Algebraic Geometry: Foliations

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Chapter 1

Introduction and motivation

Throughout these notes we work over $\mathbb{C}$ and all varieties are normal and projective.

The main topic of this course is to understand the proof of the following recent important result [CP11, CP15].

Theorem 1.1. *Let X be a smooth projective variety with K_X is pseudoeffective. Then every torsion-free quotient of $(\Omega^1_X)^{\otimes m}$ has a pseudoeffective determinant, for every positive integer m.*

Recall that a Cartier divisor D on a normal variety X is *pseudoeffective* if the class of D in $N^1(X)_{\mathbb{R}}$ is a limit of classes of effective divisors; in other words, $[D] \in N^1(X)_{\mathbb{R}}$ in the the closure of the effective cone $\mathrm{Eff}(X)$. Note that this is equivalent to $[D]$ being in the closure of the big cone $\mathrm{Big}(X)$, see [Laz04] or [LazA2].

Theorem 1.1 has already had important consequences in problems surrounding hyperbolicity and D-modules. However, our main goal in this course is an application of the result to the Minimal Model Program.

Recall that the goal of the Minimal Model Program (MMP) is a birational classification of projective varieties with mild singularities. The definition of "mild singularities" is technical, and it basically says that the canonical class of the variety behaves well under pullbacks; the details for those who are interested are in [LazA3]. The thing important for us is that if we start with a smooth variety, the operations of the MMP (*surgery operations*) produce an output which is in general no longer smooth, but has *terminal singularities*. This means that the ramification formula for K_X behaves just like the ramification formula for smooth varieties. In particular, for every m sufficiently divisible we have

$$H^0(Y, mK_Y) \simeq H^0(X, mK_X),$$

where $Y \to X$ is any resolution of X. We need two other technical facts: first, if X is a terminal variety of dimension n, then the dimension of the singular locus of X is at most $n-3$; so in particular, if X is a threefold, the singularities of X are finitely many isolated points. This is proved in [KM98, Corollary 5.18]. Second, terminal singularities are *rational*: this means that for every resolution $\pi\colon Y \to X$ we have $R^i\pi_*\mathcal{O}_Y = 0$ for all $i \geq 1$. This is proved in [KM98, Corollary 5.22]. As a consequence of this and the Leray spectral sequence, we have

$$\chi(Y,\pi^*\mathcal{F}) = \chi(X,\mathcal{F})$$

for every coherent sheaf $\mathcal{F}$ on X.

One of the main conjectures of the MMP is the following *Nonvanishing conjecture*.

Conjecture 1.2. *Let X be a smooth projective variety such that K_X is pseudoeffective. Then K_X is effective. In other words, there exists a positive integer m such that $H^0(X,mK_X) \neq 0$.*

By running the MMP, we may always modify X such that X is terminal (so we lose smoothness), but on the positive side, X is *minimal*, i.e. K_X is nef (hence we gain positivity). Here, *nef* means that K_X intersects every irreducible curve C on X nonnegatively. This concept is very close to ampleness, see [LazA2, LazA3].

Our ultimate goal in this course is to prove this conjecture in dimension 3, following [LP16]. The original wonderful proof by Miyaoka is very difficult, but does not generalise to higher dimensions. We will see how some parts of the new proof generalise to all dimensions. Hence:

Theorem 1.3. *Let X be a minimal terminal threefold. Then K_X is effective.*

One might wonder if varieties with mild singularities (so in particular terminal) behave as well as smooth varieties. This is unfortunately not the case. To illustrate this, we show that the statement is (much) easier if we replace "terminal" with "smooth" (and even then is very hard; so much so, that we take some parts of the proof for granted).

Theorem 1.4. *Let X be a smooth minimal threefold. Then K_X is effective.*

Before the proof, we need several (difficult) results without proof, some of which will also play a role in the proof in the terminal case. The first is the famous Bogomolov-Miyaoka inequality for c_2.

Theorem 1.5. *Let X be a minimal terminal variety of dimension n and let $\pi\colon Y \to X$ be a resolution of singularities. Then for any nef $\mathbb{Q}$-Cartier divisors $D_1,\dots,D_{n-2}$ we have $c_2(Y)\cdot D_1\cdot\ldots\cdot D_{n-2} \geq 0$.*

The proof is beautiful and not very difficult, once you know the theory of semistability of vector bundles and the Mehta-Ramanathan theorem, see [MP97, pp. 68–72]; some of this we will also cover in this course.

The second fact we need is that a smooth projective variety X is equipped with the *Albanese morphism* to an abelian variety A, such that $\dim A = h^1(X, \mathscr{O}_X)$ and a universal property for this morphism holds. Furthermore, if B is a subvariety of an abelian variety, and if $C \to B$ is a generically finite morphism, then $\kappa(C) \geq 0$.

The third fact is the *subadditivity of the Kodaira dimension*: if $f\colon X \to Z$ is a morphism with connected fibres between normal projective varieties such that $\dim X = 3$, and if F is a general fibre of f, then $\kappa(X) \geq \kappa(F) + \kappa(Z)$.

Now we are ready to give the proof of Theorem 1.4.

Proof of Theorem 1.4. The Riemann-Roch and Theorem 1.5 give

$$1 - h^1(X, \mathscr{O}_X) - h^3(X, \mathscr{O}_X) \leq \chi(X, \mathscr{O}_X) = -\frac{1}{24} K_X \cdot c_2(X) \leq 0,$$

hence $h^1(X, \mathscr{O}_X) \neq 0$ or $h^3(X, \mathscr{O}_X) \neq 0$. If $h^3(X, \mathscr{O}_X) \neq 0$, then $h^0(X, K_X) \neq 0$ by Serre duality, which gives the nonvanishing.

Otherwise, $h^1(X, \mathscr{O}_X) \neq 0$, and the Albanese morphism $\alpha\colon X \to A$ is non-trivial. Let $\rho\colon X \to Z$ be its Stein factorisation. Then $\kappa(Z) \geq 0$ by a fact above. Additionally, a general fibre of F is not covered by rational curves (since X is not; this is a general fact [BDPP13]). By the classification of curves and surfaces, this implies $\kappa(F) \geq 0$, and we conclude by the subadditivity of the Kodaira dimension. □

Note that the proof above fails for singular threefolds already at the beginning: we might want to take a resolution of X and repeat the procedure on it, but we may not invoke Theorem 1.5.

In order to attack Theorem 1.3, it is very convenient to introduce a new invariant, the *numerical dimension* of a nef line bundle L on an n-dimensional projective variety X. We say that L has numerical dimension d and write $\nu(X, L) = d$ if $d = \max\{k \mid L^k \not\equiv 0\}$. It is immediate that $\nu(X, L) \in \{0, 1, \dots n\}$, and $\nu(X, L) = n$ if and only if L is additionally a big line bundle by [Laz04, Theorem 2.2.16].

Now, in the context of Theorem 1.3, if $\nu(X, K_X) = 3$, we are immediately done, as big divisors are effective (this works in every dimension). If $\nu(X, K_X) = 0$, then we also have $K_X \sim_{\mathbb{Q}} 0$ by a result of Kawamata which also works in every dimension. In dimension 3, we can also prove it as follows: if $h^1(X, \mathscr{O}_X) \neq 0$, then we can argue as in the proof of Theorem 1.4 (on a resolution of X). Otherwise, if $h^1(X, \mathscr{O}_X) = 0$, then from the exponential sequence we obtain the injective map $\operatorname{Pic}(X) \to H^2(X, \mathbb{Z})$; therefore, if $K_X \equiv 0$, then $K_X \sim_{\mathbb{Q}} 0$.

Therefore, it remains to consider the cases $\nu(X, K_X) = 1$ and $\nu(X, K_X) = 2$. We start with a generalisation of the Kawamata-Viehweg vanishing.

Lemma 1.6. *Let X be a $\mathbb{Q}$-factorial projective terminal variety of dimension n and let D be a Cartier divisor on X such that $D \sim_{\mathbb{Q}} K_X + L$, where L is a nef $\mathbb{Q}$-divisor with $\nu(X,L) = k$. Then*

$$H^i\big(X, \mathscr{O}_X(D)\big) = 0 \quad \textit{for all } i > n-k.$$

Proof. The proof is by induction on n; here we prove it only for $n \leq 3$, and the proof in all dimensions is similar, but a bit more involved. If $k = n$, then this is the usual Kawamata-Viehweg vanishing [KMM87, Theorem 1-2-5 and Remark 1-2-6]. Now, assume that $k < n$ and let H be an irreducible very ample divisor on X which is general in the linear system $|H|$. Consider the exact sequence

$$0 \to \mathscr{O}_X(D) \to \mathscr{O}_X(D+H) \to \mathscr{O}_H(D+H) \to 0. \tag{1.1}$$

For $i > n-k$ we have $H^i\big(X, \mathscr{O}_X(D+H)\big) = 0$ by Kawamata-Viehweg vanishing. Since H lies in the smooth locus of X, we have

$$(D+H)|_H \sim_{\mathbb{Q}} K_H + L|_H$$

by the adjunction formula, and since $\nu(H, L|_H) = k$, we have

$$H^{i-1}\big(H, \mathscr{O}_H(D+H)\big) = 0$$

by induction. Then the result follows from the long exact sequence in cohomology associated to (1.1). □

Then we have:

Theorem 1.7. *Let X be a minimal terminal threefold with $\nu(X, K_X) = 2$. Then K_X is effective.*

Proof. Let $\pi\colon Y \to X$ be a resolution which is an isomorphism over the smooth locus of X. Since X has terminal singularities, the singular locus of X is of dimension at most $n-3$, hence

$$(\pi^* K_X)^3 = (\pi^* K_X)^2 \cdot K_Y = (\pi^* K_X) \cdot K_Y^2 = 0. \tag{1.2}$$

Indeed, we may choose a representative (in the $\mathbb{Q}$-linear equivalence class) of K_X which avoids the singularities of X, hence all three products "live" over the smooth locus of X and equal $K_X^3 = 0$.

Let m be any positive integer such that mK_X is Cartier. Then by Hirzebruch-Riemann-Roch, by (1.2) and since X has rational singularities, we obtain

$$\chi(X, \mathscr{O}_X(mK_X)) = \chi\big(Y, \mathscr{O}_Y(\pi^*(mK_X))\big) = \frac{1}{12} m (\pi^* K_X) \cdot c_2(Y) + \chi(Y, \mathscr{O}_Y). \tag{1.3}$$

By Theorem 1.5 we have

$$(\pi^* K_X) \cdot c_2(Y) \geq 0.$$

Suppose first that $(\pi^* K_X) \cdot c_2(Y) > 0$. Since

$$H^i\big(X, \mathscr{O}_X(mK_X)\big) = 0 \quad \text{for } i \geq 2 \tag{1.4}$$

by Lemma 1.6, by (1.3) we obtain $h^0\big(X, \mathscr{O}_X(mK_X)\big) > 0$ for m sufficiently divisible.

Therefore we may assume that $(\pi^* K_X) \cdot c_2(Y) = 0$, and hence

$$\chi(X, \mathscr{O}_X(mK_X)) = \chi(Y, \mathscr{O}_Y)$$

for all m sufficiently divisible. If $\chi(Y, \mathscr{O}_Y) \leq 0$, then we conclude as in the proof of Theorem 1.4. Otherwise, $\chi(Y, \mathscr{O}_Y) > 0$, and we conclude as above. □

In order to finish the proof of Theorem 1.3, it remains to consider the case $\nu(X, K_X) = 1$. This is a consequence of the content of this course, and will be done at the end (and much more). Stay tuned!

Chapter 2

Semistability with respect to movable classes

In this chapter we will study the semistability property of torsion-free coherent sheaves on a smooth projective variety X. This is very similar to the classical semistability of sheaves with respect to complete intersection curves. A big difference is that the important theorem of Mehta-Ramanathan does not work anymore in this context; however, we will find ways to get around it. The ultimate goal of this chapter are Theorem 2.27 and Corollary 2.28, which say that semistability is preserved by tensor operations.

2.1 Torsion-freeness and reflexivity

We start with the definitions of torsion-free and reflexive sheaves.

Definition 2.1. A coherent sheaf $\mathscr{F}$ on a smooth variety X is *torsion-free* if a stalk $\mathscr{F}_x$ is a torsion-free $\mathscr{O}_{X,x}$-module for every $x \in X$, and it is *reflexive* if the natural map $\mathscr{F} \to \mathscr{F}^{**}$ is an isomorphism.

It is easy to see that for a coherent sheaf $\mathscr{F}$ on X, both kernel and cokernel of a map $\mathscr{F} \to \mathscr{F}^{**}$ are torsion sheaves on X, i.e. their supports are proper subsets of X.

One can show that a torsion-free sheaf is locally free outside of a set of codimension 2, see [Kob87, Corollary 5.15], and a reflexive sheaf is locally free outside of a set of codimension 3, see [Har80, Corollary 1.4]. We will use these facts very often. Any reflexive sheaf is torsion-free.

The following simple criterion is extremely useful.

Proposition 2.2. *A coherent sheaf $\mathscr{F}$ on a smooth variety X is reflexive if and only if locally it can be included in an exact sequence*

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{G} \to 0, \tag{2.1}$$

where $\mathscr{E}$ is locally free and $\mathscr{G}$ is torsion-free.

Proof. Assume $\mathscr{F}$ is reflexive. Locally there exists a resolution of $\mathscr{F}^*$ by locally free sheaves

$$\mathscr{E}_2 \to \mathscr{E}_1 \to \mathscr{F}^* \to 0,$$

and since the functor $\mathscr{H}om(\cdot,\mathscr{O}_X)$ is contravariant left-exact, taking duals gives an exact sequence

$$0 \to \mathscr{F} \to \mathscr{E}_1^* \to \mathscr{E}_2^*.$$

Set $\mathscr{E} = \mathscr{E}_1^*$ and let $\mathscr{G}$ be the image of the map $\mathscr{E}_1^* \to \mathscr{E}_2^*$. Since $\mathscr{G}$ is a subsheaf of a locally free sheaf $\mathscr{E}_2^*$, it is torsion free.

Conversely, suppose there is (locally) an exact sequence (2.1). Then $\mathscr{F}$ is torsion-free as a subsheaf of a locally free sheaf, hence the natural map $\mathscr{F} \to \mathscr{F}^{**}$ is injective. On the other hand, dualising the sequence (2.1) twice, we get the map $\mathscr{F}^{**} \to \mathscr{E}$ which coincides (generically) with $\mathscr{F} \to \mathscr{E}$, hence is generically injective. Therefore its kernel is a torsion subsheaf of $\mathscr{F}^{**}$, hence zero since $\mathscr{F}^{**}$ is locally free. Similarly, the quotient $\mathscr{F}^{**}/\mathscr{F}$ is a torsion sheaf which is a subsheaf of torsion-free sheaf $\mathscr{G}$, hence is a zero sheaf. □

Another result we use often without explicit mention is the following [Har80, Proposition 1.6]:

Proposition 2.3. *A coherent sheaf $\mathscr{F}$ on a smooth variety X is reflexive if and only if $\mathscr{F}$ is torsion-free, and for every open subset $U \subseteq X$ and every big open subset $V \subseteq X$ the restriction map $\mathscr{F}(U) \to \mathscr{F}(U \cap V)$ is an isomorphism.*

Recall here, that an open subset $U \subseteq X$ is *big* if its complement has codimension at least 2 in X.

Definition 2.4. Let $\mathscr{F}$ be a coherent sheaf which is a subsheaf of a locally free sheaf $\mathscr{E}$. The *saturation* of $\mathscr{F}$ in $\mathscr{E}$ is the largest sheaf $\mathscr{F}_{\mathrm{sat}} \subseteq \mathscr{E}$ such that $\mathscr{F} \subseteq \mathscr{F}_{\mathrm{sat}}$, the ranks of $\mathscr{F}$ and $\mathscr{F}_{\mathrm{sat}}$ are the same, and the quotient $\mathscr{E}/\mathscr{F}_{\mathrm{sat}}$ is torsion free.

The saturation $\mathscr{F}_{\mathrm{sat}}$ in the previous definition always exists and is a reflexive sheaf, see [OSS80, Lemma 1.1.16].

2.2 Semistability

Definition 2.5. A class $\alpha \in N_1(X)_{\mathbb{R}}$ is *movable* if $\alpha \cdot D \geq 0$ for any effective Cartier divisor D. The set of movable classes forms a closed, convex cone $\mathrm{Mov}(X) \subseteq N_1(X)_{\mathbb{R}}$, called the *movable cone*. A movable class α is *big* if it lies in the interior of the movable cone.

Therefore, the movable cone is the dual cone to $\overline{\mathrm{Eff}}(X)$ with respect to the intersection pairing. There is an alternative definition of this cone, which we probably will not be using, but it might help you get an intuition for what this cone is. Recall that a *complete intersection curve* C on a smooth projective variety of dimension n is an intersection of $n-1$ very ample divisors on X. These are the curves with respect to which one usually makes the semistability theory of coherent sheaves. It is unfortunately not true that $\mathrm{Mov}(X)$ is spanned by complete intersection curves. However, one of the main results of [BDPP13] is that $\mathrm{Mov}(X)$ is the closure of the cone spanned by all classes of the form $\pi_* C$, where $\pi\colon Y \to X$ is a birational morphism from a smooth projective variety Y and C is a complete intersection curve on Y.

We come to the crucial definition of slope of a sheaf with respect to a movable class.

