| < \frac{\phi\varepsilon}{2m}$. By Claim 2.41 we have $v'_\infty \in \mathcal{Q}_i$. Pick $j \gg 0$ so that

$$\|v_j - v'_\infty\| \leq \|v_j - v_\infty\| + \|v_\infty - v'_\infty\| < \frac{\phi\varepsilon}{2m}, \tag{2.29}$$

and that $\operatorname{mult}_T(B_{j|S} - C_j) > \phi$ if $T \in \Pi_\infty$. Note that v_j is contained in the relative interior of $\mathcal{V}$ by (2.28), and $v'_\infty \in \operatorname{face}(\mathcal{C}_i, v_\infty) \subseteq \mathcal{V}$. Therefore, there exists a positive integer $m' \gg 0$ divisible by k, such that $\frac{m+m'}{k} v_j$ is integral, and such that if we define

$$v'_j = \frac{m + m'}{m'} v_j - \frac{m}{m'} v'_\infty,$$

then $v'_j \in \mathcal{V}$. Note that $\frac{m'}{k} v'_j$ is integral,

$$v_j = \frac{m'}{m + m'} v'_j + \frac{m}{m + m'} v'_\infty, \tag{2.30}$$

and

$$\|v'_j - v_j\| = \frac{m}{m'} \|v_j - v'_\infty\| < \frac{\phi\varepsilon}{2m'} \tag{2.31}$$

by (2.29). Furthermore, if $v'_\infty = (B'_\infty, C'_\infty)$, $v'_j = (B'_j, C'_j)$, and if T is a prime divisor on S such that $T \notin \Pi_\infty$, then $\operatorname{mult}_T(B'_{\infty|S} - C'_\infty) = 0$ as $v'_\infty \in \operatorname{face}(\mathcal{C}_i, v_\infty)$, hence (2.30) gives

$$\operatorname{mult}_T(B_{j|S} - C_j) = \frac{m'}{m+m'} \operatorname{mult}_T(B'_{j|S} - C'_j) \leq \operatorname{mult}_T(B'_{j|S} - C'_j). \tag{2.32}$$

Therefore, $v'_j \in \mathcal{Q}_i$ by (2.31), (2.32) and by Claim 2.41, and since v_j belongs to the interior of the segment $[v'_j, v'_\infty]$ by (2.30), the point v_j is not an extreme point of $\mathcal{Q}_i$. This is a contradiction which finishes the proof.

Step 3. It remains to prove Claim 2.41. It suffices to show

$$\operatorname{mult}_T \mathbf{Fix}_S(K_X + S + A + \Gamma + \tfrac{1}{2m}A) \leq \operatorname{mult}_T(\Gamma_{|S} - \Psi) \tag{2.33}$$

for every prime divisor $T \subseteq \operatorname{Supp}\Psi$. Indeed, then it clearly follows that

$$\Gamma_{|S} \wedge \mathbf{Fix}_S(K_X + S + A + \Gamma + \tfrac{1}{2m}A) \leq \Gamma_{|S} - \Psi,$$

hence Theorem 2.37 implies

$$|m(K_S + A_{|S} + \Psi)| + m(\Gamma_{|S} - \Psi) \subseteq |m(K_X + S + A + \Gamma)|_S \tag{2.34}$$

and

$$\mathbf{Fix}(K_S + A_{|S} + \Psi) + (\Gamma_{|S} - \Psi) \geq \mathbf{Fix}_S(K_X + S + A + \Gamma). \tag{2.35}$$

By the assumption on m from Step 0, (2.34) yields $\Gamma \in \mathcal{B}^S_A(V)$. Since $\Psi \in \mathcal{F}_i$, we have $\Psi \wedge \mathbf{Fix}(K_S + A_{|S} + \Psi) = 0$, (2.35) shows that

$$\Gamma_{|S} - \Psi \geq \Gamma_{|S} \wedge \mathbf{Fix}_S(K_X + S + A + \Gamma),$$

and finally $\Psi \leq \mathbf{\Phi}(\Gamma)$.

To conclude, we show (2.33). Since $(B, C) \in \overline{\mathcal{Q}_i}$, and $\mathcal{Q}'_i$ is dense in $\mathcal{Q}_i$ by Step 1, for every $0 < \delta < \frac{\varepsilon}{m}$ there exists a point $(B_\delta, C_\delta) \in \mathcal{Q}'_i$ such that $\|B - B_\delta\| < \frac{\delta}{2}$ and $\|C - C_\delta\| < \frac{\delta}{2}$. Since then $\|\Gamma - B_\delta\| \leq \|\Gamma - B\| + \|B - B_\delta\| < \frac{\varepsilon}{m}$, the $\mathbb{Q}$-divisors

$$H_\delta = \Gamma - B_\delta + \tfrac{1}{4m}A \qquad \text{and} \qquad G_\delta = \tfrac{\varepsilon}{m}(K_X + S + A + B_\delta) + \tfrac{1}{4m}A$$

are ample by the assumptions from Step 0. Then

$$\mathbf{B}(K_X + S + A + \Gamma + \tfrac{1}{2m}A) = \mathbf{B}(K_X + S + A + B_\delta + H_\delta + \tfrac{1}{4m}A) \subseteq \mathbf{B}(K_X + S + A + B_\delta),$$

hence $S \nsubseteq \mathbf{B}(K_X + S + A + \Gamma + \frac{1}{2m}A)$. Since

$$K_X + S + A + \Gamma + \tfrac{1}{2m}A = (1 - \tfrac{\varepsilon}{m})(K_X + S + A + B_\delta) + (G_\delta + H_\delta),$$

we have

$$\begin{aligned}\mathbf{Fix}_S(K_X+S+A+\Gamma+\tfrac{1}{2m}A) &\leq \mathbf{Fix}_S\big((1-\tfrac{\varepsilon}{m})(K_X+S+A+B_\delta)\big)\\ &= \big(1-\tfrac{\varepsilon}{m}\big)\,\mathbf{Fix}_S(K_X+S+A+B_\delta)\end{aligned}$$

Since $(B_\delta, C_\delta)\in\mathcal{Q}_i'$, Theorem 2.37 implies

$$\mathbf{Fix}_S(K_X+S+A+B_\delta)\leq B_{\delta|S}-C_\delta+\mathbf{Fix}(K_S+A_{|S}+C_\delta),$$

which together with the previous inequality yields

$$\mathbf{Fix}_S(K_X+S+A+\Gamma+\tfrac{1}{2m}A)\leq(1-\tfrac{\varepsilon}{m})(B_{\delta|S}-C_\delta)+\mathbf{Fix}(K_S+A_{|S}+C_\delta).$$