Definition 2.6. Let X be a smooth projective variety and let $\alpha \in \mathrm{Mov}(X)$. If $\mathscr{E} \neq 0$ is a torsion-free coherent sheaf on X, the *slope of $\mathscr{E}$ with respect to α*

$$\mu_\alpha(\mathscr{E}) := \frac{c_1(\mathscr{E})\cdot\alpha}{\mathrm{rk}\,\mathscr{E}}.$$

We say that $\mathscr{E}$ is *α-semistable* (respectively *α-stable*) if $\mu_\alpha(\mathscr{F}) \leq \mu_\alpha(\mathscr{E})$ for any nonzero coherent subsheaf $\mathscr{F} \subseteq \mathscr{E}$ (respectively if $\mu_\alpha(\mathscr{F}) < \mu_\alpha(\mathscr{E})$ for any nonzero coherent subsheaf $\mathscr{F} \subseteq \mathscr{E}$ with $\mathrm{rk}\,\mathscr{F} < \mathrm{rk}\,\mathscr{E}$). We define

$$\mu_\alpha^{\max}(\mathscr{E}) := \sup\left\{\mu_\alpha(\mathscr{F}) \mid 0 \neq \mathscr{F} \subseteq \mathscr{E} \text{ a coherent subsheaf}\right\}$$

and

$$\mu_\alpha^{\min}(\mathscr{E}) := \inf\left\{\mu_\alpha(\mathscr{Q}) \mid \mathscr{E} \twoheadrightarrow \mathscr{Q} \text{ a torsion-free quotient}\right\}.$$

A few remarks are in order. In this definition, $c_1(\mathscr{F})$ is the first Chern class of a coherent sheaf [Har77, p. 435]. Alternatively, one can define $c_1(\mathscr{F})$ as the *determinant of $\mathscr{F}$*,

$$\det\mathscr{F} = \Big(\bigwedge^r \mathscr{F}\Big)^{**},$$

where r is the generic rank of $\mathscr{F}$. There is an alternative definition of the determinant via locally free resolutions of $\mathscr{F}$, see [Kob87, pp. 162–166], which in particular shows that the above definitions of $c_1(\mathscr{F})$ coincide.

Remark 2.7. Let X be a smooth projective variety, let $\alpha \in \mathrm{Mov}(X)$, and let $\mathscr{E} \neq 0$ be a torsion-free coherent sheaf on X. Then there exists a big open subset X° of X such that $\mathscr{E}|_{X^\circ}$ is locally free, hence $\mu_\alpha(\mathscr{E}) = \mu_\alpha(\mathscr{E}^{**})$, $\mu_\alpha^{\max}(\mathscr{E}) = \mu_\alpha^{\max}(\mathscr{E}^{**})$ and $\mu_\alpha^{\min}(\mathscr{E}) = \mu_\alpha^{\min}(\mathscr{E}^{**})$ by Proposition 2.3.

Remark 2.8. Let X be a smooth projective variety and let $\alpha \in \mathrm{Mov}(X)$. If we have a short exact sequence

$$0 \to \mathscr{G} \to \mathscr{F} \to \mathscr{E} \to 0$$

of torsion-free coherent $\mathscr{O}_X$-modules, then clearly

$$\mathrm{rk}\mathscr{F}\mu_\alpha(\mathscr{F}) = \mathrm{rk}\mathscr{G}\mu_\alpha(\mathscr{G}) + \mathrm{rk}\mathscr{E}\mu_\alpha(\mathscr{E}).$$

Proposition 2.9. *Let X be a smooth projective variety and let $\alpha \in \mathrm{Mov}(X)$. Let $\mathscr{E}$ and $\mathscr{F}$ be torsion-free coherent sheaves on X.*

(i) If $\mathscr{F} \subseteq \mathscr{E}$ and $\mathrm{rk}\mathscr{E} = \mathrm{rk}\mathscr{F}$, then $\mu_\alpha(\mathscr{F}) \le \mu_\alpha(\mathscr{E})$. In particular, $\mu_\alpha(\mathscr{F}) \le \mu_\alpha(\mathscr{F}_{\mathrm{sat}})$.

(ii) An α-stable sheaf is α-semistable.

(iii) We have $\mu_\alpha^{\max}(\mathscr{E}) = -\mu_\alpha^{\min}(\mathscr{E}^*)$.

Proof. We have $\det\mathscr{F} \subseteq \det\mathscr{E}$, and since $\det\mathscr{E}$ and $\det\mathscr{F}$ are line bundles, there exists an effective Cartier divisor D such that

$$\det\mathscr{F} \otimes \mathscr{O}_X(D) \simeq \det\mathscr{E}.$$

In particular, we have $c_1(\mathscr{F}) + D = c_1(\mathscr{E})$. Since α is movable, we have that $D\cdot\alpha \ge 0$, and (i) follows. The claim (ii) is an obvious consequence of (i).

For (iii), since the functor $\mathscr{H}om(\cdot,\mathscr{O}_X)$ is contravariant left-exact, every torsion-free quotient $\mathscr{E}^* \twoheadrightarrow \mathscr{Q}$ gives rise to an inclusion $\mathscr{Q}^* \hookrightarrow \mathscr{E}^{**}$. Therefore,

$$\mu_\alpha^{\max}(\mathscr{E}) + \mu_\alpha(\mathscr{Q}) = \mu_\alpha^{\max}(\mathscr{E}^{**}) - \mu_\alpha(\mathscr{Q}^*) \ge 0 \quad \text{for every } \mathscr{Q},$$

hence $\mu_\alpha^{\max}(\mathscr{E}) + \mu_\alpha^{\min}(\mathscr{E}^*) \ge 0$. Conversely, every inclusion $\mathscr{F} \subseteq \mathscr{E}$ of torsion-free sheaves gives rise to a map $\mathscr{E}^* \to \mathscr{F}^*$, which is surjective on a big open subset of X on which both $\mathscr{E}$ and $\mathscr{F}$ are locally free. Therefore, by (i) we have

$$\mu_\alpha^{\min}(\mathscr{E}^*) + \mu_\alpha(\mathscr{F}) = \mu_\alpha^{\min}(\mathscr{E}^*) - \mu_\alpha(\mathscr{F}^*) \le 0 \quad \text{for every } \mathscr{F},$$

hence $\mu_\alpha^{\max}(\mathscr{E}) + \mu_\alpha^{\min}(\mathscr{E}^*) \le 0$, which proves (iii). □

Proposition 2.10. *Let X be a smooth projective variety and let $\alpha \in \mathrm{Mov}(X)$. If $\mathscr{E}$, $\mathscr{F}$ and $\mathscr{G}$ are torsion-free coherent sheaves on X, then the following holds:*

(i) if $\mathscr{F}$ is an α-semistable sheaf and if $\gamma\colon \mathscr{F} \to \mathscr{E}$ is a morphism of torsion-free $\mathscr{O}_X$-modules, then $\mu_\alpha(\gamma(\mathscr{F})) \ge \mu_\alpha(\mathscr{F})$,

(ii) if $\mathscr{F}$ is α-semistable and if $\mu_\alpha(\mathscr{F}) > \mu_\alpha^{\max}(\mathscr{E})$, then $\mathrm{Hom}(\mathscr{F},\mathscr{E}) = 0$,

(iii) if $\mathscr{F}$ and $\mathscr{E}$ are torsion free coherent sheaves such that $\mu_\alpha^{\min}(\mathscr{F}) > \mu_\alpha^{\max}(\mathscr{E})$, then $\operatorname{Hom}(\mathscr{F},\mathscr{E}) = 0$,

(iv) if $\mathscr{F} = \mathscr{E} \oplus \mathscr{G}$, then $\mathscr{F}$ is α-semistable if and only if $\mathscr{E}$ and $\mathscr{G}$ are α-semistable with $\mu_\alpha(\mathscr{E}) = \mu_\alpha(\mathscr{G})$,

(v) if $\mathscr{F}$ is a saturated subsheaf of $\mathscr{E}$ and if $\mu_\alpha(\mathscr{F}) = \mu_\alpha^{\max}(\mathscr{E})$, then

$$\mu_\alpha^{\max}(\mathscr{E}/\mathscr{F}) \le \mu_\alpha(\mathscr{F}) = \mu_\alpha^{\min}(\mathscr{F}).$$

Proof. For (i), let $\mathscr{G} = \ker\gamma$ and $\mathscr{E}' = \gamma(\mathscr{G})$. Then $\mu_\alpha(\mathscr{G}) \le \mu_\alpha(\mathscr{F})$ and $\operatorname{rk}\mathscr{F} = \operatorname{rk}\mathscr{G} + \operatorname{rk}\mathscr{E}'$, and (i) follows from Remark (2.8), whereas (ii) is an immediate consequence of (i).

For (iii), if there is a non-trivial morphism $\gamma\colon \mathscr{F} \to \mathscr{E}$, then $\mu_\alpha(\gamma(\mathscr{F})) \ge \mu_\alpha^{\min}(\mathscr{F}) > \mu_\alpha^{\max}(\mathscr{E})$, a contradiction.

For (iv), if $\mathscr{F}$ is α-semistable, the exact sequence $0 \to \mathscr{G} \to \mathscr{F} \to \mathscr{E} \to 0$ implies $\operatorname{Hom}(\mathscr{F},\mathscr{E}) \neq 0$, hence (ii) shows that $\mu_\alpha(\mathscr{F}) = \mu_\alpha^{\max}(\mathscr{E})$, and similarly $\mu_\alpha(\mathscr{F}) = \mu_\alpha^{\max}(\mathscr{G})$. Combined with Remark 2.8, this yields $\mu_\alpha(\mathscr{E}) = \mu_\alpha^{\max}(\mathscr{E}) = \mu_\alpha(\mathscr{G}) = \mu_\alpha^{\max}(\mathscr{G})$.

Conversely, assume $\mathscr{E}$ and $\mathscr{G}$ are α-semistable with $\mu_\alpha(\mathscr{E}) = \mu_\alpha(\mathscr{G})$, and assume $\mathscr{F}$ is not α-semistable. Then there exists an α-semistable subsheaf $\mathscr{A} \subseteq \mathscr{F}$ with $\mu_\alpha(\mathscr{A}) > \mu_\alpha(\mathscr{F})$; this is an easy consequence of Proposition 2.9(i) by descending induction on the rank of $\mathscr{A}$. Without loss of generality, we may assume that the map $\gamma\colon \mathscr{A} \subseteq \mathscr{F} \to \mathscr{E}$ is not zero. Then (i) shows that

$$\mu_\alpha(\mathscr{E}) \ge \mu_\alpha(\gamma(\mathscr{A})) \ge \mu_\alpha(\mathscr{A}) > \mu_\alpha(\mathscr{F}),$$

but Remark 2.8 gives $\mu_\alpha(\mathscr{E}) = \mu_\alpha(\mathscr{F})$, a contradiction.

For (v), let $\mathscr{F} \twoheadrightarrow \mathscr{Q}$ be a quotient and let $\mathscr{F}' \subseteq \mathscr{F}$ be its kernel. Then $\mu_\alpha(\mathscr{F}') \le \mu_\alpha(\mathscr{F})$, hence Remark 2.8 gives $\mu_\alpha(\mathscr{F}) \le \mu_\alpha(\mathscr{Q})$. Since this holds for any quotient $\mathscr{Q}$, we obtain $\mu_\alpha(\mathscr{F}) \le \mu_\alpha^{\min}(\mathscr{F})$. This then implies that $\mu_\alpha(\mathscr{F}) = \mu_\alpha^{\min}(\mathscr{F})$ by (iii).

Similarly, for any subsheaf $\mathscr{Q} \subseteq \mathscr{E}/\mathscr{F}$ we have $\mathscr{Q} = \mathscr{E}'/\mathscr{F}$ for some sheaf $\mathscr{E}' \subseteq \mathscr{E}$ containing $\mathscr{F}$. Then $\mu_\alpha(\mathscr{E}') \le \mu_\alpha(\mathscr{F})$, hence Remark 2.8 gives $\mu_\alpha(\mathscr{Q}) \le \mu_\alpha(\mathscr{F})$. This implies $\mu_\alpha^{\max}(\mathscr{E}/\mathscr{F}) \le \mu_\alpha(\mathscr{F})$, which finishes the proof. □

Corollary 2.11. *Let X be a smooth projective variety. If $\mathscr{H}$ is a line bundle on X and if r is a positive integer, then $\mathscr{H}^{\oplus r}$ is semistable with respect to any movable class.*

Proof. By Proposition 2.10(iv), it suffices to show that $\mathscr{H}$ is semistable with respect to any movable class. But this follows from Proposition 2.9. □

2.3 Maximal destabilisers

Lemma 2.12. *Let X be a smooth projective variety and let $\mathscr{F}_1$ and $\mathscr{F}_2$ be two different saturated subsheaves of a torsion free coherent sheaf $\mathscr{E}$ which have the same rank r. Then:*

(i) $\mathscr{F}_1 \not\subseteq \mathscr{F}_2$, $\mathscr{F}_2 \not\subseteq \mathscr{F}_1$ *and* $\operatorname{rk}(\mathscr{F}_1 + \mathscr{F}_2) > r$,

(ii) if $\mu_\alpha(\mathscr{F}_i) \geq \mu_\alpha^{\max}(\mathscr{E}) - \frac{\varepsilon}{2}$ *for* $i = 1, 2$ *and some* $\varepsilon \geq 0$*, then* $\mu_\alpha(\mathscr{F}_1 + \mathscr{F}_2) \geq \mu_\alpha^{\max}(\mathscr{E}) - \varepsilon$.

Proof. If $\mathscr{F}_1 \subseteq \mathscr{F}_2$, then we would have an inclusion $\mathscr{F}_2/\mathscr{F}_1 \subseteq \mathscr{E}/\mathscr{F}_1$ of a torsion sheaf into a torsion-free sheaf, hence $\mathscr{F}_2/\mathscr{F}_1 = 0$, a contradiction; and similarly if $\mathscr{F}_2 \subseteq \mathscr{F}_1$. In particular, $\mathscr{F}_2 \neq \mathscr{F}_1 + \mathscr{F}_2$. If $\operatorname{rk}(\mathscr{F}_1 + \mathscr{F}_2) = r$, we would have an inclusion $(\mathscr{F}_1 + \mathscr{F}_2)/\mathscr{F}_1 \subseteq \mathscr{E}/\mathscr{F}_1$ of a torsion sheaf into a torsion-free sheaf, hence $(\mathscr{F}_1 + \mathscr{F}_2)/\mathscr{F}_1 = 0$, a contradiction which shows (i).

For (ii), the exact sequence

$$0 \to \mathscr{F}_1 \cap \mathscr{F}_2 \to \mathscr{F}_1 \oplus \mathscr{F}_2 \to \mathscr{F}_1 + \mathscr{F}_2 \to 0 \tag{2.2}$$

gives

$$c_1(\mathscr{F}_1 + \mathscr{F}_2) = c_1(\mathscr{F}_1) + c_1(\mathscr{F}_2) - c_1(\mathscr{F}_1 \cap \mathscr{F}_2) \quad \text{and} \quad \operatorname{rk}(\mathscr{F}_1 + \mathscr{F}_2) = 2r - \operatorname{rk}(\mathscr{F}_1 \cap \mathscr{F}_2).$$

Then we have

$$\begin{aligned} \operatorname{rk}(\mathscr{F}_1 + \mathscr{F}_2)\mu_\alpha(\mathscr{F}_1 + \mathscr{F}_2) &= r\mu_\alpha(\mathscr{F}_1) + r\mu_\alpha(\mathscr{F}_2) - \operatorname{rk}(\mathscr{F}_1 \cap \mathscr{F}_2)\mu_\alpha(\mathscr{F}_1 \cap \mathscr{F}_2) \\ &\geq r\big(2\mu_\alpha^{\max}(\mathscr{E}) - \varepsilon\big) - \operatorname{rk}(\mathscr{F}_1 \cap \mathscr{F}_2)\mu_\alpha^{\max}(\mathscr{E}) \\ &= \operatorname{rk}(\mathscr{F}_1 + \mathscr{F}_2)\mu_\alpha^{\max}(\mathscr{E}) - r\varepsilon, \end{aligned}$$

which by (i) yields the claim. □

Given a torsion-free sheaf $\mathscr{E}$ and a movable class α, the α-slope of subsheaves $\mathscr{F} \subseteq \mathscr{E}$ cannot be arbitrarily large:

Proposition 2.13. *Let X be a smooth projective variety, let $\alpha \in \operatorname{Mov}(X)$, and let $\mathscr{E}$ be a torsion-free coherent sheaf of positive rank on X. Then $\mu_\alpha^{\max}(\mathscr{E}) < \infty$ and the supremum $\mu_\alpha^{\max}(\mathscr{E})$ is a maximum. In other words, there exists a non-zero coherent subsheaf $\mathscr{F} \subseteq \mathscr{E}$ such that $\mu_\alpha^{\max}(\mathscr{E}) = \mu_\alpha(\mathscr{F})$. Moreover, there exists an α-stable reflexive sheaf $\mathscr{F}' \subseteq \mathscr{E}$ of slope $\mu_\alpha(\mathscr{F}') = \mu_\alpha^{\max}(\mathscr{E})$.*

Proof. Let $\mathscr{H}$ be an ample line bundle on X such that the sheaf $\mathscr{E}^* \otimes \mathscr{H}$ is globally generated. This means that there is a surjective morphism $\mathscr{O}_X^{\oplus N} \to \mathscr{E}^* \otimes \mathscr{H}$ for some N, and hence a surjective morphism $(\mathscr{H}^{-1})^{\oplus N} \to \mathscr{E}^*$. Dualising, we obtain an injective morphism $\mathscr{E} \subseteq \mathscr{E}^{**} \to \mathscr{H}^{\oplus N}$. If $\mathscr{F} \subseteq \mathscr{E}$ is any coherent subsheaf, it follows from Corollary 2.11 that $\mu_\alpha(\mathscr{F}) \leq \mu_\alpha(\mathscr{H}^{\oplus N}) = \mathscr{H} \cdot \alpha$ for every $\alpha \in \operatorname{Mov}(X)$.

For the second claim, arguing by contradiction assume that $\mu_\alpha(\mathscr{F}) < \mu_\alpha^{\max}(\mathscr{E})$ for any non-zero coherent subsheaf $\mathscr{F} \subseteq \mathscr{E}$. Then there exists a sequence of pairwise different subsheaves $(\mathscr{F}_i)_{i\in\mathbb{N}}$ which are saturated in $\mathscr{E}$ and of the same rank r, such that

$$\lim_{i\to\infty} \mu_\alpha(\mathscr{F}_i) = \mu_\alpha^{\max}(\mathscr{E});$$

here we used Proposition 2.9. In addition, we can assume that the rank r is maximal among all sequences of sheaves satisfying these conditions.