If T is a component of Ψ, then T is a component of C as $(\Gamma,\Psi)\in\operatorname{face}\big(\mathcal{C}_i,(B,C)\big)$. Thus $T\subseteq\operatorname{Supp}C_\delta$ for $\delta\ll 1$, and so $\operatorname{mult}_T\mathbf{Fix}(K_S+A_{|S}+C_\delta)=0$ since $C_\delta\in\mathcal{F}_i$. Therefore

$$\begin{aligned}\operatorname{mult}_T\mathbf{Fix}_S(K_X+S+A+\Gamma+\tfrac{1}{2m}A) &\leq(1-\tfrac{\varepsilon}{m})\operatorname{mult}_T(B_{\delta|S}-C_\delta)\\ &\leq(1-\tfrac{\varepsilon}{m})\operatorname{mult}_T(B_{|S}-C)+\delta,\end{aligned}$$

and we obtain

$$\operatorname{mult}_T\mathbf{Fix}_S(K_X+S+A+\Gamma+\tfrac{1}{2m}A)\leq(1-\tfrac{\varepsilon}{m})\operatorname{mult}_T(B_{|S}-C)$$

by letting $\delta\to 0$. If $\operatorname{mult}_T(B_{|S}-C)\leq\operatorname{mult}_T(\Gamma_{|S}-\Psi)$, then clearly $(1-\frac{\varepsilon}{m})\operatorname{mult}_T(B_{|S}-C)\leq\operatorname{mult}_T(\Gamma_{|S}-\Psi)$. Otherwise, by assumption $\phi<\operatorname{mult}_T(B_{|S}-C)\leq\operatorname{mult}_T(\Gamma_{|S}-\Psi)+\frac{\phi\varepsilon}{m}$, and so

$$\begin{aligned}(1-\tfrac{\varepsilon}{m})\operatorname{mult}_T(B_{|S}-C) &\leq\operatorname{mult}_T(\Gamma_{|S}-\Psi)+\tfrac{\phi\varepsilon}{m}-\tfrac{\varepsilon}{m}\operatorname{mult}_T(B_{|S}-C)\\ &=\operatorname{mult}_T(\Gamma_{|S}-\Psi)-\tfrac{\varepsilon}{m}\big(\operatorname{mult}_T(B_{|S}-C)-\phi\big)\leq\operatorname{mult}_T(\Gamma_{|S}-\Psi).\end{aligned}$$

This proves (2.33) and finishes the proof of Proposition 2.40.

We used the following result from Diophantine approximation.

Lemma 2.42. *Let $\|\cdot\|$ be a norm on $\mathbb{R}^N$, let $\mathcal{P}\subseteq\mathbb{R}^N$ be a rational polytope and let $x\in\mathcal{P}$. Fix a positive integer k and a positive real number ε.*

Then there are finitely many $x_i\in\mathcal{P}$ and positive integers k_i divisible by k, such that k_ix_i/k are integral, $\|x-x_i\|<\varepsilon/k_i$, and x is a convex linear combination of x_i.

Proof. This is well known, see for instance [BCHM10, Lemma 3.7.7]. □

Proof of Proposition 2.40(iii)

Denote $\mathcal{P} = \mathcal{G} \cap \mathcal{B}_A^S(V)$, and recall the definition of $\mathcal{Q}_i$ from Step 0 of the proof of Proposition 2.40(ii). Let $\mathcal{P}_i \subseteq V$ be the image of $\mathcal{Q}_i$ through the first projection. Fix $B \in \mathcal{P} \cap \mathrm{Div}_{\mathbb{Q}}(X)$, and for every positive integer m such that mA, mB are integral and $S \not\subseteq \mathrm{Bs}\,|m(K_X + S + A + B)|$, denote

$$\Phi_m = B_{|S} - B_{|S} \wedge \tfrac{1}{m} \operatorname{Fix} |m(K_X + S + A + B)|_S \in \mathcal{E}_{A_{|S}}(W).$$

As in the proof of Corollary 2.39 we have

$$|m(K_S + A_{|S} + \Phi_m)| + m(B_{|S} - \Phi_m) \supseteq |m(K_X + S + A + B)|_S,$$

so

$$\operatorname{Fix} |m(K_S + A_{|S} + \Phi_m)| + m(B_{|S} - \Phi_m) \leq \operatorname{Fix} |m(K_X + S + A + B)|_S. \qquad (2.36)$$

If T is a component of Φ_m, then by definition

$$\operatorname{mult}_T \Phi_m = \operatorname{mult}_T B_{|S} - \tfrac{1}{m} \operatorname{mult}_T \operatorname{Fix} |m(K_X + S + A + B)|_S,$$

which together with (2.36) gives $\operatorname{mult}_T \operatorname{Fix} |m(K_S + A_{|S} + \Phi_m)| = 0$, and hence

$$\operatorname{mult}_T \mathbf{Fix}(K_S + A_{|S} + \Phi_m) = 0.$$

This implies $\big(B, \Phi_m\big) \in \bigcup_i \mathcal{Q}_i$, thus $B \in \bigcup_i \mathcal{P}_i$. Therefore $\mathcal{P} \cap \mathrm{Div}_{\mathbb{Q}}(X) \subseteq \bigcup_i \mathcal{P}_i$, and since $\mathcal{P} \cap \mathrm{Div}_{\mathbb{Q}}(X)$ is dense in $\mathcal{P}$ (exercise!), we have $\mathcal{P} \subseteq \bigcup_i \mathcal{P}_i$. The reverse inclusion follows by the definition of the sets $\mathcal{Q}_i'$, and this finishes the proof.

2.5.3 Proof of Theorem C

The following result contains the heart of the proof.

Proposition 2.43. *Assume Theorem A_{n-1} and Theorem B_{n-1}. Let $(X, S + \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where S and all S_i are distinct prime divisors. Let $V = \sum_{i=1}^p \mathbb{R} S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ and let A be an ample $\mathbb{Q}$-divisor on X. Let $\mathcal{G}$ be a rational polytope contained in the interior of $\mathcal{L}(V)$, and assume that $(S, B_{|S})$ is terminal for every $B \in \mathcal{G}$. Denote $\mathcal{P} = \mathcal{G} \cap \mathcal{B}_A^S(V)$.*

(i) For each $B \in \mathcal{P} \cap \mathrm{Div}_{\mathbb{Q}}(X)$, denote $\mathbf{\Phi}(B) = B_{|S} - B_{|S} \wedge \mathbf{Fix}_S(K_X + S + A + B)$. Then $\mathbf{\Phi}$ extends to a rational piecewise affine function on $\mathcal{P}$,

(ii) For every positive integer m such that mA, mB are integral and $S \not\subseteq \mathrm{Bs}\,|m(K_X + S + A + B)|$, denote

$$\Phi_m(B) = B_{|S} - B_{|S} \wedge \tfrac{1}{m} \operatorname{Fix} |m(K_X + S + A + B)|_S.$$

Then there exists a positive integer ℓ with the property that $\mathbf{\Phi}(B) = \Phi_m(B)$ for every $B \in \mathcal{P} \cap \mathrm{Div}_{\mathbb{Q}}(X)$ and every positive integer m such that mB/ℓ is integral.

This immediately implies Theorem C:

Proof of Theorem 2.8. We first prove the lemma under additional assumptions, and then treat the general case in Step 2.

Step 1. In this step we assume that all B_i lie in the interior of $\mathcal{L}(V)$ and that all $(S, B_{i|S})$ are terminal. We use functions Φ_m and $\mathbf{\Phi}$ defined in Proposition 2.43.

Let $\mathcal{G} \subseteq \mathcal{E}_{S+A}(V)$ be the convex hull of all B_i. Then $\mathcal{G}$ is contained in the interior of $\mathcal{L}(V)$, and $(S, G_{|S})$ is terminal for every $G \in \mathcal{G}$. Denote

$$\mathcal{D} = \mathbb{R}_+(K_X + S + A + \mathcal{G}).$$

Then, by Lemma 2.13(iii) it suffices to prove that $\operatorname{res}_S R(X, \mathcal{D})$ is finitely generated.

By Theorem D_n, the set $\mathcal{P} = \mathcal{G} \cap \mathcal{B}_A^S(V)$ is a rational polytope, and there exists a finite decomposition $\mathcal{P} = \bigcup \mathcal{P}_i$ into rational polytopes such that $\mathbf{\Phi}$ is rational affine on each $\mathcal{P}_i$ by Proposition 2.43, where we assume the notation from Proposition 2.43. Denote

$$\mathcal{C} = \mathbb{R}_+(K_X + S + A + \mathcal{P}) \quad \text{and} \quad \mathcal{C}_i = \mathbb{R}_+(K_X + S + A + \mathcal{P}_i),$$

and note that $\mathcal{C} = \bigcup \mathcal{C}_i$. Since $\operatorname{res}_S H^0(X, \mathcal{O}_X(D)) = 0$ for every $D \in \mathcal{D} \setminus \mathcal{C}$, and as $\mathcal{C}$ is a rational polyhedral cone, it suffices to show that $\operatorname{res}_S R(X, \mathcal{C})$ is finitely generated, and therefore, to prove that $\operatorname{res}_S R(X, \mathcal{C}_i)$ is finitely generated for each i. Hence, after replacing $\mathcal{G}$ by $\mathcal{P}_i$, we can assume that $\mathbf{\Phi}$ is rational affine on $\mathcal{G}$.

By Gordan's lemma and by definition of $\mathcal{D}$, there exist $G_i \in \mathcal{G} \cap \operatorname{Div}_{\mathbb{Q}}(X)$ and $d_i \in \mathbb{Q}_+$, with $i = 1, \dots, q$, such that

$$D_i = d_i(K_X + S + A + G_i) \quad \text{are generators of} \quad \mathcal{D} \cap \operatorname{Div}(X).$$

By Theorem 2.43, there exists a positive integer ℓ such that $\Phi_m(G) = \mathbf{\Phi}(G)$ for every $G \in \mathcal{G} \cap \operatorname{Div}_{\mathbb{Q}}(X)$ and every $m \in \mathbb{N}$ such that $mG/\ell \in \operatorname{Div}(X)$. Pick a positive integer k such that all $kd_i/\ell \in \mathbb{N}$ and $kd_iG_i/\ell \in \operatorname{Div}(X)$. For each nonzero $\alpha = (\alpha_1, \dots, \alpha_q) \in \mathbb{N}^q$, denote

$$d_\alpha = \sum \alpha_i d_i, \quad G_\alpha = \frac{1}{d_\alpha} \sum \alpha_i d_i G_i, \quad D_\alpha = \sum \alpha_i D_i = d_\alpha(K_X + S + A + G_\alpha),$$

and note that $kd_\alpha G_\alpha/\ell \in \operatorname{Div}(X)$ and $\mathbf{\Phi}(G_\alpha) = \frac{1}{d_\alpha} \sum \alpha_i d_i \mathbf{\Phi}(G_i)$. Then, by Corollary 2.39 we have

$$\begin{aligned} \operatorname{res}_S H^0(X, \mathcal{O}_X(mkD_\alpha)) &= H^0\big(S, \mathcal{O}_S(mkd_\alpha(K_S + A_{|S} + \Phi_{mkg_\alpha}(G_\alpha)))\big) \\ &= H^0\big(S, \mathcal{O}_S(mkd_\alpha(K_S + A_{|S} + \mathbf{\Phi}(G_\alpha)))\big) \end{aligned}$$

for all $\alpha \in \mathbb{N}^q$ and $m \in \mathbb{N}$, and thus

$$\operatorname{res}_S R(X; kD_1, \dots, kD_q) = R(S; kd_1D'_1, \dots, kd_qD'_q),$$

where $D'_i = K_S + A_{|S} + \mathbf{\Phi}(G_i)$. Since the last ring is a Veronese subring of the adjoint ring $R(S; D'_1, \dots, D'_q)$, it is finitely generated by Theorem A_{n-1} and by Lemma 2.13(i). Therefore $\operatorname{res}_S R(X; D_1, \dots, D_q)$ is finitely generated by Lemma 2.13(ii), and since there is the natural projection of this ring onto $\operatorname{res}_S R(X, \mathcal{D})$, this last ring is also finitely generated.

Step 2. In this step, we show that Step 1 implies the result in general.

For every i pick a $\mathbb{Q}$-divisor $G_i \in V$ such that $A - G_i$ is ample and $B_i + G_i$ is in the interior of $\mathcal{L}(V)$. Let A' be an ample $\mathbb{Q}$-divisor such that every $A - G_i - A'$ is also ample, and pick $\mathbb{Q}$-divisors $A_i \geq 0$ such that $A_i \sim_{\mathbb{Q}} A - G_i - A'$, $\lfloor A_i \rfloor = 0$, $(X, S + \sum_{i=1}^p S_i + \sum_{i=1}^m A_i)$ is log smooth, and the support of $\sum_{i=1}^m A_i$ does not contain any of the divisors $S, S_1, \dots, S_p$. Let $V' \subseteq \operatorname{Div}_{\mathbb{R}}(X)$ be the vector space spanned by V and by the components of $\sum_{i=1}^m A_i$. Let $\varepsilon > 0$ be a rational number such that

$$A'' = A' - \varepsilon \sum_{i=1}^m A_i$$

is ample, and such that

$$B'_i = B_i + G_i + A_i + \varepsilon \sum_{i=1}^m A_i$$

is in the interior of $\mathcal{L}(V')$ for every i. Note that we have

$$K_X + S + A + B_i \sim_{\mathbb{Q}} K_X + S + A'' + B'_i \quad \text{for every } i. \tag{2.37}$$

Let $B \geq 0$ be a $\mathbb{Q}$-divisor such that $\lfloor B \rfloor = 0$ and $B \geq B'_i$ for all i. By Lemma 2.38, there exists a log resolution $f\colon Y \to X$ such that

$$K_Y + T + C = f^*(K_X + S + B) + E,$$

where the $\mathbb{Q}$-divisors $C, E \geq 0$ have no common components, E is f-exceptional, $\lfloor C \rfloor = 0$, the components of C are disjoint, and $T = f_*^{-1} S \not\subseteq \operatorname{Supp} C$. Then there are $\mathbb{Q}$-divisors $0 \leq C_i \leq C$ and f-exceptional $\mathbb{Q}$-divisors $E_i \geq 0$ such that

$$K_Y + T + C_i = f^*(K_X + S + B'_i) + E_i, \tag{2.38}$$

and in particular, all pairs $(T, C_{i|T})$ are terminal. Let V° be the subspace of $\operatorname{Div}_{\mathbb{R}}(Y)$ spanned by the components of C and by all f-exceptional prime divisors. There exists a $\mathbb{Q}$-divisor $F \geq 0$ on Y such that, if we denote

$$A^\circ = f^* A'' - F \quad \text{and} \quad C_i^\circ = C_i + F, \tag{2.39}$$

then A° is ample, every C_i° is in the interior of $\mathcal{L}(V^\circ)$, and every pair $(T, C^\circ_{i|T})$ is terminal. It follows from (2.37), (2.38) and (2.39) that

$$K_Y + T + A^\circ + C_i^\circ \sim_{\mathbb{Q}} f^*(K_X + S + A + B_i) + E_i.$$