Pick a sequence (ε_n) of positive rational numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$. For each ε_n, and pick indices $i_n < j_n$ such that

$$\mu_\alpha(\mathscr{F}_{j_n}) > \mu_\alpha(\mathscr{F}_{i_n}) > \mu_\alpha^{\max}(\mathscr{E}) - \frac{\varepsilon_n}{2}.$$

Setting $\mathscr{G}_{\varepsilon_n} := \mathscr{F}_{i_n} + \mathscr{F}_{j_n}$, we have $\operatorname{rk}\mathscr{G}_{\varepsilon_n} > r$ and $\mu_\alpha(\mathscr{G}_{\varepsilon_n}) \geq \mu_\alpha^{\max}(\mathscr{E}) - \varepsilon$ by Lemma 2.12, which contradicts the choice of r and proves the first claim.

Now, let $\mathscr{F}_1$ be a non-zero coherent subsheaf $\mathscr{F}_1 \subseteq \mathscr{E}$ such that $\mu_\alpha^{\max}(\mathscr{E}) = \mu_\alpha(\mathscr{F}_1)$. If $\mathscr{F}_1$ is not stable, then there exists a sheaf $\mathscr{F}_2 \subseteq \mathscr{F}_1$ such that $\mu_\alpha^{\max}(\mathscr{E}) = \mu_\alpha(\mathscr{F}_2)$ and $\operatorname{rk}\mathscr{F}_2 < \operatorname{rk}\mathscr{F}_1$. After finitely many iterations of this process, we reach the desired sheaf. □

Corollary 2.14. *Let X be a smooth projective variety, let $\alpha \in \operatorname{Mov}(X)$, and let $\mathscr{E}$ be a torsion-free coherent sheaf of positive rank on X. Then there exists a unique sheaf $\mathscr{F} \subseteq \mathscr{E}$, maximal with respect to the inclusion, such that $\mu_\alpha(\mathscr{F}) = \mu_\alpha^{\max}(\mathscr{E})$. The sheaf $\mathscr{F}$ is the "maximal destabilising subsheaf", and it is clearly semistable and saturated in $\mathscr{E}$.*

Proof. By Proposition 2.13, there exists a saturated sheaf $\mathscr{F}_1 \subseteq \mathscr{E}$ of maximal slope, and whose rank r_1 is maximal among all such subsheaves. If $\mathscr{F}_2$ is any other saturated subsheaf of maximal slope and of rank r_2, from the exact sequence (2.2) we obtain $c_1(\mathscr{F}_1+\mathscr{F}_2) = c_1(\mathscr{F}_1)+c_1(\mathscr{F}_2)-c_1(\mathscr{F}_1\cap\mathscr{F}_2)$ and $\operatorname{rk}(\mathscr{F}_1+\mathscr{F}_2) = r_1+r_2-\operatorname{rk}(\mathscr{F}_1\cap\mathscr{F}_2)$. Then we have

$$\begin{aligned}\operatorname{rk}(\mathscr{F}_1 + \mathscr{F}_2)\mu_\alpha(\mathscr{F}_1 + \mathscr{F}_2) &= r_1\mu_\alpha(\mathscr{F}_1) + r_2\mu_\alpha(\mathscr{F}_2) - \operatorname{rk}(\mathscr{F}_1 \cap \mathscr{F}_2)\mu_\alpha(\mathscr{F}_1 \cap \mathscr{F}_2)\\ &\geq (r_1 + r_2)\mu_\alpha^{\max}(\mathscr{E}) - \operatorname{rk}(\mathscr{F}_1 \cap \mathscr{F}_2)\mu_\alpha^{\max}(\mathscr{E}) = \operatorname{rk}(\mathscr{F}_1 + \mathscr{F}_2)\mu_\alpha^{\max}(\mathscr{E}),\end{aligned}$$

hence $\mu_\alpha(\mathscr{F}_1 + \mathscr{F}_2) = \mu_\alpha^{\max}(\mathscr{E})$, and consequently the α-slope of the saturation $(\mathscr{F}_1 + \mathscr{F}_2)_{\mathrm{sat}}$ is also maximal by Proposition 2.9. Therefore the rank of $(\mathscr{F}_1 + \mathscr{F}_2)_{\mathrm{sat}}$ must be r_1 by assumption, which contradicts Lemma 2.12 unless $(\mathscr{F}_1 + \mathscr{F}_2)_{\mathrm{sat}} = \mathscr{F}_1$ and hence $\mathscr{F}_2 \subseteq \mathscr{F}_1$. □

Corollary 2.15. *Let X be a smooth projective variety, let $\alpha \in \operatorname{Mov}(X)$, and let $\mathscr{E}$ be a torsion-free coherent sheaf of positive rank on X. Then there exists a unique*

"Harder-Narasimhan-filtration", that is, a filtration $0 = \mathscr{E}_0 \subsetneq \mathscr{E}_1 \subsetneq \cdots \subsetneq \mathscr{E}_r = \mathscr{E}$ *where each quotient* $\mathscr{Q}_i := \mathscr{E}_i/\mathscr{E}_{i-1}$ *is torsion-free,* α*-semistable, and where the sequence of slopes* $\mu_\alpha(\mathscr{Q}_i)$ *is strictly decreasing.*

Proof. The proof of the existence is by induction on the rank of $\mathscr{E}$. Let $\mathscr{E}_1$ be the maximal destabilising subsheaf of $\mathscr{E}$. By induction, $\mathscr{E}/\mathscr{E}_1$ has a Harder-Narasimhan filtration $0 = \mathscr{G}_0 \subseteq \mathscr{G}_1 \subseteq \cdots \subseteq \mathscr{G}_{r-1} = \mathscr{E}/\mathscr{E}_1$, and denote by $\mathscr{E}_{i+1} \subseteq \mathscr{E}$ the preimage of $\mathscr{G}_i$. It remains to show that $\mu_\alpha(\mathscr{E}_1) > \mu_\alpha(\mathscr{E}_2/\mathscr{E}_1)$: indeed, otherwise we would have $\mu_\alpha(\mathscr{E}_2) > \mu_\alpha(\mathscr{E}_1)$ by Remark 2.8, which would contradict the maximality of $\mathscr{E}_1$.

For uniqueness, assume that we have two Harder-Narasimhan filiations $\mathscr{E}_\bullet$ and $\mathscr{E}'_\bullet$ of $\mathscr{E}$. Without loss of generality, we may assume that $\mu_\alpha(\mathscr{E}'_1) \geq \mu_\alpha(\mathscr{E}_1)$, and let j be the minimal index such that $\mathscr{E}'_1 \subseteq \mathscr{E}_j$. Then the map $\mathscr{E}'_1 \to \mathscr{E}_j \to \mathscr{E}_j/\mathscr{E}_{j-1}$ is a non-zero morphism between semistable sheaves, hence

$$\mu_\alpha(\mathscr{E}_j/\mathscr{E}_{j-1}) \geq \mu_\alpha(\mathscr{E}'_1) \geq \mu_\alpha(\mathscr{E}_1) \geq \mu_\alpha(\mathscr{E}_j/\mathscr{E}_{j-1})$$

by the properties of the Harder-Narasimhan filtration and by Proposition 2.10(i). In particular, $\mu_\alpha(\mathscr{E}_1) = \mu_\alpha(\mathscr{E}_j/\mathscr{E}_{j-1})$ and thus $j = 1$ by the definition of the Harder-Narasimhan filtration. Therefore, $\mathscr{E}'_1 \subseteq \mathscr{E}_1$ and $\mu_\alpha(\mathscr{E}'_1) \leq \mu_\alpha(\mathscr{E}_1)$ by the semistability of $\mathscr{E}_1$. Reversing the roles of $\mathscr{E}_1$ and $\mathscr{E}'_1$, we obtain $\mathscr{E}'_1 = \mathscr{E}_1$. By induction on the rank, the uniqueness holds for the Harder-Narasimhan filtrations of $\mathscr{E}/\mathscr{E}_1$, hence $\mathscr{E}_i/\mathscr{E}_1 = \mathscr{E}'_i/\mathscr{E}_1$ and consequently $\mathscr{E}_i = \mathscr{E}'_i$, which was to be proved. □

Corollary 2.16. *Let* X *be a smooth projective variety, let* $\alpha \in \operatorname{Mov}(X)$*, and let* $\mathscr{E}$ *be a torsion-free coherent sheaf of positive rank on* X*. If* $\mathscr{E}$ *is* α*-semistable, then there exists a "Jordan-Hölder-filtration", that is, a filtration* $0 = \mathscr{E}_0 \subsetneq \mathscr{E}_1 \subsetneq \cdots \subsetneq \mathscr{E}_r = \mathscr{E}$ *where each quotient* $\mathscr{Q}_i := \mathscr{E}_i/\mathscr{E}_{i-1}$ *is torsion-free,* α*-stable, and with slopes* $\mu_\alpha(\mathscr{Q}_i) = \mu_\alpha(\mathscr{E})$*.*

Proof. Exercise! □

Remark 2.17. Let X be a smooth projective variety, let $\alpha \in \operatorname{Mov}(X)$, and let $\mathscr{E}$ be a torsion-free coherent sheaf of positive rank on X. Combining Harder-Narasimhan and Jordan-Hölder filtrations, one obtains a "refined Harder-Narasimhan-filtration" $0 = \mathscr{E}_0 \subsetneq \mathscr{E}_1 \subsetneq \cdots \subsetneq \mathscr{E}_r = \mathscr{E}$ where each quotient $\mathscr{Q}_i := \mathscr{E}_i/\mathscr{E}_{i-1}$ is torsion-free, α-stable, and where the sequence of slopes $\mu_\alpha(\mathscr{Q}_i)$ is non-increasing.

2.4 Openness of semistability

Definition 2.18. Let X be a smooth projective variety and let $\alpha \in \operatorname{Mov}(X)$ be a movable class. If $\mathscr{E}$ is any torsion-free coherent sheaf of $\mathscr{O}_X$-modules with $\operatorname{rk}\mathscr{E} \geq 2$, write

$$\mu_\alpha^{\max,sc}(\mathscr{E}) := \sup\left\{\mu_\alpha(\mathscr{F}) \mid 0 \neq \mathscr{F} \subseteq \mathscr{E} \text{ coherent with } \operatorname{rk}\mathscr{F} < \operatorname{rk}\mathscr{E}\right\}.$$

Clearly $\mu_\alpha^{\max,sc}(\mathscr{E}) \le \mu_\alpha^{\max}(\mathscr{E})$.

Proposition 2.19. *Let X be a smooth projective variety and let $\alpha \in \mathrm{Mov}(X)$ be a movable class which is big or rational. Then there exists a coherent subsheaf $\mathscr{F} \subseteq \mathscr{E}$ such that $\mathrm{rk}\,\mathscr{F} < \mathrm{rk}\,\mathscr{E}$ and $\mu_\alpha^{\max,sc}(\mathscr{E}) = \mu_\alpha(\mathscr{F})$. In particular, $\mathscr{E}$ is α-stable if and only if $\mu_\alpha^{\max,sc}(\mathscr{E}) < \mu_\alpha^{max}(\mathscr{E})$*

Proof. Step 1. We first deal with the case when α is big. If $\|\cdot\|$ is any norm on the vector space $N^1(X)_\mathbb{R}$, there exists a constant $C > 0$ such that

$$D \cdot \alpha \ge C \cdot \|D\| \quad \text{for } D \in \overline{\mathrm{Eff}}(X). \tag{2.3}$$

Since the statement of the proposition is invariant under taking tensor product of $\mathscr{E}$ with line bundles, we may replace $\mathscr{E}$ by a tensor product with a sufficiently ample line bundle and assume that there exist a positive integer N and a surjection $\mathscr{O}_X^{\oplus N} \to \mathscr{E}$. In particular, for any torsion-free quotient $\mathscr{E} \to \mathscr{Q}$ of positive rank, $c_1(\mathscr{Q}) = \det \mathscr{Q}$ is pseudoeffective.

Assume that the number $\mu_\alpha^{\max,sc}(\mathscr{E})$ is not attained. Then there exists a sequence of saturated subsheaves $\mathscr{F}_j \subseteq \mathscr{E}$ of the same rank such that the sequence of slopes $\mu_\alpha(\mathscr{F}_j)$ is strictly increasing and converges to $\mu_\alpha^{\max,sc}(\mathscr{E})$. Denote $\mathscr{Q}_j = \mathscr{E}/\mathscr{F}_j$. The it follows from Remark 2.8 that the set $\{\mu_\alpha(\mathscr{Q}_j) \mid j \in \mathbb{N}\}$ is infinite, hence the set $\{c_1(\mathscr{Q}_j) \mid j \in \mathbb{N}\}$ is infinite since α is big and all $c_1(\mathscr{Q}_j)$ are pseudoeffective. In particular, (2.3) implies that the sequence $\big(\mu_\alpha(\mathscr{Q}_j)\big)_{j\in\mathbb{N}}$ is unbounded, and hence so is $\big(\mu_\alpha(\mathscr{F}_j)\big)_{j\in\mathbb{N}}$, a contradiction.

Step 2. If α is rational, pick a positive integer m such that $m\alpha$ is integral. Then for every subsheaf $\mathscr{F} \subseteq \mathscr{E}$ of positive rank we have $\mu_\alpha(\mathscr{F}) \in \frac{1}{m(\mathrm{rk}\,\mathscr{E})!}\mathbb{Z}$, and the claim is obvious. ☐

We need the following version of the so called Grothendieck lemma.

Theorem 2.20. *Let X be a smooth projective variety and let $\beta \in \mathrm{Mov}(X)$ be a big class. Further, let $\mathscr{E}$ be a torsion-free coherent sheaf on X and let $c \in \mathbb{R}$. Then the set*

$$S_c = \{c_1(\mathscr{F}) \mid \mathscr{F} \subseteq \mathscr{E} \textit{ any positive-rank subsheaf with } \mu_\beta(\mathscr{F}) \ge c\} \subseteq N^1(X)_\mathbb{Q}$$

is finite.

Proof. Choose a sufficiently ample bundle $\mathscr{H}$ and an embedding $\mathscr{E} \hookrightarrow \mathscr{H}^{\oplus r}$ as in the proof of Proposition 2.13. It suffices to show the claim when $\mathscr{E} = \mathscr{H}^{\oplus r}$. Since slopes behave linearly under twists by line bundles, we may further assume that $\mathscr{E} = \mathscr{O}_X^{\oplus r}$.

For $\mathscr{F} \in S_c$, consider its saturation $\mathscr{F}_{\mathrm{sat}} \subseteq \mathscr{E}$ and the torsion-free quotient $\mathscr{Q} = \mathscr{E}/\mathscr{F}_{\mathrm{sat}}$. Then $\mathscr{Q}$ is globally generated, hence $\det \mathscr{Q} \simeq \mathscr{O}_X(D')$ for some effective

Cartier divisor D'. This implies $\det\mathscr{F}_{\text{sat}} \simeq \mathscr{O}_X(-D')$, and since $\det\mathscr{F} \hookrightarrow \det\mathscr{F}_{\text{sat}}$, there exists an effective Cartier divisor D such that $\det\mathscr{F} \simeq \mathscr{O}_X(-D)$. We then have

$$c \le \mu_\beta(\mathscr{F}) = -\frac{1}{\operatorname{rk}\mathscr{F}} D \cdot \beta \le 0.$$

Since β is a big class, the set $\{L \in \overline{\operatorname{Eff}}(X) \mid 0 \le L \cdot \beta \le rc\}$ is compact and the result follows. □

We now show that stability is an open property, at least within the interior of the movable cone.

Definition 2.21. Let X be a smooth projective variety and let $\mathscr{E}$ be a non-trivial torsion-free sheaf on X. Define

$$\operatorname{Stab}(\mathscr{E}) := \{\alpha \in \operatorname{Mov}(X) \mid \mathscr{E} \text{ is } \alpha\text{-stable}\}.$$

This set is clearly convex.

Theorem 2.22. *Let X be a smooth projective variety and let $\mathscr{E}$ be a non-trivial torsion-free sheaf on X. If $\alpha \in \operatorname{Stab}(\mathscr{E})$ is big, then $\operatorname{Stab}(\mathscr{E})$ contains an open neighbourhood $U \subseteq \operatorname{Mov}(X)$.*

Proof. The statement if clear if $\operatorname{rk}\mathscr{E} = 1$, hence we assume that $\operatorname{rk}\mathscr{E} \ge 2$. Choose a bounded open neighbourhood V of α in $\operatorname{Mov}(X)$. By the proof of Proposition 2.13, there exists an ample line bundle $\mathscr{A}$ such that $\mu_\beta^{\max}(\mathscr{E}) \le c_1(\mathscr{A}) \cdot \beta$ for every $\beta \in \operatorname{Mov}(X)$. Setting $\mathscr{H} = \mathscr{A}^{\otimes \operatorname{rk}\mathscr{E}}$, it is easy to check that

$$\mu_\beta^{\max}(\mathscr{E}) - \mu_\beta(\mathscr{E}) \le \big(c_1(\mathscr{H}) - c_1(\mathscr{E})\big) \cdot \beta \tag{2.4}$$

for every $\beta \in \operatorname{Mov}(X)$. Since $\mu_\alpha^{\max,sc}(\mathscr{E}) < \mu_\alpha(\mathscr{E})$ by Proposition 2.19, then (2.4) implies that there exists a rational number $0 < e \ll 1$ such that for all $0 \le \varepsilon \le e$ and all $\beta \in V$ we have

$$\varepsilon\big(\mu_\beta^{\max}(\mathscr{E}) - \mu_\beta(\mathscr{E})\big) < (1-\varepsilon)\big(\mu_\alpha(\mathscr{E}) - \mu_\alpha^{\max,sc}(\mathscr{E})\big).$$

Therefore, for any coherent subsheaf $\mathscr{F} \subseteq \mathscr{E}$ with $\operatorname{rk}\mathscr{F} < \operatorname{rk}\mathscr{E}$, any $\beta \in V$ and any $0 < \varepsilon \le e$ this implies

$$\begin{aligned}\mu_{(1-\varepsilon)\alpha+\varepsilon\beta}(\mathscr{F}) &= (1-\varepsilon)\mu_\alpha(\mathscr{F}) + \varepsilon\mu_\beta(\mathscr{F}) \le (1-\varepsilon)\mu_\alpha^{\max,sc}(\mathscr{E}) + \varepsilon\mu_\beta^{\max}(\mathscr{E}) \\ &< (1-\varepsilon)\mu_\alpha(\mathscr{E}) + \varepsilon\mu_\beta(\mathscr{E}) = \mu_{(1-\varepsilon)\alpha+\varepsilon\beta}(\mathscr{E}).\end{aligned}$$

We set $U := e(V - \alpha) + \alpha$. □

Theorem 2.23. *Let X be a smooth projective variety, let $\mathscr{E}$ be a non-trivial torsion-free sheaf on X, and let $\alpha \in \operatorname{Stab}(\mathscr{E})$ and $\beta \in \operatorname{Mov}(X)^\circ$. Then:*

(i) there exists a positive rational number e such that $\alpha + \varepsilon\beta \in \operatorname{Stab}(\mathscr{E})$ for any $\varepsilon \in [0,e]$,

(ii) there exists a sequence of big rational classes $\beta_i \in \operatorname{Stab}(\mathscr{E})$ with $\lim \beta_i = \alpha$.