Since the ring

$$\operatorname{res}_T R(Y; K_Y + T + A^\circ + C_1^\circ, \ldots, K_Y + T + A^\circ + C_m^\circ)$$

is finitely generated by Step 1, we conclude by Lemma 2.13(iii). $\square$

Proof of Theorem 2.43(i)

Step 1. For (i), fix a prime divisor $T \in W$, and consider the map $\mathbf{\Phi}_T \colon \mathcal{P} \cap \operatorname{Div}_{\mathbb{Q}}(X) \to [0,1]$ defined by

$$\mathbf{\Phi}_T(B) = \operatorname{mult}_T \mathbf{\Phi}(B) \quad \text{for every} \quad B \in \mathcal{P} \cap \operatorname{Div}_{\mathbb{Q}}(X).$$

In order to show that $\mathbf{\Phi}$ extends to a rational piecewise affine function on $\mathcal{P}$, it suffices to prove that each function $\mathbf{\Phi}_T$ extends to a rational piecewise affine function on $\mathcal{P}$.

Let $\mathcal{R}_T$ be the closure of the set

$$\mathcal{R}_T' = \{B \in \mathcal{P} \cap \operatorname{Div}_{\mathbb{Q}}(X) \mid \mathbf{\Phi}_T(B) \neq 0\} \subseteq \mathcal{P} \cap \operatorname{Div}_{\mathbb{Q}}(X).$$

Note that

$$\mathbf{\Phi}_T(B) \neq 0 \quad \Rightarrow \quad \mathbf{\Phi}_T(B) = \operatorname{mult}_T \big(B_{|S} - \mathbf{Fix}_S(K_X + S + A + B)\big),$$

and since $\mathbf{Fix}_S$ is a convex map on $\mathcal{P}$, the set $\mathcal{R}_T$ is convex, and $\mathbf{\Phi}_T$ is concave on $\mathcal{R}_T$. Now it is clear that $\mathbf{\Phi}_T$ extends to a rational piecewise affine on $\mathcal{P}$ if and only if:

(a) $\mathcal{R}_T$ is a rational polytope, and

(b) $\mathbf{\Phi}_T$ extends to a rational piecewise affine function on $\mathcal{R}_T$.

Step 2. In this step we show (a). Let $\mathcal{Q}_i'$ be the sets as in Proposition 2.40(ii), let $\mathcal{Q}_i$ be the convex hull of $\mathcal{Q}_i'$, and let $\mathcal{P}_i \subseteq V$ be the image of $\mathcal{Q}_i$ through the first projection. Recall from the proof of Proposition 2.40(iii) that each $\mathcal{P}_i$ is a rational polytope and $\mathcal{P} = \bigcup \mathcal{P}_i$.

We show that $\mathcal{R}_T$ is a union of some of the sets $\mathcal{P}_i$: this then implies that $\mathcal{R}_T$ is a rational polytope since it is convex.

Let B be any rational point of $\mathcal{R}'_T$. From the proof of Proposition 2.40(iii) we have $\big(B, \Phi_m(B)\big) \in \bigcup \mathcal{Q}_i$ for every m sufficiently divisible, hence by compactness,

$$\big(B, \mathbf{\Phi}(B)\big) \in \mathcal{Q}_i \quad \text{for some } i.$$

Since $\operatorname{mult}_T \mathbf{\Phi}(B) > 0$, the image of the polytope $\mathcal{Q}_i$ through the second projection is not zero, which implies that $\operatorname{mult}_T C > 0$ for every rational point (B, C) in the relative interior of $\mathcal{Q}_i$. It is enough to show that for every such a point (B, C) we have $\operatorname{mult}_T \mathbf{\Phi}(B) > 0$: indeed, by looking at the first projection, this then implies that every rational point in the relative interior of $\mathcal{P}_i$ belongs to $\mathcal{R}_T$, hence $\mathcal{P}_i \subseteq \mathcal{R}_T$ as the set of such points is dense in $\mathcal{P}_i$.

To prove the claim, fix a rational point (B, C) in the relative interior of $\mathcal{Q}_i$. Note that this implies $(B, C) \in \mathcal{Q}'_i$, so Theorem 2.37 gives

$$\mathbf{Fix}(K_S + A_{|S} + C) + (B_{|S} - C) \geq \mathbf{Fix}_S(K_X + S + A + B).$$

On the other hand, $\operatorname{mult}_T C > 0$ yields $\operatorname{mult}_T \mathbf{Fix}(K_S + A_{|S} + C) = 0$ by the definition of the set $\mathcal{Q}'_i$, and thus

$$\operatorname{mult}_T \big(B_{|S} - \mathbf{Fix}_S(K_X + S + A + B)\big) \geq \operatorname{mult}_T C > 0.$$

In particular, $\mathbf{\Phi}_T(B) = \operatorname{mult}_T \big(B_{|S} - \mathbf{Fix}_S(K_X + S + A + B)\big)$, which shows the claim.

Step 3. In this step we show (b). Let (B_j, C_j) be the extreme points of all $\mathcal{Q}_i$ for which $\mathcal{P}_i \subseteq \mathcal{R}_T$. Since $\mathcal{Q}_i$ is the convex hull of $\mathcal{Q}'_i$, it follows that $(B_j, C_j) \in \bigcup \mathcal{Q}'_i$, and in particular

$$\operatorname{mult}_T C_j \leq \operatorname{mult}_T \mathbf{\Phi}(B_j) = \mathbf{\Phi}_T(B_j). \tag{2.40}$$

Fix a rational point $B \in \mathcal{R}_T$. Then $\big(B, \mathbf{\Phi}(B)\big) \in \mathcal{Q}_i$ for some i by the proof of Proposition 2.40(iii), hence there exist $r_j \in \mathbb{R}_+$ such that

$$\sum r_j = 1 \quad \text{and} \quad \big(B, \mathbf{\Phi}(B)\big) = \sum r_j (B_j, C_j).$$

Thus $\mathbf{\Phi}_T(B) = \operatorname{mult}_T \mathbf{\Phi}(B) = \sum r_j \operatorname{mult}_T C_j$, so by concavity of $\mathbf{\Phi}_T$ and by (2.40) we have

$$\sum r_j \mathbf{\Phi}_T(B_j) \leq \mathbf{\Phi}_T(B) = \sum r_j \operatorname{mult}_T C_j \leq \sum r_j \mathbf{\Phi}_T(B_j).$$

Therefore

$$\mathbf{\Phi}_T(B_j) = \operatorname{mult}_T C_j \in \mathbb{Q} \quad \text{for any } j \text{ and } \mathbf{\Phi}_T(B) = \sum r_j \mathbf{\Phi}_T(B_j).$$

Now by the following lemma, $\mathbf{\Phi}_T$ extends to a rational piecewise affine map on $\mathcal{R}_T$.

Lemma 2.44. *Let $\mathcal{P} \subseteq \mathbb{R}^N$ be a rational polytope, and denote $\mathcal{P}_\mathbb{Q} = \mathcal{P} \cap \mathbb{Q}^N$. Let $f\colon \mathcal{P}_\mathbb{Q} \to \mathbb{R}$ be a bounded convex function, and assume that there exist $x_1, \dots, x_q \in \mathcal{P}_\mathbb{Q}$ such that:*

(i) $f(x_i) \in \mathbb{Q}$ for all i,

(ii) for any $x \in \mathcal{P}_\mathbb{Q}$ there exists $(r_1, \dots, r_q) \in \mathbb{R}^q_+$ such that

$$\sum r_i = 1, \quad x = \sum r_i x_i \quad \textit{and} \quad f(x) = \sum r_i f(x_i).$$

Then f can be extended to a rational piecewise affine function on $\mathcal{P}$.

Proof. Pick $C \in \mathbb{Q}_+$ such that $-C \leq f(x) \leq C$ for all $x \in \mathcal{P}_\mathbb{Q}$. Let $\mathcal{Q} \subseteq \mathbb{R}^{N+1}$ be the convex hull of all the points $\big(x_i, f(x_i)\big)$ and (x_i, C), and set

$$\mathcal{Q}' = \{(x, y) \in \mathcal{P}_\mathbb{Q} \times \mathbb{R} \mid f(x) \leq y \leq C\}.$$

We first claim that $\mathcal{Q} \cap \mathbb{Q}^{N+1} = \mathcal{Q}' \cap \mathbb{Q}^{N+1}$, and in particular $\mathcal{Q} = \overline{\mathcal{Q}'}$. Indeed, since f is convex, and all $\big(x_i, f(x_i)\big)$ and (x_i, C) are contained in $\mathcal{Q}'$, it follows that $\mathcal{Q} \cap \mathbb{Q}^{N+1} \subseteq \mathcal{Q}'$. Conversely, fix $(u, v) \in \mathcal{Q}' \cap \mathbb{Q}^{N+1}$. Then there exists $t \in [0,1]$ such that $v = tf(u) + (1-t)C$, and as $u \in \mathcal{P}_\mathbb{Q}$, there exist $r_i \in \mathbb{R}_+$ such that $\sum r_i = 1$, $u = \sum r_i x_i$ and $f(u) = \sum r_i f(x_i)$. Therefore

$$(u, v) = \sum t r_i \big(x_i, f(x_i)\big) + \sum (1-t) r_i (x_i, C),$$

and hence $(u, v) \in \mathcal{Q}$, which proves the claim. Now, define $F\colon \mathcal{P} \to [-C, C]$ as

$$F(x) = \min\{y \in [-C, C] \mid (x, y) \in \mathcal{Q}\}.$$

Then F extends f, and it is rational piecewise affine as $\mathcal{Q}$ is a rational polytope. $\square$

Proof of Proposition 2.43(ii)

From Proposition 2.43(i) we have $\mathbf{\Phi}(B) \in \mathrm{Div}_\mathbb{Q}(S)$ for every $P \in \mathcal{P} \cap \mathrm{Div}_\mathbb{Q}(X)$, and by subdividing $\mathcal{P}$, we may assume that $\mathbf{\Phi}$ extends to a rational affine map on $\mathcal{P}$. By Theorem 1.10 there exists a positive integer k with the property that

$$\mathbf{Fix}(K_S + A_{|S} + E) = \frac{1}{m} \operatorname{Fix} |m(K_S + A_{|S} + E)|$$

for every rational $E \in \mathcal{E}_{A_{|S}}(W)$ and every $m \in \mathbb{N}$ such that mA/k and mE/k are integral. By Gordan's lemma, the monoid $\mathbb{R}_+(S + \mathcal{P}) \cap \mathrm{Div}(X)$ is finitely generated, and let $b_i(S + B_i)$ be its generators for some $b_i \in \mathbb{Q}_+$ and $B_i \in \mathcal{P} \cap \mathrm{Div}_\mathbb{Q}(X)$. Pick a positive integer w such that $w b_i \mathbf{\Phi}(B_i) \in \mathrm{Div}(S)$ for every i, and set $\ell = wk$.

Fix $B \in \mathcal{P} \cap \mathrm{Div}_{\mathbb{Q}}(X)$ and a positive integer m such that $mB/\ell \in \mathrm{Div}(X)$. Then there are non-negative integers α_i are such that

$$m(S+B)/\ell = \sum \alpha_i b_i (S + B_i).$$

In particular, we have $m/\ell = \sum \alpha_i b_i$, and therefore

$$m\mathbf{\Phi}(B)/\ell = \sum \alpha_i b_i \mathbf{\Phi}(B_i)$$

since $\mathbf{\Phi}$ is an affine map. Hence $m\mathbf{\Phi}(B)/k = \sum \alpha_i w b_i \mathbf{\Phi}(B_i) \in \mathrm{Div}(S)$, so

$$\mathbf{Fix}(K_S + A_{|S} + \mathbf{\Phi}(B)) = \frac{1}{m} \mathrm{Fix}\, |m(K_S + A_{|S} + \mathbf{\Phi}(B))|$$

by the choice of k. Recall that $\big(B, \mathbf{\Phi}(B)\big) \in \bigcup_i \mathcal{Q}_i$ by the proof of Proposition 2.40(iii), hence $\mathbf{\Phi}(B) \wedge \mathbf{Fix}\, |m(K_S + A_{|S} + \mathbf{\Phi}(B))| = 0$. In particular,

$$\mathbf{\Phi}(B) \wedge \mathrm{Fix}\, |m(K_S + A_{|S} + \mathbf{\Phi}(B))| = 0. \tag{2.41}$$

Now Theorem 2.37 gives

$$\begin{aligned}\mathrm{Fix}\, |m(K_S + A_{|S} + \mathbf{\Phi}(B))| + m(B_{|S} - \mathbf{\Phi}(B)) &\geq \mathrm{Fix}\, |m(K_X + S + A + B)|_S \\ &\geq m(B_{|S} \wedge \tfrac{1}{m} \mathrm{Fix}\, |m(K_X + S + A + B)|_S) = m(B_{|S} - \Phi_m(B)).\end{aligned}$$

This together with (2.41) implies $\Phi_m(B) \geq \mathbf{\Phi}(B)$. But, by definition, $\mathbf{\Phi}(B) \geq \Phi_m(B)$, and (ii) follows.

2.6 Proof of the Extension theorem

In this section we prove Theorem 2.37.

We will need the following easy consequence of Kawamata-Viehweg vanishing:

Lemma 2.45. *Let (X, B) be a log smooth projective pair of dimension n, where B is a $\mathbb{Q}$-divisor such that $\lfloor B \rfloor = 0$. Let A be a nef and big $\mathbb{Q}$-divisor.*

(i) Let S be a smooth prime divisor such that $S \not\subseteq \mathrm{Supp}\, B$. If $G \in \mathrm{Div}(X)$ is such that $G \sim_{\mathbb{Q}} K_X + S + A + B$, then $|G_{|S}| = |G|_S$.