Proof. Note that (ii) follows by combining (i) with Theorem 2.22, so we only need to show (i). By Theorem 2.20, the set

$$S := \{(c_1(\mathscr{F}), \operatorname{rk}\mathscr{F}) \mid \mathscr{F} \subseteq \mathscr{E} \text{ such that } \mu_\beta(\mathscr{F}) \geq \mu_\beta(\mathscr{E})\}$$

is finite, hence there exist subsheaves $\mathscr{F}_j \subseteq \mathscr{E}$ such that

$$S = \{(c_1(\mathscr{F}_1), \operatorname{rk}\mathscr{F}_1), \dots, (c_1(\mathscr{F}_n), \operatorname{rk}\mathscr{F}_n)\}.$$

For $\varepsilon \in [0,1]$, denote $\Phi(\varepsilon) = \max\{\mu_{(1-\varepsilon)\alpha+\varepsilon\beta}(\mathscr{F}_j) \mid 1 \leq j \leq n\}$. Then Φ is a continuous function in ε and $\Phi(0) < \mu_\alpha(\mathscr{E})$, hence there exists a positive rational number e such that $\Phi(\varepsilon) < \mu_{(1-\varepsilon)\alpha+\varepsilon\beta}(\mathscr{E})$ for all $\varepsilon \in [0,e]$. Consider a subsheaf $\mathscr{F} \subseteq \mathscr{E}$ with $\operatorname{rk}\mathscr{F} < \operatorname{rk}\mathscr{E}$. If $(c_1(\mathscr{F}), \operatorname{rk}\mathscr{F}) \notin S$, then $\mu_\alpha(\mathscr{F}) < \mu_\alpha(\mathscr{E})$ and $\mu_\beta(\mathscr{F}) < \mu_\beta(\mathscr{E})$ imply

$$\mu_{(1-\varepsilon)\alpha+\varepsilon\beta}(\mathscr{F}) < \mu_{(1-\varepsilon)\alpha+\varepsilon\beta}(\mathscr{E}) \quad \text{for } \varepsilon \in [0,e],$$

whereas if $(c_1(\mathscr{F}), \operatorname{rk}\mathscr{F}) \in S$, then there exists $\mathscr{F}_j$ such that $c_1(\mathscr{F}) = c_1(\mathscr{F}_j)$ and $\operatorname{rk}\mathscr{F} = \operatorname{rk}\mathscr{F}_j$, hence

$$\mu_{(1-\varepsilon)\alpha+\varepsilon\beta}(\mathscr{F}) = \mu_{(1-\varepsilon)\alpha+\varepsilon\beta}(\mathscr{F}_j) < \mu_{(1-\varepsilon)\alpha+\varepsilon\beta}(\mathscr{E}) \quad \text{for } \varepsilon \in [0,e].$$

This finishes the proof. □

2.5 Tensor products of semistable sheaves

In this section we prove the main results of the chapter.

First some notation. Given two coherent $\mathscr{O}_X$-modules $\mathscr{A}$ and $\mathscr{B}$ on a smooth variety X, and any resolution of singularities $\pi\colon \widetilde{X} \to X$, denote by:

(a) $\mathscr{A} \boxtimes \mathscr{B} := (\mathscr{A} \otimes \mathscr{B})^{**}$ the *reflexive tensor product* of $\mathscr{A}$ and $\mathscr{B}$,

(b) $\operatorname{Sym}^{[q]}\mathscr{A} := (\operatorname{Sym}^q \mathscr{A})^{**}$ the q-th *reflexive symmetric power* of $\mathscr{A}$,

(c) $\bigwedge^{[q]}\mathscr{A} := (\bigwedge^q \mathscr{A})^{**}$ the q-th *reflexive exterior power* of $\mathscr{A}$,

(d) $\pi^{[*]}\mathscr{A} = (\pi^*\mathscr{A})^{**}$.

We note for the later use the following result.

Lemma 2.24. *Let $\pi\colon X \to Y$ be a flat morphism between smooth varieties, and let $\mathscr{F}$ be a torsion-free sheaf on Y.*

(a) If $\mathscr{F}$ is reflexive, then $\pi^*\mathscr{F}$ is reflexive.

(b) If π is finite, then $\operatorname{Sym}^{[q]}\pi^*\mathscr{F} = \pi^*\operatorname{Sym}^{[q]}\mathscr{F}$ for every positive integer q.

Proof. We start with an observation. Let $\mathscr{Q}$ be a torsion-free sheaf on X. Then, locally, there exists a locally free sheaf $\mathscr{E}$ and an injection $\mathscr{Q} \to \mathscr{E}$. Since π is flat, the induced map $\pi^*\mathscr{Q} \to \pi^*\mathscr{E}$ is also injective, hence $\pi^*\mathscr{Q}$ is torsion free since $\pi^*\mathscr{E}$ is locally free.

Now, locally there exist a locally free sheaf $\mathscr{E}$, a torsion-free sheaf $\mathscr{Q}$ and an exact sequence

$$0 \to \mathscr{F} \to \mathscr{E} \to \mathscr{Q} \to 0.$$

Since π is flat, we obtain the induced exact sequence

$$0 \to \pi^*\mathscr{F} \to \pi^*\mathscr{E} \to \pi^*\mathscr{Q} \to 0.$$

Since $\pi^*\mathscr{Q}$ is torsion-free by above, this implies that $\pi^*\mathscr{F}$ is reflexive, which proves (a).

For (b), let Y° be a big open subset of Y on which $\mathscr{F}$ is locally free. Then the sheaves $\operatorname{Sym}^{[q]}\pi^*\mathscr{F}$ and $\pi^*\operatorname{Sym}^{[q]}\mathscr{F}$ coincide on the big open subset $\pi^{-1}(Y^\circ)$ of X. Since they are both reflexive by (a), they coincide on the whole X. □

If X is a projective variety and if $\pi\colon \widetilde{X} \to X$ is any resolution of singularities, then it is easy to see that a pullback of a movable class on X is again movable on $\widetilde{X}$.

Lemma 2.25. *Let X be a smooth projective variety and let $\alpha \in \operatorname{Mov}(X)$ be a movable class. If $\mathscr{F}$ and $\mathscr{G}$ are torsion-free coherent sheaves of positive rank on X. Let $\pi\colon \widetilde{X} \to X$ be any resolution of singularities. Then the following holds.*

(i) The sheaf $\mathscr{F}$ is α-stable if and only if $\pi^{[*]}\mathscr{F}$ is $\pi^*\alpha$-stable.

(ii) The sheaf $\mathscr{F}\boxtimes\mathscr{G}$ is α-semistable if and only if $\pi^{[*]}\mathscr{F}\boxtimes\pi^{[*]}\mathscr{G}$ is $\pi^*\alpha$-semistable.

Proof. For (i), let $\mathscr{E}$ be a torsion-free subsheaf of $\pi^{[*]}\mathscr{F}$. Then $\mathscr{E}$ and $\pi^*\pi_*\mathscr{E}$ agree away from the exceptional locus, hence if E_i are the prime exceptional divisors on $\widetilde{X}$, there integers λ_i such that

$$c_1(\mathscr{E}) = \pi^* c_1(\pi_*\mathscr{E}) + \sum \lambda_i E_i.$$

Therefore $\mu_{\pi^*\alpha}(\mathscr{E}) = \mu_\alpha(\pi_*\mathscr{E})$. Conversely, let $\mathscr{G}$ be a torsion-free subsheaf of $\mathscr{F}$. Then $\mu_{\pi^*\alpha}(\pi^{[*]}\mathscr{G}) = \mu_\alpha(\pi_*\pi^{[*]}\mathscr{G})$ by what we just proved. Since $\pi_*\pi^{[*]}\mathscr{G}$ and $\mathscr{G}$ agree on a big open subset of X, we have $c_1(\pi_*\pi^{[*]}\mathscr{G}) = c_1(\mathscr{G})$, and hence $\mu_\alpha(\pi_*\pi^{[*]}\mathscr{G}) = \mu_\alpha(\mathscr{G})$. Now (i) follows (exercise), and (ii) is similar. □

Theorem 2.26. *Let X be a smooth projective variety and let $\alpha \in \operatorname{Mov}(X)$ be a movable class. If $\mathscr{F}$ and $\mathscr{G}$ are α-stable torsion-free coherent sheaves of positive rank on X, then $\mathscr{F} \boxtimes \mathscr{G}$ is α-semistable.*

Proof. By [Ros68, Theorem 3.5], there is a smooth projective variety $\widetilde{X}$ and a birational morphism $\pi\colon \widetilde{X} \to X$ such that $\pi^{[*]}\mathscr{F}$ and $\pi^{[*]}\mathscr{G}$ are both locally free. By Lemma 2.25, it suffices to establish $\pi^*\alpha$-semistability of $\pi^{[*]}\mathscr{F} \otimes \pi^{[*]}\mathscr{G}$. Replacing X by $\widetilde{X}$, we may hence assume that the sheaves $\mathscr{F}$ and $\mathscr{G}$ are locally free.

Let $\mathscr{A} \subseteq \mathscr{F} \otimes \mathscr{G}$ be a proper subsheaf. By Theorem 2.23, there exists a sequence of big, rational classes $\beta_i \in \operatorname{Stab}(\mathscr{F}) \cap \operatorname{Stab}(\mathscr{G})$ with $\lim \beta_i = \alpha$. A result of Toma[1] [CP11, Proposition 6.1] implies

$$\mu_\alpha(\mathscr{A}) = \lim_{i\to\infty} \mu_{\beta_i}(\mathscr{A}) \leq \lim_{i\to\infty} \mu_{\beta_i}(\mathscr{F} \otimes \mathscr{G}) = \mu_\alpha(\mathscr{F} \otimes \mathscr{G}),$$

which was to be shown. □

Theorem 2.27. *Let X be a smooth projective variety and let $\alpha \in \operatorname{Mov}(X)$ be a movable class. If $\mathscr{F}$ and $\mathscr{G}$ are torsion-free coherent sheaves of positive rank on X, then:*

(i) $\mu_\alpha^{\max}(\mathscr{F} \boxtimes \mathscr{G}) = \mu_\alpha^{\max}(\mathscr{F}) + \mu_\alpha^{\max}(\mathscr{G})$,

(ii) if $\mathscr{F}$ and $\mathscr{G}$ are α-semistable, then $\mathscr{F} \boxtimes \mathscr{G}$ is α-semistable.

Proof. Step 1. Since numerical classes and slopes are unaffected when modifying $\mathscr{F}$ and $\mathscr{G}$ along a subset of codimension at least two, by replacing these sheaves by their double duals we may assume that $\mathscr{F}$ and $\mathscr{G}$ are reflexive.

Combining the Harder-Narasimhan filtration and a Jordan-Hölder-filtration as in Remark 2.17, choose a filtration of $\mathscr{F}$, say $0 = \mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \cdots \subseteq \mathscr{F}_k = \mathscr{F}$ such that where each quotient $\mathscr{Q}_i := \mathscr{F}_i/\mathscr{F}_{i-1}$ is torsion-free, α-stable, and where the sequence of slopes $\mu_\alpha(\mathscr{Q}_i)$ is non-increasing. There exists a big open subset $X^\circ \subseteq X$ on which all sheaves $\mathscr{F}$, $\mathscr{G}$, $\mathscr{F}_i$ and $\mathscr{Q}_i$ are locally free.

Taking reflexive tensor products with $\mathscr{G}$, we obtain a filtration of $\mathscr{F} \boxtimes \mathscr{G}$:

$$0 = \mathscr{F}_0 \boxtimes \mathscr{G} \subseteq \mathscr{F}_1 \boxtimes \mathscr{G} \subseteq \cdots \subseteq \mathscr{F}_k \boxtimes \mathscr{G} = \mathscr{F} \boxtimes \mathscr{G}.$$

Note that $(\mathscr{F}_{i+1} \otimes \mathscr{G})/(\mathscr{F}_i \otimes \mathscr{G})$ coincides with $\mathscr{Q}_{i+1} \otimes \mathscr{G}$ on X° for each i, hence $\mathscr{F}_i \boxtimes \mathscr{G}$ and its saturation in $\mathscr{F}_{i+1} \boxtimes \mathscr{G}$ coincide on X°, and thus coincide on X. In particular,

(a) the quotient $(\mathscr{F}_{i+1} \boxtimes \mathscr{G})/(\mathscr{F}_i \boxtimes \mathscr{G})$ is torsion-free,

(b) $\big((\mathscr{F}_{i+1} \boxtimes \mathscr{G})/(\mathscr{F}_i \boxtimes \mathscr{G})\big)^{**} = \mathscr{Q}_{i+1} \boxtimes \mathscr{G}$, and

(c) $\mu_\alpha(\mathscr{Q}_{i+1} \boxtimes \mathscr{G}) = \mu_\alpha(\mathscr{Q}_{i+1}) + \mu_\alpha(\mathscr{G}) \leq \mu_\alpha(\mathscr{Q}_1) + \mu_\alpha(\mathscr{G}) \leq \mu_\alpha^{\max}(\mathscr{F}) + \mu_\alpha^{\max}(\mathscr{G})$.

[1]The proof of this result uses the Yang-Mills theory and is beyond the scope of this course.

Step 2. In this step we prove (i). Let $\mathscr{G}_1$ be the maximal destabiliser of $\mathscr{G}$, and note that $\mathscr{F}_1 \boxtimes \mathscr{G}_1 \subseteq \mathscr{F} \boxtimes \mathscr{G}$. By the construction of the refined Harder-Narasimhan filtration, we have $\mu_\alpha(\mathscr{F}_1) = \mu_\alpha^{\max}(\mathscr{F})$, hence

$$\mu_\alpha(\mathscr{F}_1 \boxtimes \mathscr{G}_1) = \mu_\alpha(\mathscr{F}_1) + \mu_\alpha(\mathscr{G}_1) = \mu_\alpha^{\max}(\mathscr{F}) + \mu_\alpha^{\max}(\mathscr{G}),$$

which shows that $\mu_\alpha^{\max}(\mathscr{F} \boxtimes \mathscr{G}) \geq \mu_\alpha^{\max}(\mathscr{F}) + \mu_\alpha^{\max}(\mathscr{G})$. Now, let $\mathscr{A}$ be the maximal destabiliser of $\mathscr{F} \boxtimes \mathscr{G}$, and assume that $\mu_\alpha(\mathscr{A}) > \mu_\alpha^{\max}(\mathscr{F}) + \mu_\alpha^{\max}(\mathscr{G})$. We derive a contradiction in Steps 3 and 4.

Step 3. Assume first that $\mathscr{G}$ is α-stable. By (a) and (b), we have the composition

$$\phi\colon \mathscr{A} \hookrightarrow \mathscr{F}_k \boxtimes \mathscr{G} \to (\mathscr{F}_k \boxtimes \mathscr{G})/(\mathscr{F}_{k-1} \boxtimes \mathscr{G}) \hookrightarrow \big((\mathscr{F}_k \boxtimes \mathscr{G})/(\mathscr{F}_{k-1} \boxtimes \mathscr{G})\big)^{**} = \mathscr{Q}_k \boxtimes \mathscr{G} \quad (2.5)$$

By Theorem 2.26, $\mathscr{Q}_k \boxtimes \mathscr{G}$ is α-semistable, and therefore the inequality (c) and Proposition 2.10(ii) imply that ϕ must be zero, hence $\mathscr{A} \subseteq \mathscr{F}_{k-1} \boxtimes \mathscr{G}$. Continuing this process analogously, we obtain $\mathscr{A} = 0$, a contradiction.

Step 4. By (a) and (b), we have the map ϕ as in (2.5). Since $\mathscr{Q}_k$ is α-stable, Step 3 shows that

$$\mu_\alpha^{\max}(\mathscr{Q}_k \boxtimes \mathscr{G}) = \mu_\alpha^{\max}(\mathscr{Q}_k) + \mu_\alpha^{\max}(\mathscr{G}) \leq \mu_\alpha^{\max}(\mathscr{F}) + \mu_\alpha^{\max}(\mathscr{G}).$$

Therefore Proposition 2.10 implies that ϕ must be zero, hence $\mathscr{A} \subseteq \mathscr{F}_{k-1} \boxtimes \mathscr{G}$. Continuing this process analogously, we obtain $\mathscr{A} = 0$, a contradiction which finishes the proof of (i).

Step 5. Now for (ii), by (i) we have

$$\mu_\alpha^{\max}(\mathscr{F} \boxtimes \mathscr{G}) = \mu_\alpha^{\max}(\mathscr{F}) + \mu_\alpha^{\max}(\mathscr{G}) = \mu_\alpha(\mathscr{F}) + \mu_\alpha(\mathscr{G}) = \mu_\alpha(\mathscr{F} \boxtimes \mathscr{G}),$$

which finishes the proof. □

Corollary 2.28. *Let X be a smooth projective variety and let $\alpha \in \operatorname{Mov}(X)$ be a movable class. If $\mathscr{F}$ is a torsion-free coherent sheaf of positive rank on X and q is a positive integer, then:*

(i) $\mu_\alpha^{\max}\big(\operatorname{Sym}^{[q]}\mathscr{F}\big) = q\mu_\alpha^{\max}(\mathscr{F})$ *and* $\mu_\alpha^{\max}\big(\bigwedge^{[q]}\mathscr{F}\big) = q\mu_\alpha^{\max}(\mathscr{F})$,

(ii) if $\mathscr{F}$ is α-semistable, then $\operatorname{Sym}^{[q]}\mathscr{F}$ *and* $\bigwedge^{[q]}\mathscr{F}$ *are α-semistable.*

Proof. Let $\mathscr{A}$ be the maximal destabiliser of $\mathscr{F}$, and let X° be a big open subset of X such that $\mathscr{F}|_{X^\circ}$ and $\mathscr{A}|_{X^\circ}$ are locally free. Then the direct sum decompositions

$$\mathscr{F}|_{X^\circ}^{\otimes q} = \operatorname{Sym}^q \mathscr{F}|_{X^\circ} \oplus \big(\mathscr{F}|_{X^\circ}^{\otimes q} / \operatorname{Sym}^q \mathscr{F}|_{X^\circ}\big) \quad\text{and}\quad \mathscr{F}|_{X^\circ}^{\otimes q} = \bigwedge^q \mathscr{F}|_{X^\circ} \oplus \big(\mathscr{F}|_{X^\circ}^{\otimes q} / \bigwedge^q \mathscr{F}|_{X^\circ}\big)$$

extend to

$$\mathscr{F}^{\boxtimes q} = \mathrm{Sym}^{[q]} \mathscr{F} \oplus \left(\mathscr{F}^{\otimes q} / \mathrm{Sym}^q \mathscr{F}\right)^{**} \tag{2.6}$$

and

$$\mathscr{F}^{\boxtimes q} = \bigwedge^{[q]} \mathscr{F} \oplus \left(\mathscr{F}^{\otimes q} / \bigwedge^q \mathscr{F}\right)^{**}. \tag{2.7}$$

In particular,

$$\mu_\alpha^{\max}\left(\mathrm{Sym}^{[q]} \mathscr{F}\right) \leq \mu_\alpha^{\max}\left(\mathscr{F}^{\boxtimes q}\right) = q\mu_\alpha^{\max}(\mathscr{F})$$

and

$$\mu_\alpha^{\max}\left(\bigwedge^{[q]} \mathscr{F}\right) \leq \mu_\alpha^{\max}\left(\mathscr{F}^{\boxtimes q}\right) = q\mu_\alpha^{\max}(\mathscr{F})$$

by Theorem 2.27(i). On the other hand, note that

$$c_1\left(\mathrm{Sym}^{[q]} \mathscr{A}\right) = \binom{r+q-1}{q-1} c_1(\mathscr{A}) \quad \text{and} \quad c_1\left(\bigwedge^{[q]} \mathscr{A}\right) = \binom{r-1}{q-1} c_1(\mathscr{A}),$$

and

$$\mathrm{rk}\left(\mathrm{Sym}^{[q]} \mathscr{A}\right) = \binom{r+q-1}{q} \quad \text{and} \quad \mathrm{rk}\left(\bigwedge^{[q]} \mathscr{A}\right) = \binom{r}{q}.$$

Since $\mathrm{Sym}^{[q]} \mathscr{A} \subseteq \mathrm{Sym}^{[q]} \mathscr{F}$ and $\bigwedge^{[q]} \mathscr{A} \subseteq \bigwedge^{[q]} \mathscr{F}$, the relations above give

$$q\mu_\alpha^{\max}(\mathscr{F}) = q\mu_\alpha(\mathscr{A}) = \mu_\alpha\left(\mathrm{Sym}^{[q]} \mathscr{A}\right) \leq \mu_\alpha^{\max}\left(\mathrm{Sym}^{[q]} \mathscr{F}\right)$$

and

$$q\mu_\alpha^{\max}(\mathscr{F}) = q\mu_\alpha(\mathscr{A}) = \mu_\alpha\left(\bigwedge^{[q]} \mathscr{A}\right) \leq \mu_\alpha^{\max}\left(\bigwedge^{[q]} \mathscr{F}\right),$$

which shows (i). We deduce (ii) from (2.6) and from Theorem 2.27(ii) and Proposition 2.10(iv). □

Chapter 3

Weak positivity

Corollary 3.1. *Let X be a smooth projective variety and let $f : X \to Z$ be a morphism with connected fibred such that Z is smooth and such that K_F is pseudoeffective for a general fibre F of f. Then the divisor*

$$K_{X/Z} - \mathrm{Ram}(f)$$

is pseudoeffective.

Chapter 4

Foliations

In this chapter we study the main objects of this course: the foliations. The main result is Theorem 4.21, which gives a criterion for when a foliation (which is by definition an analytic object) actually comes from an algebraic construction. We will then use it to give a quick proof of Theorem 1.1, as promised in the introduction.

4.1 Preliminary definitions

We start with a definition.

Definition 4.1. Let X be a smooth variety. A *(singular) foliation* is a saturated subsheaf $\mathscr{F} \subseteq T_X$ which is closed under the Lie bracket, i.e. $[\mathscr{F}, \mathscr{F}] \subseteq \mathscr{F}$. The singularity locus $\mathrm{Sing}(\mathscr{F})$ of $\mathscr{F}$ is the subset of X on which $\mathscr{F}$ is not locally free, and it has codimension at least 2 in X. A *leaf* of $\mathscr{F}$ is the maximal connected, locally closed submanifold $L \subseteq X_{\mathrm{reg}}$ such that $T_L = \mathscr{F}|_L$.

Recall that the Lie bracket is a map $[\cdot,\cdot]\colon T_X \times T_X \to T_X$ such that for any two vector fields – or two local sections – $f_1, f_2 \in T_X$ and a local section $s \in \mathscr{O}_X$ we have $[f_1, f_2](s) = f_1\big(f_2(s)\big) - f_2\big(f_1(s)\big)$.

Remark 4.2. Let X be a smooth variety and let $\mathscr{F} \subseteq T_X$ be a saturated subsheaf. The map $[\cdot,\cdot]\colon \mathscr{F} \times \mathscr{F} \to T_X$ is not $\mathscr{O}_X$-bilinear. Indeed, for two local sections $f_1, f_2 \in \mathscr{F}$ and a local section $s \in \mathscr{O}_X$ we have

$$[sf_1, f_2] = s[f_1, f_2] - f_2(s) f_1 \quad \text{and} \quad [f_1, sf_2] = s[f_1, f_2] + f_1(s) f_2.$$

This implies that the induced map $[\cdot,\cdot]\colon \mathscr{F} \times \mathscr{F} \to T_X/\mathscr{F}$ is $\mathscr{O}_X$-bilinear and anti-commutative, hence induces an $\mathscr{O}_X$-linear map $\bigwedge^2 \mathscr{F} \to T_X/\mathscr{F}$, hence an $\mathscr{O}_X$-linear map $\big(\bigwedge^2 \mathscr{F}\big)/(\text{torsion}) \to T_X/\mathscr{F}$. Therefore, $\mathscr{F}$ is a foliation if and only if this map is the zero map.

The remark allows us to produce a general example of a foliation.

Lemma 4.3. *Let X be a smooth variety, let $\alpha \in \operatorname{Mov}(X)$ and let $\mathscr{F} \subseteq T_X$ be the maximal destabiliser with respect to α. If $\mu_\alpha(\mathscr{F}) > 0$, then $\mathscr{F}$ is a foliation.*

Proof. Since $\mathscr{F}$ is reflexive, Proposition 2.9(iii) and Remark 2.7 give

$$\mu^{\min}(\mathscr{F}) = -\mu^{\max}(\mathscr{F}^*) \quad \text{and} \quad \mu_\alpha^{\min}\Big(\bigwedge^{[2]}\mathscr{F}\Big) = -\mu_\alpha^{\max}\Big(\bigwedge^{[2]}\mathscr{F}^*\Big). \tag{4.1}$$

By Remark 2.7, by Proposition 2.10(v) we have

$$\begin{aligned}\mu_\alpha^{\min}\Big(\big(\bigwedge^2\mathscr{F}\big)/(\text{torsion})\Big) &= \mu_\alpha^{\min}\Big(\bigwedge^{[2]}\mathscr{F}\Big) = -\mu_\alpha^{\max}\Big(\bigwedge^{[2]}\mathscr{F}^*\Big) = -2\mu^{\max}(\mathscr{F}^*)\\ &= 2\mu_\alpha^{\min}(\mathscr{F}) = 2\mu_\alpha(\mathscr{F}) > \mu_\alpha(\mathscr{F}) \geq \mu_\alpha^{\max}(T_X/\mathscr{F}).\end{aligned}$$

Therefore, any morphism $\big(\wedge^2\mathscr{F}\big)/(\text{torsion}) \to T_X/\mathscr{F}$ is the zero map by Proposition 2.10(ii), and the conclusion follows from Remark 4.2. □

A central result is the following theorem of Frobenius from differential geometry, which is beyond the scope of this course.

Theorem 4.4. *Let X be a smooth variety of dimension n and let $\mathscr{F} \subseteq T_X$ be a singular foliation of rank r. Then for every point $x \in X \setminus \operatorname{Sing}(\mathscr{F})$ there exists an analytic neighbourhood $U \simeq \mathbb{C}^r \times \mathbb{C}^{n-r}$ of x in $X \setminus \operatorname{Sing}(\mathscr{F})$ such that the vectors $p_1^*(\partial/\partial x_i)$ form a basis of $\mathscr{F}|_U$, where $x_1, \ldots, x_r$ are the local coordinates on $\mathbb{C}^r$ and $p_1 \colon U \to \mathbb{C}^r$ is the first projection.*

Note that, in the context of this theorem, the sheaf $\mathscr{F}|_U$ can be identified with the relative tangent bundle of the smooth fibration $p_2 \colon U \to \mathbb{C}^q$, where $q = n - r$. The integer q is the *codimension of $\mathscr{F}$*.

The theorem of Frobenius shows that through every point $x \in X \setminus \operatorname{Sing}(\mathscr{F})$ there exists a neighbourhood U of x and a submanifold $N \subseteq U$ such that $\mathscr{F}|_U$ is identified with the tangent bundle T_N: this is just (locally) the fibre of the projection p_2 above containing x. The manifold N is locally a *leaf* passing through x. It is clear that this construction glues, which motivates the following definition.

Definition 4.5. Let X be a smooth variety of dimension n and let $\mathscr{F} \subseteq T_X$ be a singular foliation of rank r. A *leaf* of $\mathscr{F}$ is the maximal connected, locally closed submanifold $L \subseteq X \setminus \operatorname{Sing}(\mathscr{F})$ such that $T_L = \mathscr{F}|_L$.

Example 4.6. To have an idea of what is going on here, it is instructive to consider real manifolds. For instance, let $T = \mathbb{R}^2/\mathbb{Z}^2$ be the torus and consider the one-dimensional foliation on T given by parallel lines. If their slope is rational, then each leaf is diffeomorphic to S^1. If the slope is irrational, then each leaf is diffeomorphic to $\mathbb{R}$ and moreover standard results from diophantine approximation show that the leafs are all dense in T. Here is to note that when the slope is irrational, the subspace topology induced from T does not agree with the topology that makes the family of lines a manifold.

Using Frobenius' theorem, we can construct an *analytic graph* of a foliation:

Lemma 4.7. *Let $\mathscr{F}$ be a foliation of rank r on a smooth variety X of dimension n and denote $X^\circ = X \setminus \operatorname{Sing}(\mathscr{F})$. Let $\Delta \subseteq X^\circ \times X^\circ$ be the diagonal, and let p_1 and p_2 be the projections of $X^\circ \times X^\circ$ onto the factors.*

Then there exists a smooth locally closed analytic submanifold $V \subseteq X^\circ \times X^\circ$ containing Δ such that $p_2|_V$ is smooth and such that its fibres are analytic open subsets of the leaves of the foliation $p_1^*\mathscr{F}|_{X^\circ}$ passing through points of Δ. Moreover, $N_{\Delta/V} \simeq \mathscr{F}|_{X^\circ}$. The analytic germ of V along Δ is the analytic graph *of the foliation $\mathscr{F}$.*

Proof. Applying the theorem of Frobenius to the smooth foliation $p_1^*\mathscr{F}|_{X^\circ} \subseteq p_1^*T_{X^\circ} \subseteq p_1^*T_{X^\circ} \oplus p_2^*T_{X^\circ} = T_{X^\circ \times X^\circ}$, for every point $x \in \Delta$ there exists an open neighbourhood $U \simeq \mathbb{C}^r \times \mathbb{C}^{n-r} \times W$ for some $W \simeq \mathbb{C}^r \times \mathbb{C}^{n-r} \subseteq X^\circ$ such that $p_1^*\mathscr{F}|_U$ is spanned by the vectors $\partial/\partial x_i$, where x_i are the coordinates on $\mathbb{C}^r$. Consider the set

$$V|_U = \left\{(x, y, z, y) \in U \mid x \in \mathbb{C}^r, (z, y) \in W\right\} \simeq \mathbb{C}^r \times \Delta|_U.$$

It is clear that these sets glue to give a locally closed manifold V such that $N_{\Delta/V} \simeq \mathscr{F}|_{X^\circ}$. □

4.2 Algebraic integrability of foliations

Definition 4.8. Let $\mathscr{F}$ be a foliation of rank r on a smooth variety X of dimension n. A leaf L of $\mathscr{F}$ is *algebraic* if it is open in its Zariski closure $\overline{L}^{\mathrm{Zar}}$, and if $\dim L = \dim \overline{L}^{\mathrm{Zar}}$. A foliation $\mathscr{F}$ on X is *algebraically integrable* if every leaf passing through a general point of X is algebraic.

Lemma 4.9. *Let $\mathscr{F}$ be a foliation of rank r on a smooth variety X of dimension n, and let $V \subseteq X \times X$ be any locally closed analytic manifold whose germ is an analytic graph of $\mathscr{F}$. If $\dim \overline{V}^{\mathrm{Zar}} = n + r$, then $\mathscr{F}$ is algebraically integrable.*

Proof. Consider the projection $\pi = p_2|_{\overline{V}^{\mathrm{Zar}}} : \overline{V}^{\mathrm{Zar}} \to X$. Since $\Delta \subseteq \overline{V}^{\mathrm{Zar}}$, the morphism π is surjective, and the general fibre of π has dimension $\dim \overline{V}^{\mathrm{Zar}} - n$. If $\dim \overline{V}^{\mathrm{Zar}} = n + r$, let $X_0 \subseteq X$ be the set such that each fibre over a point in X_0 has dimension r. In particular, for each leaf L passing through a point in $\pi^{-1}(X_0)$, the Zariski closure $\overline{L}$ of $L \cap \pi^{-1}(X_0)$ has dimension r. Since there are uncountably many such cycles

$\overline{L}$ and Chow(X) has countably many irreducible components, there exists a closed subvariety W of Chow(X) such that the universal cycle over W dominates X, and the subset of points in W parametrizing such $\overline{L}$ (viewed as reduced and irreducible cycles in X) is Zariski dense in W. Let $U \subseteq W \times X$ be the universal cycle over W. Then as in the proof of Lemma 4.12 below, one can show that $\overline{L} \cap (X \setminus \mathrm{Sing}(\mathscr{F}))$ is a leaf of $\mathscr{F}$ for all such $\overline{L}$, and the conclusion follows. □

Now we come to the first major result of this section [BM01, CP15], which shows that in a favourable situation a foliation is automatically algebraically integrable.

Theorem 4.10. *Let X be a smooth projective variety of dimension n and let $\mathscr{F} \subseteq T_X$ be a foliation of rank r. Assume that there exists $\alpha \in \mathrm{Mov}(X)$ such that $\mu_\alpha^{\min}(\mathscr{F}) > 0$. Then $\mathscr{F}$ is algebraically integrable.*

Proof. Let $X^\circ = X \setminus \mathrm{Sing}(\mathscr{F})$, and note that X° is a big open subset of X. Let $V \subseteq X^\circ \times X^\circ$ be a locally closed analytic manifold as in Lemma 4.7 containing the diagonal Δ. By Lemma 4.9, it suffices to show that $\dim \overline{V}^{\mathrm{Zar}} = n+r$. Since clearly $\dim \overline{V}^{\mathrm{Zar}} \geq n+r$, it suffices to prove the converse inequality.