(ii) Let $f \colon X \to Y$ be a birational morphism to a projective variety Y, and let $U \subseteq X$ be an open set such that $f_{|U}$ is an isomorphism and U intersects at most one irreducible component of B. Let H' be a very ample divisor on Y and let $H = f^*H'$. If $F \in \mathrm{Div}(X)$ is such that $F \sim_{\mathbb{Q}} K_X + (n+1)H + A + B$, then $|F|$ is basepoint free at every point of U.

Proof. Considering the exact sequence

$$0 \to \mathcal{O}_X(G-S) \to \mathcal{O}_X(G) \to \mathcal{O}_S(G) \to 0,$$

Kawamata-Viehweg vanishing implies $H^1(X, \mathcal{O}_X(G-S)) = 0$. In particular, the map $H^0(X, \mathcal{O}_X(G)) \to H^0(S, \mathcal{O}_S(G))$ is surjective. This proves (i).

We prove (ii) by induction on n. Let $x \in U$ be a closed point, and pick a general element $T \in |H|$ which contains x. Then by the assumptions on U, it follows that $(X, T+B)$ is log smooth, and since $F_{|T} \sim_{\mathbb{Q}} K_T + nH_{|T} + A_{|T} + B_{|T}$, by induction $F_{|T}$ is free at x. Considering the exact sequence

$$0 \to \mathcal{O}_X(F-T) \to \mathcal{O}_X(F) \to \mathcal{O}_T(F) \to 0,$$

Kawamata-Viehweg vanishing implies that $H^1(X, \mathcal{O}_X(F-T)) = 0$. In particular, the map $H^0(X, \mathcal{O}_X(F)) \to H^0(T, \mathcal{O}_T(F))$ is surjective, and (ii) follows. $\square$

Lemma 2.46. *Let $(X, S+B)$ be a projective pair, where X is smooth, S is a smooth prime divisor and B is a $\mathbb{Q}$-divisor such that $S \not\subseteq \operatorname{Supp} B$. Let A be a nef and big $\mathbb{Q}$-divisor on X. Assume that $D \in \operatorname{Div}(X)$ is such that $D \sim_{\mathbb{Q}} K_X + S + A + B$, and let $\Sigma \in |D_{|S}|$. Let $\Phi \in \operatorname{Div}_{\mathbb{Q}}(S)$ be such that the pair (S, Φ) is klt and $B_{|S} \leq \Sigma + \Phi$.*

Then $\Sigma \in |D|_S$.

Proof. Let $f\colon Y \to X$ be a log resolution of the pair $(X, S+B)$, and write $T = f_*^{-1}S$. Then there are $\mathbb{Q}$-divisors $\Gamma \geq 0$ and $E \geq 0$ on Y with no common components such that $T \not\subseteq \operatorname{Supp}\Gamma$, E is f-exceptional, and

$$K_Y + T + \Gamma = f^*(K_X + S + B) + E.$$

Let $C = \Gamma - E$ and

$$G = f^*D - \lfloor C \rfloor = f^*D - \lfloor \Gamma \rfloor + \lceil E \rceil. \tag{2.42}$$

Then

$$G - (K_Y + T + \{C\}) \sim_{\mathbb{Q}} f^*(K_X + S + A + B) - (K_Y + T + C) = f^*A$$

is nef and big, and Lemma 2.45(i) implies that

$$|G_{|T}| = |G|_T. \tag{2.43}$$

Moreover, since $E \geq 0$ is f-exceptional, we have

$$\begin{aligned} |G|_T + \lfloor \Gamma \rfloor_{|T} &= |f^*D - \lfloor \Gamma \rfloor + \lceil E \rceil|_T + \lfloor \Gamma \rfloor_{|T} \\ &\subseteq |f^*D + \lceil E \rceil|_T = |f^*D|_T + \lceil E \rceil_{|T}. \end{aligned} \tag{2.44}$$

Denote $g = f_{|T} \colon T \to S$. Then

$$K_T + C_{|T} = g^*(K_S + B_{|S}) \quad \text{and} \quad K_T + \Psi = g^*(K_S + \Phi),$$

for some $\mathbb{Q}$-divisor Ψ on T, and note that $\lfloor \Psi \rfloor \leq 0$ since (S, Φ) is klt. Therefore

$$g^*(B_{|S} - \Phi) = C_{|T} - \Psi. \tag{2.45}$$

By assumption we have that $B_{|S} \leq \Sigma + \Phi$, that $g^*\Sigma$ is integral, and that the support of $C + T$ has normal crossings, so this together with (2.45) gives

$$\begin{aligned} g^*\Sigma \geq g^*\Sigma + \lfloor \Psi \rfloor &= \lfloor g^*\Sigma + \Psi \rfloor \geq \lfloor g^*(B_{|S} - \Phi) + \Psi \rfloor \\ &= \lfloor C_{|T} \rfloor = \lfloor C \rfloor_{|T} = (f^*D)_{|T} - G_{|T}. \end{aligned}$$

Denote

$$R = G_{|T} - (f^*D)_{|T} + g^*\Sigma.$$

Then $R \geq 0$ by the above, and $g^*\Sigma \in |(f^*D)_{|T}|$ implies $R \in |G_{|T}| = |G|_T$ by (2.43). Therefore $R + \lfloor \Gamma \rfloor_{|T} \in |f^*D|_T + \lceil E \rceil_{|T}$ by (2.44), and this together with (2.42) yields

$$g^*\Sigma = R + (f^*D)_{|T} - G_{|T} = R + \lfloor \Gamma \rfloor_{|T} - \lceil E \rceil_{|T} \in |f^*D|_T,$$

hence the claim follows. $\square$

Lemma 2.47. *Let $(X, S + B + D)$ be a log smooth projective pair, where S is a prime divisor, B is a $\mathbb{Q}$-divisor such that $\lfloor B \rfloor = 0$ and $S \not\subseteq \operatorname{Supp} B$, and $D \geq 0$ is a $\mathbb{Q}$-divisor such that D and $S + B$ have no common components. Let P be a nef $\mathbb{Q}$-divisor and denote $\Delta = S + B + P$. Assume that*

$$K_X + \Delta \sim_{\mathbb{Q}} D.$$

Let k be a positive integer such that kP and kB are integral, and write $\Omega = (B+P)_{|S}$.

Then there is a very ample divisor H such that for all divisors $\Sigma \in |k(K_S + \Omega)|$ and $U \in |H_{|S}|$, and for every positive integer l we have

$$l\Sigma + U \in |lk(K_X + \Delta) + H|_S.$$

Proof. For any $m \geq 0$, let $l_m = \lfloor \frac{m}{k} \rfloor$ and $r_m = m - l_m k \in \{0, 1, \ldots, k-1\}$, define $B_m = \lceil mB \rceil - \lceil (m-1)B \rceil$, and set $P_m = kP$ if $r_m = 0$, and otherwise $P_m = 0$. Let

$$D_m = \sum_{i=1}^{m} (K_X + S + P_i + B_i) = m(K_X + S) + l_m kP + \lceil mB \rceil,$$

and note that D_m is integral and

$$D_m = l_m k(K_X + \Delta) + D_{r_m}. \tag{2.46}$$

By Serre vanishing, we can pick a very ample divisor H on X such that:

(i) $D_j + H$ is ample and basepoint free for every $0 \leq j \leq k-1$,

(ii) $|D_k + H|_S = |(D_k + H)_{|S}|$.