To this end, fix an ample line bundle $\mathscr{L}$ on $X \times X$, and let L be the restriction of $\mathscr{L}$ to the diagonal in $X \times X$. It suffices to show that there exists a constant $C > 0$ such that

$$h^0\left(\overline{V}^{\mathrm{Zar}}, \mathscr{L}^{\otimes k}\right) \leq Ck^{n+r} \quad \text{for all } k \geq 0.$$

Note that the restriction of (holomorphic) sections gives the inclusion

$$H^0\left(\overline{V}^{\mathrm{Zar}}, \mathscr{L}^{\otimes k}\right) \to H^0(V, \mathscr{L}^{\otimes k}).$$

Indeed, if a section in $H^0\left(\overline{V}^{\mathrm{Zar}}, \mathscr{L}^{\otimes k}\right)$ vanishes on V, then it vanishes on $\overline{V}^{\mathrm{Zar}}$, since these sections are algebraic by Serre's GAGA theorems. Hence, it is enough to prove that there exists a constant $C > 0$ such that

$$h^0(V, \mathscr{L}^{\otimes k}) \leq Ck^{n+r} \quad \text{for all } k \geq 0. \tag{4.2}$$

If $\mathscr{I}$ is the ideal of Δ in V, then for all non-negative integers m and k we have

$$0 \to \mathscr{L}^{\otimes k}|_V \otimes \mathscr{I}^{m+1} \to \mathscr{L}^{\otimes k}|_V \otimes \mathscr{I}^m \to L^{\otimes k}|_\Delta \otimes \mathscr{I}^m/\mathscr{I}^{m+1} \to 0.$$

Since $\mathscr{I}^m/\mathscr{I}^{m+1} = \mathrm{Sym}^m(\mathscr{I}/\mathscr{I}^2) = \mathrm{Sym}^m N^*_{\Delta/V} \simeq \mathrm{Sym}^m \mathscr{F}^*|_{X^\circ}$ by Lemma 4.7, and as $\Delta \simeq X^\circ$, we obtain

$$h^0(V, \mathscr{L}^{\otimes k}) \leq \sum_{m \geq 0} h^0\big(X^\circ, L^{\otimes k} \otimes \mathrm{Sym}^m \mathscr{F}^*|_{X^\circ}\big) = \sum_{m \geq 0} h^0\big(X, L^{\otimes k} \otimes \mathrm{Sym}^{[m]} \mathscr{F}^*\big), \tag{4.3}$$

where the last equality follows by the analytic analogue of Proposition 2.3, see [Kob87, Proposition V.5.21]. Then by (4.2) it suffices to show that there exists a constant $C>0$ such that

$$\sum_{m\geq 0} h^0\big(X, L^{\otimes k}\otimes \operatorname{Sym}^{[m]}\mathscr{F}^*\big) \leq C k^{n+r} \quad \text{for all } k\geq 0. \tag{4.4}$$

Denote $N = \left\lceil \frac{L\cdot\alpha}{\mu_\alpha^{\min}(\mathscr{F})}\right\rceil$. Since

$$\mu_\alpha^{\max}\big(L^{\otimes k}\otimes \operatorname{Sym}^{[m]}\mathscr{F}^*\big) = kL\cdot\alpha - m\mu_\alpha^{\min}(\mathscr{F})$$

by Theorem 2.27(i), Corollary 2.28(i) and Proposition 2.9(iii), for $m>kN$ we have

$$H^0\big(X, L^{\otimes k}\otimes \operatorname{Sym}^{[m]}\mathscr{F}^*\big) \simeq \operatorname{Hom}\big(\mathscr{O}_X, L^{\otimes k}\otimes \operatorname{Sym}^{[m]}\mathscr{F}^*\big) = 0, \tag{4.5}$$

Proposition 2.10(ii), hence the sum on the left hand side of (4.4) is finite.

Consider the natural projection $p\colon \operatorname{Proj}_X(\operatorname{Sym}\mathscr{F}^*)\to X$. Let Y' be the normalisation of the irreducible component of $\operatorname{Proj}_X(\operatorname{Sym}\mathscr{F}^*)$ which contains $p^{-1}(X^\circ)$, and let $Y\to Y'$ be a resolution of singularities. Let D be a pullback of $\mathscr{O}_{\operatorname{Proj}_X(\operatorname{Sym}\mathscr{F}^*)}(1)$ to Y and denote by $\pi\colon Y\to X$ denote the composite morphism. Then by [Nak04, Lemma III.5.10(c)] there exists a divisor M on Y such that for every positive integer m we have

$$(\pi_*\mathscr{O}_Y(mD))^{**}\simeq \pi_*\mathscr{O}_Y(mM).$$

Since $\pi_*\mathscr{O}_Y(mD)|_{X^\circ}\simeq \operatorname{Sym}^m\mathscr{F}^*|_{X^\circ}$, this implies $\operatorname{Sym}^{[m]}\mathscr{F}^*\simeq \pi_*\mathscr{O}_Y(mM)$ for all positive integers m, hence

$$H^0\big(X, L^{\otimes k}\otimes \operatorname{Sym}^{[m]}\mathscr{F}^*\big)\simeq H^0\big(Y, \pi^*L^{\otimes k}\otimes \mathscr{O}_Y(mM)\big). \tag{4.6}$$

Pick an ample line bundle A on Y such that $A\otimes \pi^*L^{-1}$ and $A\otimes\mathscr{O}_Y(-M)$ are ample. Then by (4.5) and (4.6) we have

$$\begin{aligned}\sum_{m\geq 0} h^0\big(X, L^{\otimes k}\otimes \operatorname{Sym}^{[m]}\mathscr{F}^*\big) &\leq \sum_{m=0}^{kN} h^0\big(Y, \pi^*L^{\otimes k}\otimes\mathscr{O}_Y(mM)\big)\\ &\leq (kN+1)h^0\big(Y, A^{\otimes k(N+1)}\big)\leq (kN+1)C'k^{n+r-1},\end{aligned}$$

where C' is a positive constant. This implies (4.4). □

The main example of algebraically integrable foliations is contained in the following definition.

Definition 4.11. Let $f\colon X\to Y$ be a morphism of normal projective varieties. Then the kernel of the differential $df\colon T_X\to f^*T_Y$ defines a foliation $\mathscr{F}$ on X, and we say that $\mathscr{F}$ is *induced by* f. It is clear that this foliation is algebraically integrable.

The following result proves a kind of a converse:

Lemma 4.12. *Let X be a smooth projective variety and let $\mathscr{F}$ an algebraically integrable foliation on X. Then there is a unique irreducible closed subvariety W of* Chow(X) *whose general point parametrizes the closure of a general leaf of $\mathscr{F}$ (viewed as a reduced and irreducible cycle in X). In other words, if $U \subseteq W \times X$ is the universal cycle with projections $\pi\colon U \to W$ and $e\colon U \to X$, then e is birational and $e\big(\pi^{-1}(w)\big) \subseteq X$ is the closure of a leaf of $\mathscr{F}$ for a general point $w \in W$.*

$$\begin{array}{ccc} U & \xrightarrow{\ \pi\ } & W \\ {\scriptstyle e}\big\downarrow & & \\ X & & \end{array}$$

Then there exists a foliation $\widehat{\mathscr{F}}$ on the normalisation $\nu\colon U^\nu \to U$ induced by $\pi\circ\nu$ and which coincides with $\mathscr{F}$ on $(e\circ\nu)^{-1}(X^\circ)$, where X° is a big open subset of X.

Proof. Since $\mathscr{F}$ has uncountably many leaves and Chow(X) has countably many irreducible components, there exists a closed subvariety W of Chow(X) such that:

(a) the universal cycle over W dominates X, and

(b) the subset W' of points in W parametrizing leaves of $\mathscr{F}$ (viewed as reduced and irreducible cycles in X) is Zariski dense in W.

Let $U \subseteq W \times X$ be the universal cycle over W, denote by $p\colon W\times X \to W$ and $q\colon W \times X \to X$ the projections, and set $\pi = p|_U$ and $e = q|_U$. It is clear that e is birational.

Let $X^\circ \subseteq X$ be a big open subset such that $\mathscr{F}|_{X^\circ}$ and $(T_X/\mathscr{F})|_{X^\circ}$ are locally free, and denote $W^\circ \subseteq \pi\big(e^{-1}(X^\circ)\big)$. We claim that for every point $w \in W^\circ$, $L_w := e\big(\pi^{-1}(w)\big) \subseteq X$ is the closure of a leaf of $\mathscr{F}$; by assumption, this holds on a dense subset of W°. If r is the rank of $\mathscr{F}$, the map $\eta_X\colon \Omega^1_X \to \mathscr{F}^*$ induces a map

$$\widetilde{\eta}_U\colon \Omega^r_{W\times X}\big|_U = \bigwedge^r \big(p^*\Omega^1_W \oplus q^*\Omega^1_X\big)\big|_U \to e^*\Omega^r_X \to e^*\det\mathscr{F}^*.$$

Since η_X is surjective on X°, the map $\bigwedge^r \eta_X$ is surjective on X°. Therefore, if K is the kernel of the natural surjective map $\theta\colon \Omega^r_{W\times X}\big|_U \twoheadrightarrow \Omega^r_U$, then the composite map $K \to \Omega^r_{W\times X}\big|_U \to e^*\det\mathscr{F}^*$ vanishes on a dense subset $\pi^{-1}(W')$ of U: indeed, every element of the dual K^* represents normal vectors to U inside $W\times X$, and in particular is orthogonal to each $\pi^{-1}(w)$ for $w \in W'$; since $\pi^{-1}(w)$ is a closure of a leaf of $\mathscr{F}$, the claim follows by the construction of $\widetilde{\eta}_U$. Since $e^*\mathscr{F}^*$ is a line bundle, hence torsion-free, this implies that this last map vanishes everywhere, hence we obtain a factorisation

$$\begin{array}{ccc} \Omega^r_{W\times X}\big|_U & \xrightarrow{\ \widetilde{\eta}_U\ } & e^*\det\mathscr{F}^* \\ {\scriptstyle \theta}\big\downarrow & \nearrow{\scriptstyle \widehat{\eta}_U} & \\ \Omega^r_U. & & \end{array}$$

Analogously, by using that the restriction of $\mathscr{F}$ to a leaf is the tangent bundle of the leaf, we have a factorisation

$$\begin{array}{ccc} \Omega^r_U & \xrightarrow{\widehat{\eta}_U} & e^*\det\mathscr{F}^* \\ \downarrow & \nearrow_{\eta_U} & \\ \Omega^r_{U/W}, & & \end{array} \tag{4.7}$$

and note that η_U is an isomorphism on a dense subset of U. Fix a point $w \in W^\circ$, let $x \in L_w \cap X^\circ$, and let $v_1,\dots,v_r$ be the local generators of $\mathscr{F}$ on an affine neighbourhood $V \subseteq X^\circ$ of x. Then it suffices to show that $v_i(x) \in T_{L_w,x}$ for all $i \in \{1,\dots,r\}$. Observe that $\eta_U|_V : \Omega^r_V \to (e^*\det\mathscr{F}^*)|_V$ is given by

$$\begin{array}{ccc} H^0(V,\Omega^r_V) & \longrightarrow & H^0(V,(e^*\det\mathscr{F}^*)|_V) \\ \alpha & \longmapsto & \alpha(v_1,\dots,v_r)\omega, \end{array}$$

where $\omega \in H^0(V,(e^*\det\mathscr{F}^*)|_V)$ is such that $\omega(v_1,\dots,v_r)=1$. Let f be a local function on V vanishing on $L_w \cap V$, and let β be any local $(r-1)$-differential form on V. Then be restricting (4.7) to $L_w \cap V$ we obtain $(df \wedge \beta)(v_1,\dots,v_r)=0$, hence $df(v_i)=0$ for any $i \in \{1,\dots,r\}$. But this yields $v_i(x) \in T_{L_w,x}$ for all $i \in \{1,\dots,r\}$, which was to be proved.

The last claim is clear from the construction. □

We will need the following lemma later.

Lemma 4.13. *Let $f : X \to Y$ be a morphism with connected fibres between normal projective varieties and let $\mathscr{F}$ be the foliation on X induced by f. Assume $\mathscr{G}$ is another foliation on X such that $\mathscr{F} \subseteq \mathscr{G}$. Then for a general point $y \in Y$ there exists an analytic open neighbourhood V of y in Y and an exact sequence*

$$0 \to \mathscr{F}|_U \to \mathscr{G}|_U \to \mathscr{O}_U^{\oplus q} \to 0$$

for some positive integer q, where $U = f^{-1}(V)$.

Proof. Consider an open subset $Y^\circ \subseteq Y$ such that the morphism f is smooth over Y°, consider a point $x \in f^{-1}(Y^\circ)$ which belongs to the regular locus of both $\mathscr{F}$ and $\mathscr{G}$ and let $y = f(x)$. Let F_y be the fibre of f over y. By the theorem of Frobenius, there exists an open neighbourhood U of x in X such that $\mathscr{G}|_U$ can be identified with the kernel of the differential of the projection map $\pi : \mathbb{C}^{\mathrm{rk}\mathscr{G}} \times \mathbb{C}^{\dim X - \mathrm{rk}\mathscr{G}} \to \mathbb{C}^{\dim X - \mathrm{rk}\mathscr{G}}$. Since $\mathscr{F} \subseteq \mathscr{G}$, the fibre G_x of π containing x is covered by open subsets of the fibres of f (i.e. by open subsets of the leaves of $\mathscr{F}$ in a neighbourhood U of x). In other words, we may write $\mathbb{C}^{\mathrm{rk}\mathscr{G}} = \mathbb{C}^{\mathrm{rk}\mathscr{F}} \times \mathbb{C}^{\mathrm{rk}\mathscr{G}-\mathrm{rk}\mathscr{F}}$ such that the fibres of the map $\pi|_{\mathbb{C}^{\mathrm{rk}\mathscr{G}-\mathrm{rk}\mathscr{F}}}$ are open subsets of the leaves of $\mathscr{F}$. Hence there exists a submanifold $N_x \simeq \mathbb{C}^{\mathrm{rk}\mathscr{G}-\mathrm{rk}\mathscr{F}}$ of G_x which contains x and is transversal to $F_x \cap U$. The images of all these manifolds in Y define a foliation $\mathscr{H}$ on the open neighbourhood $f(U)$ of y in Y: indeed, this is the subsheaf generated by the tangent vectors to $f(N_x)$ for every $x \in U$. By possibly shrinking U, we may assume that $U = f^{-1}(f(U))$ and that $\mathscr{H} \simeq \mathscr{O}_{f(U)}^{\oplus q}$ for some positive integer q. It is clear that this implies the statement. □

4.3 Canonical bundle of a foliation

Let X be a smooth projective variety and let $\mathscr{F} \subseteq T_X$ be a foliation. Then the *canonical class* of $\mathscr{F}$ is any Cartier divisor $K_{\mathscr{F}}$ on X such that $\mathscr{O}_X(K_{\mathscr{F}}) := \det \mathscr{F}^*$; we already saw this sheaf in action in the proof of Lemma 4.12. The sheaf $N_{\mathscr{F}} := (T_X/\mathscr{F})^{**}$ is the *normal sheaf* of $\mathscr{F}$ and is of rank q. On the big open subset $X^\circ := X \setminus \mathrm{Sing}(\mathscr{F}) \subseteq X$ we have the short exact sequence

$$0 \to \mathscr{F}|_{X^\circ} \to T_{X^\circ} \to N_{\mathscr{F}}|_{X^\circ} \to 0,$$

hence

$$\mathscr{O}_X(K_{\mathscr{F}}) \simeq \mathscr{O}_X(K_X) \otimes \det N_{\mathscr{F}}.$$

Definition 4.14. Let $\pi\colon X \to Y$ be a morphism between smooth varieties or an equidimensional morphism of normal varieties. The *ramification divisor of* π is

$$\mathrm{Ram}(\pi) := \sum_{D \subseteq Y} \big(\pi^*D - (\pi^*D)_{\mathrm{red}}\big),$$

where D runs through all the prime divisors on Y. The set $\pi\big(\mathrm{Ram}(\pi)\big)$ is the *branch locus of* π.

Note that if $\pi\colon X \to Y$ is an equidimensional morphism of normal varieties, and if D is a Weil $\mathbb{Q}$-divisor on Y, then the pullback π^*D is defined as the unique $\mathbb{Q}$-divisor on X whose restriction to $\pi^{-1}(Y_{\mathrm{reg}})$ is $\big(\pi|_{\pi^{-1}(Y_{\mathrm{reg}})}\big)^*\big(D|_{Y_{\mathrm{reg}}}\big)$.

Lemma 4.15. *Let $f: X \to Y$ be an equidimensional dominant morphism of smooth varieties, and let $\mathscr{F}$ be a foliation on X induced by f. Then*

$$K_{\mathscr{F}} \sim_{\mathbb{Q}} K_{X/Y} - \mathrm{Ram}(f).$$

Proof. By removing a codimension 2 subset of Y, we may assume that the branch locus of f is the disjoint union of prime divisors in Y; note that this does not affect the result. We have the exact sequence

$$0 \to \mathscr{F} \to T_X \to \mathscr{Q} \to 0,$$

where $\mathscr{Q} \subseteq f^*T_Y$. Let D be a prime divisor on Y such that $f^*D = kD'$ for a prime divisor D', and fix points $P \in D'$ and $Q \in D$ such that $f(P) = Q$. We may choose local coordinates $x_1, \ldots, x_n$ around P and $y_1, \ldots, y_m$ around Q such that $y_i = x_i$ for $i = 1, \ldots, m-1$, and $y_m = ux_m^k$, where u is a unit in $\mathscr{O}_{X,P}$ and k is a positive integer; here x_m and y_m are the local equations of D' and D, respectively. Then from the definition of the differential map between tangent spaces we obtain that $\mathscr{Q}$ is generated by the vectors

$$\frac{\partial}{\partial y_1}, \ldots, \frac{\partial}{\partial y_{m-1}}, x_m^{k-1}\frac{\partial}{\partial y_m},$$

hence $\det \mathscr{Q} = (f^*\omega_Y)^* \otimes \mathscr{O}_X(-(k-1)D')$. The conclusion easily follows from the exact sequence above. $\square$

Recall that if we have a morphism $f\colon X \to Y$ between varieties, a divisor E on X is *f-exceptional* if $\operatorname{codim}_Y f(E) \geq 2$.

Lemma 4.16. *Let $f : X \to Y$ be a morphism of projective varieties. Then there exists a birational morphism $\tau\colon Y' \to Y$ from a smooth variety Y' and a resolution X' of the main component of $X \times_Y Y'$ so that we have the induced diagram*

$$\begin{array}{ccc} X' & \xrightarrow{f'} & Y' \\ {\scriptstyle \tau'}\downarrow & & \downarrow{\scriptstyle \tau} \\ X & \xrightarrow{f} & Y \end{array}$$

and such that every f'-exceptional divisor is also τ'-exceptional.

Proof. By Hironaka's resolution of singularities and by [Ray72] there exists a birational morphism $\tau\colon Y' \to Y$ from a smooth variety such that, if X'' is the main component of $X \times_Y Y'$ so that we have the induced diagram

$$\begin{array}{ccc} X'' & \xrightarrow{f''} & Y' \\ {\scriptstyle \tau''}\downarrow & & \downarrow{\scriptstyle \tau} \\ X & \xrightarrow{f} & Y, \end{array}$$

then f'' is flat, and in particular equidimensional. Letting X' be any resolution of X'', the result follows. $\square$

With notation from Lemma 4.12, note that the proof shows that X and U are isomorphic on $X^\circ = X \setminus \operatorname{Sing}(\mathscr{F})$. By blowing up normalisations of U and W, there exists a modification $\varphi\colon \widehat{X} \to X$ and a fibration $f : \widehat{X} \to Z$ with $\widehat{X}$ and Z smooth, and the foliation $\widehat{\mathscr{F}}$ induced by f. The foliation $\widehat{\mathscr{F}}$ agrees with $\mathscr{F}$ on $\varphi^{-1}(X^\circ)$, and we call $\widehat{\mathscr{F}}$ the *pullback of $\mathscr{F}$ to $\widehat{X}$*. This shows that

$$\pi_* K_{\widehat{\mathscr{F}}} = K_{\mathscr{F}} \quad \text{and} \quad K_{\widehat{\mathscr{F}}} - \pi^* K_{\mathscr{F}} \text{ is a } \pi\text{-exceptional divisor.} \tag{4.8}$$

Then we have:

Proposition 4.17. *Let X be a smooth projective variety and let $\mathscr{F}$ be an algebraically integrable foliation on X. Then there exists a modification $\varphi\colon \widehat{X} \to X$ and a morphism with connected fibres $f\colon \widehat{X} \to Z$ with $\widehat{X}$ and Z smooth, such that all f-exceptional divisors are φ-exceptional, and we have*

$$K_{\mathscr{F}} \sim_{\mathbb{Q}} \varphi_*\big(K_{\widehat{X}/Z} - \mathrm{Ram}(f)\big).$$

Proof. The proof follows by combining (the proof of) Lemma 4.16, Lemma 4.15 and (4.8). □

4.4 MRC fibration

We say that a smooth variety is *rationally connected* if every two general points can be connected by a rational curve; Fano manifolds are examples of rationally connected varieties by [KMM92]. Here we need an important result of [Cam92, KMM92], which says that it is possible to take a quotient of a variety by an equivalence relation, in which each two general points can be connected by a chain of rational curves.

Theorem 4.18. *Let X be a smooth projective variety. Then there exists a dominant rational map $\pi\colon X \dashrightarrow Z$ to a projective variety Z and an open subset $X' \subseteq X$ such that the following holds:*

(a) there exists an open subset $Z' \subseteq Z$ such that the induced map $\pi|_{X'}\colon X' \to Z'$ is a proper morphism,

(b) a general fibre of π is irreducible and rationally connected,

(c) all rational curves which meet a general fibre F of π are contained in F.

Any such a map π is called a maximal rationally connected fibration *or* MRC fibration*, and is unique up to birational equivalence.*

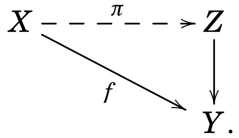
The name of these maps comes from the following universal property: if $\widehat{\pi}\colon X \dashrightarrow \widehat{Z}$ is another *rationally connected fibration* (i.e. a map satisfying (a) and (b) in the definition above), then there exists a map $\zeta\colon \widehat{Z} \dashrightarrow Z$ such that $\pi = \zeta \circ \widehat{\pi}$.

An important property of MRC fibrations is contained in the following main result of [GHS03], combined with the main result of [BDPP13]:

Theorem 4.19. *Let $\pi\colon X \dashrightarrow Z$ an MRC fibration. Then K_Z is pseudoeffective.*

The following is a refinement of Theorem 4.18 in the relative setting, which follows from [Kol96, Theorem 5.9] by using Chow's lemma and resolution of singularities.

Theorem 4.20. *Let $f: X \to Y$ be a morphism between smooth projective varieties. Then there exists a dominant rational map $\pi: X \dashrightarrow Z$ to a smooth projective variety Z over Y such that for a general point $y \in Y$, the induced map $\pi_y: X_y \dashrightarrow Z_y$ is an MRC fibration of X_y, where $X_y \subseteq X$ and $Z_y \subseteq Z$ are the fibres over y.*



Any such a map π is called a relative maximal rationally connected fibration *or* relative MRC fibration *of f.*

4.5 Rational connectedness of a foliation

The following is the main result of this section [CP15].

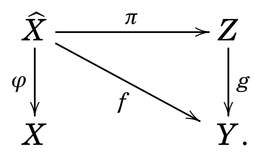
Theorem 4.21. *Let X be a smooth projective variety and let $\mathscr{F} \subseteq T_X$ be a foliation. If there exists $\alpha \in \mathrm{Mov}(X)$ such that $\mu_\alpha^{\min}(\mathscr{F}) > 0$, then $\mathscr{F}$ is an algebraically integrable foliation with rationally connected leaves.*

Proof. Algebraic integrability was shown in Theorem 4.10, hence it remains to show the rational connectedness of the leaves of $\mathscr{F}$.

Consider a birational model $\varphi: \widehat{X} \to X$ on which the foliation $\widehat{\mathscr{F}}$ is induced by a fibration $f: \widehat{X} \to Y$ to a smooth variety Y. In particular, by the proof of Lemma 2.25 we have

$$\mu_{\varphi^*\alpha}^{\min}\big(\widehat{\mathscr{F}}\big) = \mu_\alpha^{\min}(\mathscr{F}) > 0. \tag{4.9}$$

Assume that a general fibre of f is not rationally connected. By Theorem 4.20, there exists a relative MRC fibration of f:



Possibly blowing up $\widehat{X}$ and Z further, by Lemma 4.16 we may assume that:

(a) π is a morphism,

(b) all f-exceptional divisors are φ-exceptional,

(c) all π-exceptional divisors are φ-exceptional, and

(d) the map π is flat over a big open subset $Z^\circ \subseteq Z$.

We have the exact sequence

$$0 \to \widehat{\mathscr{F}} \to T_{\widehat{X}} \xrightarrow{df} f^*T_Y.$$

Let $\mathscr{G}$ be the foliation induced by g, so that we have the exact sequence

$$0 \to \mathscr{G} \to T_Z \xrightarrow{dg} g^*T_Y.$$

Note that $df = (\pi^* dg) \circ d\pi$, and let $\mathscr{G}' \subseteq \pi^* T_Z$ be the image of $\widehat{\mathscr{F}}$ under $d\pi$, so that by (4.9) we have

$$\mu_{\varphi^*\alpha}(\mathscr{G}') > 0. \tag{4.10}$$

Since $(\pi^* dg)(\mathscr{G}') = 0$, by (d) the sheaves $\mathscr{G}'$ and $\pi^*\mathscr{G}$ coincide over $\pi^{-1}(Z^\circ)$. In particular, $c_1(\mathscr{G}')$ and $c_1(\pi^*\mathscr{G})$ coincide away from an exceptional set of φ by (c), which together with (4.10) gives

$$\mu_{\varphi^*\alpha}(\pi^*\mathscr{G}) > 0. \tag{4.11}$$

Note that $\det(\pi^*\mathscr{G})$ and $\pi^*\mathscr{O}_Z(-K_{\mathscr{G}})$ coincide away from an exceptional set of π, hence they coincide away from an exceptional set of φ by (c), so that

$$\mu_{\varphi^*\alpha}(\pi^*\mathscr{G}) = -\frac{1}{\operatorname{rk}\mathscr{G}}\pi^* K_{\mathscr{G}} \cdot \varphi^*\alpha. \tag{4.12}$$

Now, the same proof as that of Lemma 4.15 shows that

$$K_{\mathscr{G}} = K_{Z/Y} - \operatorname{Ram}(g) + E,$$

where the divisor E is g-exceptional, thus π^*E is φ-exceptional by (b). For a general fibre F of g the divisor K_F is pseudoeffective by Theorem 4.19, hence the divisor $K_{Z/Y} - \operatorname{Ram}(g)$ is pseudoeffective by Corollary 3.1. Therefore,

$$\pi^* K_{\mathscr{G}} \cdot \varphi^*\alpha = \pi^*\big(K_{Z/Y} - \operatorname{Ram}(g) + E\big) \cdot \varphi^*\alpha = \pi^*\big(K_{Z/Y} - \operatorname{Ram}(g)\big) \cdot \varphi^*\alpha \geq 0,$$

which contradicts (4.11) and (4.12), and finishes the proof. □

Theorem 4.21 allows the following characterisation of uniruled varieties.

Corollary 4.22. *A smooth projective variety X is uniruled if and only if there exists $\alpha \in \operatorname{Mov}(X)$ such that $\mu_\alpha^{\max}(T_X) > 0$.*

Proof. If there exists a movable class α such that $\mu_\alpha^{\max}(T_X) > 0$, consider the maximal destabiliser $\mathscr{F}$ of T_X for α. Then $\mathscr{F}$ is a foliation with $\mu_\alpha^{\min}(\mathscr{F}) > 0$ by Lemma 4.3 and Proposition 2.10(v), and Theorem 4.21 then implies that X is uniruled.

Conversely, if X is uniruled, then K_X is not pseudoeffective by [BDPP13], hence there exists $\alpha \in \operatorname{Mov}(X)$ such that

$$\mu_\alpha(T_X) = -\frac{1}{\dim X} K_X \cdot \alpha > 0.$$

This finishes the proof. □

Finally, we obtain a generalisation of our main Theorem 1.1.

Theorem 4.23. *Let X be a smooth projective variety, let $\mathscr{F}$ be a foliation with $K_{\mathscr{F}}$ pseudoeffective and let m be a positive integer. Then every quotient of $(\mathscr{F}^*)^{\boxtimes m}$ has a pseudoeffective determinant. In particular, if K_X is pseudoeffective, every quotient of $(\Omega_X^1)^{\otimes m}$ has a pseudoeffective determinant.*

Proof. Arguing by contradiction, assume that there exists $\alpha \in \operatorname{Mov}(X)$ such that $\mu_\alpha^{\min}\big((\mathscr{F}^*)^{\boxtimes m}\big) < 0$. Then by Proposition 2.9 and Theorem 2.27 we have

$$0 < -\mu_\alpha^{\min}\big((\mathscr{F}^*)^{\boxtimes m}\big) = \mu_\alpha^{\max}(\mathscr{F}^{\boxtimes m}) = m\mu_\alpha^{\max}(\mathscr{F}).$$

The maximal destabiliser $\mathscr{G}$ of $\mathscr{F}$ for the class α is therefore a foliation by Lemma 4.3 with $\mu^{\min}(\mathscr{G}) > 0$ by Proposition 2.10(v). Therefore, $\mathscr{G}$ is algebraically integrable with rationally connected leaves by Theorem 4.21. By blowing up X, we may assume that $\mathscr{G}$ is induced by a fibration $f\colon X \to Z$. By Lemma 4.13 for a general point $z \in Z$ there exist an analytic open neighbourhood V of z such that, denoting $U = f^{-1}(V)$, we have the short exact sequence

$$0 \to \mathscr{G}|_U \to \mathscr{F}|_U \to \mathscr{O}_U^{\oplus q} \to 0$$

for some positive integer q. In particular, for a general fibre F of f we have $K_{\mathscr{F}}|_F \sim K_{\mathscr{G}}|_F \sim K_F$. However, for a very general fibre F the restriction $K_{\mathscr{F}}|_F$ is pseudoeffective, hence so is K_F: indeed, there exist effective divisors E_i such that the numerical class of $K_{\mathscr{F}}$ is the limit of the numerical classes of E_i. Then for each i, a general fibre F of f is not contained in $\operatorname{Supp} E_i$, hence $E_i|_F$ is also effective.

This implies that F is not uniruled by [BDPP13], which contradicts Theorem 4.21. □

Chapter 5

Nonvanishing for threefolds and other applications

Recall from Chapter 1 that what remains in order to finish to proof of Theorem 1.3 is to consider the case $\nu(X,K_X)=1$. We first make some additional simplifications.

Let X be a minimal terminal threefold. We may assume that $h^1(X,\mathscr{O}_X)=0$ and $h^3(X,\mathscr{O}_X)=0$: otherwise, if $\pi\colon Y\to X$ is a resolution, then $h^1(Y,\mathscr{O}_Y)\neq 0$ or $h^3(Y,\mathscr{O}_Y)\neq 0$ as X has rational singularities. In the first case we may conclude as in the proof of Theorem 1.4, and in the second case the Serre duality gives $h^0(Y,K_Y)\neq 0$, hence $\kappa(X)\geq 0$, as desired. In particular, we may assume that $\chi(X,\mathscr{O}_X)=h^0(X,\mathscr{O}_X)+h^2(X,\mathscr{O}_X)>0$.

Therefore, to complete the proof of Theorem 1.3, we have to show:

Theorem 5.1. *Let X be a minimal terminal threefold such that $\nu(X,K_X)=1$ and $\chi(X,\mathscr{O}_X)>0$. Then K_X is effective.*

In fact, what we will prove in this chapter is much stronger [LP16]:

Theorem 5.2. *Let X be a minimal terminal n-fold such that $\nu(X,K_X)=1$ and $\chi(X,\mathscr{O}_X)\neq 0$. Then K_X is effective.*

5.1 Hodge index theorem

We will need the following version of the Hodge index theorem [Rei97].

Theorem 5.3. *Let X be a smooth projective surface, and let D_1 and D_2 be two $\mathbb{R}$-divisors on X. If there exist $\lambda,\mu\in\mathbb{R}$ such that $(\lambda D_1+\mu D_2)^2>0$, then $D_1^2D_2^2\leq(D_1D_2)^2$, with equality if and only if $\alpha D_1+\beta D_2\equiv 0$ for some real α and β, not both zero.*

Proof. Fix an ample divisor H on X. If $HD_1 = HD_2 = 0$, then $(\lambda D_1 + \mu D_2)H = 0$, hence $(\lambda D_1 + \mu D_2)^2 \leq 0$ by the standard formulation of the Hodge index theorem, a contradiction. Thus, without loss of generality we may assume that $HD_1 \neq 0$. Setting $\alpha = -HD_2/HD_1$, then $(\alpha D_1 + D_2)H = 0$, hence again $(\alpha D_1 + D_2)^2 \leq 0$. Consider the real quadratic form

$$f(x,y) = (xD_1 + yD_2)^2 = x^2D_1^2 + 2xyD_1D_2 + y^2D_2^2.$$

Since $f(\lambda,\mu) > 0$ and $f(\alpha,1) \leq 0$, the discriminant $\mathscr{D} = (D_1D_2)^2 - D_1^2D_2^2$ must be non-negative, which proves the first claim. Moreover,

$$\mathscr{D} = 0 \quad \text{if and only if} \quad f(x,y) = D_1^2\big(x + \tfrac{D_1D_2}{D_1^2}y\big)^2,$$

hence $f(x,y) \geq 0$ for all x, y since $f(\lambda,\mu) > 0$. In particular we have $(\alpha D_1 + D_2)^2 = 0$, which implies $\alpha D_1 + D_2 \equiv 0$ by the Hodge index theorem. Conversely, if $\alpha D_1 + D_2 \equiv 0$, then immediately $\mathscr{D} = 0$. □

Another consequence of the Hodge index theorem is:

Lemma 5.4. *Let X be a smooth projective surface, and let L and M be divisors on X such that*

$$L^2 = M^2 = L \cdot M = 0.$$

If L and M are not numerically trivial, then L and M are numerically proportional.

Proof. Let H be an ample divisor on X. By the Hodge index theorem we have $\lambda = L \cdot H \neq 0$ and $\mu = M \cdot H \neq 0$, and set $D = \lambda M - \mu L$. Then $D^2 = D \cdot H = 0$, hence $D \equiv 0$ again by the Hodge index theorem. □

5.2 Nakayama-Zariski decomposition

Recall first the Zariski decomposition on surfaces.

Theorem 5.5. *Let X be a smooth projective surface and let D be a pseudoeffective $\mathbb{Q}$-divisor on X. Then there is a unique decomposition $D = P + N$, where P and N are $\mathbb{Q}$-divisors (the* positive *and* negative *parts of D) such that*

(i) P is nef,

(ii) $N = \sum_{i=1}^{r} n_iN_i \geq 0$, and if $N \neq 0$, then the $(r \times r)$-matrix $(N_i \cdot N_j)$ is negative definite,

(iii) $P \cdot N_i = 0$ for every $i = 1, \dots, r$.

This theorem is an extremely useful tool to study the geometry of surfaces. Unfortunately, an analogous statement fails in higher dimensions. However, Nakayama [Nak04] defined a generalisation of this decomposition to higher dimensional varieties, which still enjoys some pleasurable properties.

First, for an effective divisor D on a smooth projective variety X, i.e. for divisor such that $|kD| \neq \emptyset$ for some positive integer k, and for every prime divisor Γ on X we define

$$o_\Gamma(D) = \inf\{\mathrm{mult}_\Gamma D' \mid D \sim_{\mathbb{Q}} D' \geq 0\}.$$

Then we define:

Definition 5.6. Let X be a smooth projective variety, let A be an ample $\mathbb{Q}$-divisor, and let Γ be a prime divisor. If $D \in \mathrm{Div}_{\mathbb{R}}(X)$ is pseudo-effective, set

$$\sigma_\Gamma(D) = \lim_{\varepsilon \downarrow 0} o_\Gamma(D + \varepsilon A) \qquad \text{and} \qquad N_\sigma(D) = \textstyle\sum_\Gamma \sigma_\Gamma(D) \cdot \Gamma,$$

where the sum runs over all prime divisors Γ on X.