We claim that for all divisors $\Sigma \in |k(K_S + \Omega)|$ and $U_m \in |(D_{r_m} + H)_{|S}|$ we have

$$l_m\Sigma + U_m \in |D_m + H|_S.$$

The case $r_m = 0$ immediately implies the lemma.

We prove the claim by induction on m. The case $m = k$ is covered by (ii). Now let $m > k$, and pick a rational number $0 < \delta \ll 1$ such that $D_{r_{m-1}} + H + \delta B_m$ is ample. Note that $0 \leq B_m \leq \lceil B \rceil$, that $(X, S + B + D)$ is log smooth, and that D and $S + B$ have no common components. Thus, there exists a rational number $0 < \varepsilon \ll 1$ such that, if we define

$$F = (1 - \varepsilon\delta)B_m + l_{m-1}k\varepsilon D, \tag{2.47}$$

then $(X, S + F)$ is log smooth, $\lfloor F \rfloor = 0$ and $S \not\subseteq \operatorname{Supp} F$. In particular, if W is a general element of the free linear system $|(D_{r_{m-1}} + H)_{|S}|$ and

$$\Phi = F_{|S} + (1 - \varepsilon)W, \tag{2.48}$$

then (S, Φ) is klt.

By induction, there is a divisor $\Upsilon \in |D_{m-1} + H|$ such that $S \not\subseteq \operatorname{Supp} \Upsilon$ and

$$\Upsilon_{|S} = l_{m-1}\Sigma + W.$$

Denoting $C = (1 - \varepsilon)\Upsilon + F$, by (2.47) we have

$$C \sim_{\mathbb{Q}} (1 - \varepsilon)(D_{m-1} + H) + (1 - \varepsilon\delta)B_m + l_{m-1}k\varepsilon D, \tag{2.49}$$

and (2.48) yields

$$C_{|S} = (1 - \varepsilon)\Upsilon_{|S} + F_{|S} \leq l_{m-1}\Sigma + \Phi \leq (l_m\Sigma + U_m) + \Phi. \tag{2.50}$$

By the choice of δ and since P_m is nef, the $\mathbb{Q}$-divisor

$$A = \varepsilon(D_{r_{m-1}} + H + \delta B_m) + P_m \tag{2.51}$$

is ample. Then by (2.46), (2.51) and (2.49) we have

$$\begin{aligned} D_m + H &= K_X + S + D_{m-1} + B_m + P_m + H \\ &= K_X + S + (1 - \varepsilon)D_{m-1} + l_{m-1}k\varepsilon(K_X + \Delta) + \varepsilon D_{r_{m-1}} + B_m + P_m + H \\ &\sim_{\mathbb{Q}} K_X + S + A + (1 - \varepsilon)D_{m-1} + l_{m-1}k\varepsilon D + (1 - \varepsilon\delta)B_m + (1 - \varepsilon)H \\ &\sim_{\mathbb{Q}} K_X + S + A + C, \end{aligned}$$

and thus $l_m\Sigma + U_m \in |D_m + H|_S$ by (2.50) and Lemma 2.46. $\square$

Proof of Theorem 2.37. Let $f\colon Y \to X$ be a log resolution of the pair $(X, S+B)$ and of the linear system $|qm(K_X + \Delta + \frac{1}{m}A)|$, and write $T = f_*^{-1}S$. Then there are $\mathbb{Q}$-divisors $B', E \geq 0$ on Y with no common components, such that E is f-exceptional and

$$K_Y + T + B' = f^*(K_X + S + B) + E.$$

Note that

$$K_T + B'_{|T} = g^*(K_S + B_{|S}) + E_{|T},$$

and since $(Y, T + B' + E)$ is log smooth and B' and E do not have common components, it follows that $B'_{|T}$ and $E_{|T}$ do not have common components, and in particular, $E_{|T}$ is g-exceptional and $g_*B'_{|T} = B_{|S}$. Let $\Gamma = T + f^*A + B'$, and define

$$F_q = \tfrac{1}{qm}\operatorname{Fix}|qm(K_Y + \Gamma + \tfrac{1}{m}f^*A)|, \quad B'_q = B' - B' \wedge F_q, \quad \Gamma_q = T + B'_q + f^*A.$$

Since $(Y, T + B' + F_q)$ is log smooth, $\operatorname{Mob}\big(qm(K_Y + \Gamma + \frac{1}{m}f^*A)\big)$ is basepoint free, and $T \not\subseteq \operatorname{Bs}(K_Y + \Gamma + \frac{1}{m}f^*A)$, by Bertini's theorem there exists a $\mathbb{Q}$-divisor $D \geq 0$ such that

$$K_Y + \Gamma_q + \tfrac{1}{m}f^*A \sim_{\mathbb{Q}} D,$$

the pair $(Y, T + B'_q + D)$ is log smooth, and D does not contain any component of $T + B'_q$. Let $g = f_{|T}\colon T \to S$. Since (S, C) is canonical, there is a g-exceptional $\mathbb{Q}$-divisor $F \geq 0$ on T such that

$$K_T + C' = g^*(K_S + C) + F,$$

where $C' = g_*^{-1}C$. We claim that $C' \leq B'_{q|T}$. Assuming the claim, let us show how it implies the theorem.

By Lemma 2.47, there exists a very ample divisor H on Y such that for all divisors $\Sigma' \in |qm(K_T + (B'_q + (1 + \frac{1}{m})f^*A)_{|T})|$ and $U \in |H_{|T}|$, and for every positive integer p we have

$$p\Sigma' + U \in |pqm(K_Y + \Gamma_q + \tfrac{1}{m}f^*A) + H|_T.$$

Pick an f-exceptional $\mathbb{Q}$-divisor $G \geq 0$ such that $\lfloor B' + \frac{1}{m}G \rfloor = 0$ and $f^*A - G$ is ample. In particular, $(T, (B' + \frac{1}{m}G)_{|T})$ is klt. Let $W_1 \in |q(f^*A)_{|T}|$ and $W_2 \in |H_{|T}|$ be general sections. Pick a positive integer $k \gg 0$ such that, if we denote $l = kq$, $W = kW_1 + W_2$ and $\Phi = B'_{|T} + \frac{1}{m}G_{|T} + \frac{1}{l}W$, then the $\mathbb{Q}$-divisor

$$A_0 = \frac{1}{m}(f^*A - G) - \frac{m-1}{ml}H \tag{2.52}$$

is ample and the pair (T, Φ) is klt.