There are several remarks in order. Nakayama shows several things: (a) if D is big, then $\sigma_\Gamma(D) = o_\Gamma(D)$, (b) the definition of $\sigma_\Gamma(D)$ does not depend on the choice of A, (c) if $D \equiv D'$, then $\sigma_\Gamma(D) = \sigma_\Gamma(D')$, (d) $N_\sigma(D)$ is a divisor, i.e. has finitely many components.

Set $P_\sigma(D) = D - N_\sigma(D)$. Then one can also show that $P_\sigma(D)$ is a pseudoeffective divisor, and if X is a surface, then the decomposition $D = P_\sigma(D) + N_\sigma(D)$ recovers the Zariski decomposition. Note that the divisors $P_\sigma(D)$ and $N_\sigma(D)$ in general have real coefficients.

Remark 5.7. Unfortunately, $P_\sigma(D)$ is in general not nef. However, if S is the intersection of $\dim X - 2$ general very ample divisors on X, then the divisor $P_\sigma(D)|_S$ is nef on S by [Nak04, Remark III.2.8 and paragraph after Corollary V.1.5]. We will need this fact in the proof of Theorem 5.8 below.

5.3 Divisors of numerical dimension 1

The following is a simple, but key technical observation needed in the proof of Theorem 5.2.

Theorem 5.8. *Let X be a projective $\mathbb{Q}$-factorial variety of dimension n, and let L be a nef divisor on X such that $\nu(X,L) = 1$. Assume that there exist a pseudoeffective $\mathbb{Q}$-divisor F and a non-zero $\mathbb{Q}$-divisor $D \geq 0$ on X such that*

$$D + F \sim_{\mathbb{Q}} L.$$

Then there exists a $\mathbb{Q}$-divisor $E \geq 0$ such that $L \equiv E$.

Proof. Let $f: Y \to X$ be a resolution of X, and denote $L' = f^*L$, $D' = f^*D$ and $F' = f^*F$, so that $D' + F' \sim_{\mathbb{Q}} L'$. Let $P = P_\sigma(F')$ and $N = N_\sigma(F') \geq 0$, so that we have the Nakayama-Zariski decomposition

$$F' = P + N.$$

Assume first that $P \not\equiv 0$. Let S be a surface in Y cut out by $n-2$ general hyperplane sections. Then $P|_S$ is nef by Remark 5.7, and in particular

$$(P|_S)^2 \geq 0. \tag{5.1}$$

On the other hand, since $\nu(Y, L') = 1$, we have

$$0 = (L'|_S)^2 = L'|_S \cdot P|_S + L'|_S \cdot N|_S + L'|_S \cdot D'|_S,$$

hence

$$L'|_S \cdot P|_S = L'|_S \cdot N|_S = L'|_S \cdot D'|_S = 0.$$

Now Theorem 5.3 implies $(P|_S)^2 \leq 0$, and hence $(P|_S)^2 = 0$ by (5.1). Then Lemma 5.4 yields $P|_S \equiv \lambda L'|_S$ for some real number $\lambda > 0$, and hence $P \equiv \lambda L'$ by the Lefschetz hyperplane section theorem. Note that $D' \not\equiv 0$ implies $\lambda < 1$. Therefore, setting

$$E' = \frac{1}{1-\lambda}(N + D') \geq 0,$$

we obtain

$$L' \equiv E'.$$

Let $E_1, \ldots, E_r$ be the components of E' and let $\pi\colon \operatorname{Div}_{\mathbb{R}}(Y) \to N^1(Y)_{\mathbb{R}}$ be the standard projection. Then $\pi^{-1}\big(\pi(L')\big) \cap \sum \mathbb{R}_+ E_i$ is a rational affine subspace of $\sum \mathbb{R} E_i \subseteq \operatorname{Div}(Y)_{\mathbb{R}}$ which contains E', hence there exists a rational point

$$0 \leq E'' \in \pi^{-1}\big(\pi(L')\big) \cap \sum \mathbb{R}_+ E_i.$$

Setting $E = f_* E''$, we have $L \equiv E$ and $E \geq \varepsilon D$, which proves the result in the case $P \not\equiv 0$.

If $P \equiv 0$, denote $E' = N + D' \geq 0$, so that $L' \equiv E'$. We conclude as above. □

Corollary 5.9. *Let X be a $\mathbb{Q}$-factorial projective terminal variety such that K_X is nef and $\nu(X, K_X) = 1$. Assume that there exist a pseudoeffective $\mathbb{Q}$-divisor F and a non-zero $\mathbb{Q}$-divisor $D \geq 0$ on X such that $K_X \sim_{\mathbb{Q}} D + F$. Then $\kappa(X, K_X) \geq 0$.*

Proof. By Theorem 5.8 applied to $L = K_X$, there exists an effective $\mathbb{Q}$-divisor E on X such that $K_X \equiv E$. By [CKP12, Theorem 0.1] we have $\kappa(X, K_X) \geq \kappa(X, E)$, and the result follows. □

5.4 Singular metrics

We need a few facts about singular metrics on line bundles and associated multiplier ideals. A good general source for these ideas is [Dem01].

Definition 5.10. Let L be a holomorphic line bundle on a complex manifold X of dimension n. A *singular hermitian metric* on L is a metric which is given in every trivialization $\theta\colon L|_U \simeq U \times \mathbb{C}$ by

$$\|\xi\| = |\theta(\xi)|e^{-\varphi(x)}, \quad x \in U,\ \xi \in L_x,$$

where $\varphi \in L^1_{\mathrm{loc}}(U)$ (locally integrable function on U), called the *weight* of the metric with respect to the trivialization θ. The *curvature current* of L is given formally by the closed $(1,1)$-current $\Theta_h(L) = dd^c\varphi$ on U; the assumption $\varphi \in L^1_{\mathrm{loc}}(U)$ guarantees that $\Theta_h(L)$ exists in the sense of distribution theory. A $(1,1)$-current Θ is *semi-positive* if for every choice of smooth $(1,0)$-forms $\alpha_1, \ldots, \alpha_{n-1}$ on X the distribution $\Theta \wedge i\alpha_1 \wedge \overline{\alpha}_1 \wedge \cdots \wedge i\alpha_{n-1} \wedge \overline{\alpha}_{n-1}$ is a positive measure.

Definition 5.11. A function $\varphi\colon U \to [-\infty, \infty)$ defined on an open set $U \subseteq \mathbb{C}^n$ is *plurisubharmonic* if it is upper semi-continuous, and for every $a \in U$ and $\xi \in \mathbb{C}^n$ satisfying $|\xi| < \inf\{|a-x| \mid x \in \mathbb{C}^n \setminus U\}$, the function φ satisfies the mean value inequality

$$\varphi(a) \le \frac{1}{2\pi}\int_0^{2\pi} \varphi\left(a + e^{i\theta}\xi\right) d\theta.$$

If X is an n-dimensional complex manifold, a function $\varphi\colon X \to [-\infty, \infty)$ is *plurisubharmonic* if there exists an open cover $X = \bigcup_{i \in I} U_i$ by coordinate patches such that $\varphi|_{U_i}$ is plurisubharmonic on U_i for every i.

There is a dictionary between positivity statements in algebraic geometry and the corresponding differential-geometric counterparts. A line bundle L on a projective variety X with a fixed Kähler form ω is:

(a) pseudoeffective if and only if there exists a singular metric h on L whose weight is a plurisubharmonic function if and only if there exists a singular metric h on L such that $\Theta_h(L) \ge 0$,

(b) ample if and only if there exists a smooth metric h on L such that $\Theta_h(L) > 0$,

(c) nef if and only if for each $\varepsilon > 0$ there exists a smooth metric h_ε on L such that $\Theta_{h_\varepsilon}(L) \ge -\varepsilon\omega$,

(d) big if and only if there exists a singular metric h on L and $\varepsilon > 0$ such that $\Theta_h(L) \ge \varepsilon\omega$.

We also need the following very useful definition.

Definition 5.12. Let φ be a plurisubharmonic function on a complex manifold X. Then the *multiplier ideal sheaf* $\mathscr{I}(\varphi) \subseteq \mathscr{O}_X$ is defined by

$$\Gamma(U, \mathscr{I}(\varphi)) = \{f \in \mathscr{O}_X(U) \mid |f|^2 e^{-2\varphi} \in L^1_{\mathrm{loc}}(U)\}$$

for every open set $U \subseteq X$.

Therefore, $\mathscr{I}(\varphi)$ collects all the locally L^2-integrable holomorphic sections of X with respect to a metric whose weight is φ. It is well-known that $\mathscr{I}(\varphi)$ is an analytic coherent ideal sheaf on X; in particular, by GAGA theorems, if X is projective, then this sheaf is always an algebraic coherent sheaf.

One of the reasons why multiplier ideals are very useful to us is the following important result [DPS01, Theorem 0.1], which is an extension of the hard Lefschetz theorem.

Theorem 5.13. *Let X be a compact Kähler manifold of dimension n with a Kähler form ω. Let $\mathscr{L}$ be a pseudoeffective line bundle on X with a singular hermitian metric h such that $\Theta_h(\mathscr{L}) \geq 0$. Then for every non-negative integer q the morphism*

$$H^0\big(X, \Omega_X^{n-q} \otimes \mathscr{L} \otimes \mathscr{I}(h)\big) \xrightarrow{\omega^q \wedge \bullet} H^q\big(X, \Omega_X^n \otimes \mathscr{L} \otimes \mathscr{I}(h)\big)$$

is surjective.

5.5 Proof of Theorem 5.2

We have all the tools to prove Theorem 5.2. We start with the following important step; the need for the sheaves appearing in the following result is justified by Theorem 5.13.

Theorem 5.14. *Let X be a minimal $\mathbb{Q}$-factorial projective terminal variety of dimension n. Assume that $\nu(X, K_X) = 1$. Let $\pi\colon Y \to X$ be a resolution of X, and assume that for infinitely many $m \neq 0$ sufficiently divisible and for some $0 \leq p \leq n$ we have*

$$H^0\big(Y, \Omega_Y^p \otimes \mathscr{O}_Y(m\pi^* K_X)\big) \neq 0.$$

Then $\kappa(X, K_X) \geq 0$.

Proof. We first note that we have $K_X \not\equiv 0$ by hypothesis.

Arguing by contradiction, assume that there exists $p \geq 1$ and an infinite set $\mathscr{T} \subseteq \mathbb{Z}$ such that

$$H^0\big(Y, \Omega_Y^p \otimes \mathscr{O}_Y(m\pi^* K_X)\big) \neq 0$$

for all $m \in \mathscr{T}$. Denote $Z = \mathbb{P}(\Omega_Y^p)$ with the projection $f\colon Z \to Y$. First note that

$$H^0\big(Y, \Omega_Y^p \otimes \mathscr{O}_Y(m\pi^* K_X)\big) \simeq H^0\big(Z, \mathscr{O}_Z(1) \otimes f^* \mathscr{O}_Y(m\pi^* K_X)\big),$$

hence it is immediate that there are only finitely many negative integers in $\mathscr{T}$ since $K_X \not\equiv 0$. Therefore, we may assume that $\mathscr{T} \subseteq \mathbb{N}$.

Every nontrivial global section of the sheaf $\Omega_Y^p \otimes \mathscr{O}_Y(m\pi^* K_X)$ gives an inclusion $\mathscr{O}_Y(-m\pi^* K_X) \to \Omega_Y^p$, and let $\mathscr{F} \subseteq \Omega_Y^p$ be the image of the map

$$\bigoplus_{m \in \mathscr{T}} \mathscr{O}_Y(-m\pi^* K_X) \to \Omega_Y^p.$$

Then $\mathscr{F}$ is quasi-coherent by [Har77, Proposition II.5.7], and therefore a torsion free coherent sheaf as it is a subsheaf of the torsion free coherent sheaf Ω_Y^p. Let r be the rank of $\mathscr{F}$. We may assume that there exist infinitely many r-tuples $(m_1, \dots, m_r)$ such that the image of the map

$$\mathscr{O}_Y(-m_1\pi^* K_X) \oplus \cdots \oplus \mathscr{O}_Y(-m_r\pi^* K_X) \to \mathscr{F} \tag{5.2}$$

has rank r: indeed, if this is not the case, we replace $\mathscr{T}$ by a suitable infinite subset, and the rank of $\mathscr{F}$ is smaller than r. Taking determinants in (5.2) yields inclusions

$$\mathscr{O}_Y\big(-(m_1 + \cdots + m_r)\pi^* K_X\big) \to \det \mathscr{F} \subseteq \bigwedge^r \Omega_Y^p. \tag{5.3}$$

There is a Cartier divisor F_Y such that $\mathscr{O}_Y(-F_Y)$ is the saturation of $\det \mathscr{F}$ in $\bigwedge^r \Omega_Y^p$. Then by (5.3) there exists an infinite set $\mathscr{S} \subseteq \mathbb{N}$ such that

$$H^0(Y, m\pi^* K_X - F_Y) \neq 0 \quad \text{for all } m \in \mathscr{S}. \tag{5.4}$$

Consider the exact sequence

$$0 \to \mathscr{O}_Y(-F_Y) \to \bigwedge^r \Omega_Y^p \to \mathscr{Q} \to 0.$$

Since $\mathscr{O}_Y(-F_Y)$ is saturated in $\bigwedge^r \Omega_Y^p$, the sheaf $\mathscr{Q}$ is torsion free, and hence $\tilde{F}_Y = c_1(\mathscr{Q})$ is pseudoeffective by Theorem 1.1. From the above exact sequence, there exists a positive integer ℓ such that $\ell K_Y \sim \tilde{F}_Y - F_Y$.

From (5.4), for every $m \in \mathscr{S}$ we obtain an effective divisor $\tilde{N}_{m+\ell}$ such that $\tilde{N}_{m+\ell} \sim m\pi^* K_X - F_Y$, and hence

$$\tilde{N}_{m+\ell} + \tilde{F}_Y \sim m\pi^* K_X + \ell K_Y. \tag{5.5}$$

Denote $N_{m+\ell} = \pi_* \tilde{N}_{m+\ell}$ and $F = \pi_* \tilde{F}_Y$; note that $N_{m+\ell}$ is effective and that F is pseudoeffective. Pushing forward the relation (5.5) to X, we get

$$N_{m+\ell} + F \sim_{\mathbb{Q}} (m+\ell) K_X. \tag{5.6}$$

Now Corollary 5.9 gives a contradiction. □

Finally, we have:

Proof of Theorem 5.2. Assume first that there exist a resolution $\pi\colon Y \to X$, such that for every positive integer m such that mK_X is Cartier, and for every singular metric h_m on $\pi^*\mathscr{O}_X(mK_X)$ with semipositive curvature current, we have $\mathscr{I}(h_m) = \mathscr{O}_Y$. Arguing by contradiction, assume that $\kappa(X,K_X) = -\infty$. Then by Theorem 5.14, for all $p \geq 0$ and for all m sufficiently divisible we have

$$H^0\big(Y,\Omega_Y^p \otimes \pi^*\mathscr{O}_X(mK_X)\big) = 0.$$

Theorem 5.13 implies that for all $p \geq 0$ and for all $m > 0$ sufficiently divisible,

$$H^p\big(Y,\mathscr{O}_Y(K_Y + m\pi^*K_X))\big) = 0,$$

or by Serre duality:

$$H^{n-p}\big(Y,\mathscr{O}_Y(-m\pi^*K_X))\big) = 0.$$

Then we immediately have

$$\chi\big(Y,\mathscr{O}_Y(-m\pi^*K_X)\big) = 0 \tag{5.7}$$

for all $m > 0$ sufficiently divisible. Since the Euler characteristic is a polynomial in m, for $m = 0$ we obtain $\chi(Y,\mathscr{O}_Y) = 0$. Since X has rational singularities, this implies $\chi(X,\mathscr{O}_X) = 0$, a contradiction.

Hence, it remains to consider the case when there exists a resolution $\pi\colon Y \to X$, a positive integer m such that mK_X is Cartier, and a singular metric h on $\pi^*\mathscr{O}_X(mK_X)$ with semipositive curvature current such that $\mathscr{I}(h) \neq \mathscr{O}_Y$. Let $V \subseteq Y$ be the subspace defined by $\mathscr{I}(h)$, and let y be a closed point in V with ideal sheaf $\mathscr{I}_y$ in y. Let $\mu\colon \widehat{Y} \to Y$ be the blow-up of Y at y and let $E = \pi^{-1}(y)$ be the exceptional divisor. Let $\widehat{h}$ be the induced metric on $L := (\pi \circ \mu)^*\mathscr{O}_X(mK_X)$. By [Dem01, Proposition 14.3], we have

$$\mathscr{I}(\widehat{h}) \subseteq \mu^{-1}\mathscr{I}(h)\cdot\mathscr{O}_{\widehat{Y}} \subseteq \mu^{-1}\mathscr{I}_y\cdot\mathscr{O}_{\widehat{Y}} = \mathscr{O}_{\widehat{Y}}(-E).$$

By [DEL00, Theorem 1.10] there exists an ample line bundle G on $\widehat{Y}$ such that

$$\mathscr{O}_{\widehat{Y}}(G + kL) \otimes \mathscr{I}(\widehat{h}^{\otimes k})$$

is globally generated for all $k \geq 1$. Since

$$\mathscr{I}(\widehat{h}^{\otimes k}) \subseteq \mathscr{I}(\widehat{h})^k \subseteq \mathscr{O}_{\widehat{Y}}(-kE),$$

where the first inclusion follows from [DEL00, Theorem 2.6], for all $k \geq 1$ we have

$$H^0\big(\widehat{Y}, G + k(L-E)\big) \neq 0.$$

Hence $L - E = \lim_{m\to\infty} \frac{1}{m}\big(m(L-E)+G\big)$ is pseudoeffective. Then Corollary 5.9 implies $\kappa(X,K_X) = \kappa\big(\widehat{Y},K_{\widehat{Y}}\big) = \kappa\big(\widehat{Y},L\big) \geq 0.$ □

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