Fix $\Sigma \in |m(K_S + A_{|S} + C)|$. Since $C' \leq B'_{q|T}$ by the claim, it is easy to check that

$$qg^*\Sigma + qm(F + B'_{q|T} - C') + W_1 \in |qm(K_T + (B'_q + (1 + \tfrac{1}{m})f^*A)_{|T})|.$$

Then, by the choice of H, there exists $\Upsilon \in |lm(K_Y + \Gamma_q + \frac{1}{m}f^*A) + H|$ such that $T \nsubseteq \operatorname{Supp}\Upsilon$ and

$$\Upsilon_{|T} = lg^*\Sigma + lm(F + B'_{q|T} - C') + W.$$

Denoting

$$B_0 = \frac{m-1}{ml}\Upsilon + (m-1)(\Gamma - \Gamma_q) + B' + \frac{1}{m}G, \tag{2.53}$$

relations (2.52) and (2.53) imply

$$\begin{aligned} m(K_Y + \Gamma) &= K_Y + T + (m-1)(K_Y + \Gamma + \tfrac{1}{m}f^*A) + \tfrac{1}{m}f^*A + B' \qquad (2.54)\\ &\sim_{\mathbb{Q}} K_Y + T + \tfrac{m-1}{ml}\Upsilon + (m-1)(\Gamma - \Gamma_q) + \tfrac{1}{m}f^*A - \tfrac{m-1}{ml}H + B'\\ &= K_Y + T + A_0 + B_0. \end{aligned}$$

Noting that $\Gamma - \Gamma_q = B' - B'_q$, we have

$$\begin{aligned} B_{0|T} = \tfrac{m-1}{m}g^*\Sigma &+ (m-1)\big(F + B'_{q|T} - C' + (\Gamma - \Gamma_q)_{|T}\big) \qquad (2.55)\\ &+ \tfrac{m-1}{ml}W + B'_{|T} + \tfrac{1}{m}G_{|T} \leq g^*\Sigma + m(F + B'_{|T} - C') + \Phi, \end{aligned}$$

and since $g^*\Sigma + m(F + B'_{|T} - C') \in |m(K_Y + \Gamma)_{|T}|$, by (2.54), (2.55) and Lemma 2.46 we obtain

$$g^*\Sigma + m(F + B'_{|T} - C') \in |m(K_Y + \Gamma)|_T.$$

Pushing forward by g yields $\Sigma + m(B_{|S} - C) \in |m(K_X + \Delta)|_S$ and the lemma follows.

Now we prove the claim stated above. Since $\operatorname{Mob}\big(qm(K_Y + \Gamma + \frac{1}{m}f^*A)\big)$ is basepoint free and T is not a component of F_q, it follows that $\frac{1}{qm}\operatorname{Fix}|qm(K_Y + \Gamma + \frac{1}{m}f^*A)|_T = F_{q|T}$ and

$$B'_{q|T} = B'_{|T} - (B' \wedge F_q)_{|T} = B'_{|T} - B'_{|T} \wedge \tfrac{1}{qm}\operatorname{Fix}|qm(K_Y + \Gamma + \tfrac{1}{m}f^*A)|_T.$$

Furthermore, we have

$$g_*\operatorname{Fix}|qm(K_Y + \Gamma + \tfrac{1}{m}f^*A)|_T = \operatorname{Fix}|qm(K_X + \Delta + \tfrac{1}{m}A)|_S,$$

so

$$g_*C' = C \leq B_{|S} - B_{|S} \wedge \tfrac{1}{qm}\operatorname{Fix}|qm(K_X + \Delta + \tfrac{1}{m}A)|_S = g_*B'_{q|T}.$$

Therefore $C' \leq B'_{q|T}$, since $B'_{q|T} \geq 0$ and $C' = g_*^{-1}C$. $\square$

Lemma 2.48. *Let X be a smooth projective variety and let S be a smooth prime divisor on X. Let D be a $\mathbb{Q}$-divisor such that $S \not\subseteq \mathrm{Bs}(D)$, and let A be an ample $\mathbb{Q}$-divisor. Then*

$$\frac{1}{q}\operatorname{Fix}|q(D+A)|_S \leq \mathbf{Fix}_S(D)$$

for any sufficiently divisible positive integer q.

Proof. Let P be a prime divisor on S and let $\gamma = \operatorname{mult}_P \mathbf{Fix}_S(D)$. It is enough to show that

$$\operatorname{mult}_P \frac{1}{q}\operatorname{Fix}|q(D+A)|_S \leq \gamma$$

for some sufficiently divisible positive integer q.

Assume first that $\gamma > 0$. Let $\varepsilon > 0$ be a rational number such that $\varepsilon D + A$ is ample, and pick a positive integer m such that

$$\frac{1-\varepsilon}{m}\operatorname{mult}_P \operatorname{Fix}|mD|_S \leq \gamma.$$

Let q be a sufficiently divisible positive integer such that the divisor $q(\varepsilon D + A)$ is very ample, and such that m divides $q(1-\varepsilon)$. Then

$$\begin{aligned}\frac{1}{q}\operatorname{mult}_P \operatorname{Fix}|q(D+A)|_S &= \frac{1}{q}\operatorname{mult}_P \operatorname{Fix}|q(1-\varepsilon)D + q(\varepsilon D + A)|_S \\ &\leq \frac{1}{q}\operatorname{mult}_P \operatorname{Fix}|q(1-\varepsilon)D|_S \leq \frac{1-\varepsilon}{m}\operatorname{mult}_P \operatorname{Fix}|mD|_S \leq \gamma.\end{aligned}$$

Now assume that $\gamma = 0$. Let $n = \dim X$ and let H be a very ample divisor on X. Pick a positive integer q such that qA and qD are integral, and such that

$$C = qA - K_X - S - nH \tag{2.56}$$

is ample. Then there exists a $\mathbb{Q}$-divisor $D' \geq 0$ such that $D' \sim_{\mathbb{Q}} D$, $S \not\subseteq \operatorname{Supp} D'$ and $\operatorname{mult}_P(D'_{|S}) < \frac{1}{q}$. Let $f\colon Y \longrightarrow X$ be a log resolution of $(X, S + D')$ which is obtained as a sequence of blowups along smooth centres. Let $T = f_*^{-1}S$, and let $E \geq 0$ be the f-exceptional integral divisor such that

$$K_Y + T = f^*(K_X + S) + E.$$

Then, denoting $F = qf^*(D+A) - \lfloor qf^*D' \rfloor + E$, by (2.56) we have

$$F \sim_{\mathbb{Q}} qf^*A + \{qf^*D'\} + E = K_Y + T + f^*(nH + C) + \{qf^*D'\},$$

and in particular $|F_{|T}| = |F|_T$ by Lemma 2.45(i). Denote $g = f_{|T}\colon T \to S$ and let $P' = g_*^{-1}P$. Since $F_{|T} \sim_{\mathbb{Q}} K_T + g^*(nH_{|S}) + g^*(C_{|S}) + \{qf^*D'\}_{|T}$ and g is an isomorphism at the generic point of P', Lemma 2.45(ii) implies that the base locus of $|F_{|T}|$ does not contain P'. In particular, if $V \in |F|$ is a general element, then $P \not\subseteq \operatorname{Supp} f_*V$.

Let $U = V + \lfloor qf^*D' \rfloor \in |qf^*(D+A)+E|$. Since E is f-exceptional, this implies that $f_*U \in |q(D+A)|$, and since $f_*\lfloor qf^*D' \rfloor \leq qD'$, we have

$$\operatorname{mult}_P(f_*U)_{|S} = \operatorname{mult}_P(f_*V)_{|S} + \operatorname{mult}_P(f_*\lfloor qf^*D' \rfloor)_{|S} \leq \operatorname{mult}_P qD'_{|S} < 1.$$

Thus, $\operatorname{mult}_P(f_*U)_{|S} = 0$ and the lemma follows. $\square$

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