On the existence and number of good models of algebraic varieties

Habilitationsschrift

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Contents

The results of this manuscript

The results of this work have been mostly published in several papers of mine with my coauthors. In particular, the results in Chapter 2 are proved in my paper [DL14] with Tobias Dorsch. The results in Chapter 3, apart from Section 3.2, are proved in my paper [LP13] with Thomas Peternell. Theorem 3.11 and the results in Chapter 4 are proved in my paper [CL14] with Paolo Cascini. Proposition 1.11 and Proposition 1.12 are proved in my survey paper [Laz13]. Theorem 3.13 is new. Most of the other discussion in Section 3.2 and the results in Section 5 are proved in my paper [KKL12] with Anne-Sophie Kaloghiros and Alex Küronya.

[CL14] P. Cascini and V. Lazić, *On the number of minimal models of a log smooth threefold*, J. Math. Pures Appl. 102 (2014), 597–616.

[DL14] T. Dorsch and V. Lazić, *A note on the abundance conjecture*, arXiv:1406.6554.

[KKL12] A.-S. Kaloghiros, A. Küronya and V. Lazić, *Finite generation and geography of models*, to appear in "Minimal models and extremal rays", Advanced Studies in Pure Mathematics, Mathematical Society of Japan, Tokyo, arXiv:1202.1164

[Laz13] V. Lazić, *Around and beyond the canonical class*, Birational Geometry, Rational Curves, and Arithmetic (Fedor Bogomolov, Brendan Hassett, Yuri Tschinkel, eds.), Simons Symposia, Springer New York, 2013, pp. 171–203.

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Zusammenfassung

Als ‘Minimal Model Programm’ bezeichnet man in der algebraischen Geometrie höherdimensionaler Varietäten ein Verfahren, welches algebraische Varietäten in ihre Grundbausteine zerlegt. Es handelt sich um ein zentrales Projekt der algebraischen Geometrie, welches seit der Fields-Medaille von Mori in 1990 für den dreidimensionalen Fall stetig weiterentwickelt wurde. Aber gerade in den letzten zehn Jahren hat das Gebiet enorme Fortschritte gemacht. Viele wichtige Resultate wurden erzielt, die unser Verständnis wesentlich vertieft haben.

Wir wissen heute, dass das Minimal Model Program für glatte projektive Varietäten X zum Ziel führt (‘terminiert’), falls der kanonische Divisor K_X gross oder nicht pseudo-effektiv ist. Insbesondere hat jede Varietät allgemeinen Typs ein birationales Modell, für welches der kanonische Divisor semiample ist. Darüberhinaus ist bekannt, dass die Anzahl solcher guten Modelle immer endlich ist.

Das zentrale ausstehende Problem der birationalen Geometrie ist es nun, die in den letzten Jahren entwickelte Theorie auf allgemeine Varietäten auszudehnen, das heißt die Existenz guter Modelle für möglichst viele Varietäten zu beweisen, die nicht von allgemeinem Typ sind. Diese Habilitationsschrift stellt sich als Aufgabe, mit den folgenden vier Resultaten zur Lösung dieses Problems beizutragen:

(a) Die Existenz guter Modelle für klt Paare (X,Δ) mit pseudo-effektivem log-kanonischen Divisor $K_X+\Delta$ ist die wichtigste ausstehende Vermutung im Minimal Model Program für projektive klt Paare in verschwindender Charakteristik. Es ist wohl bekannt, dass die Existenz guter Modelle die Abundance-Vermutung impliziert, welche behauptet, dass auf einem minimalen Modell der kanonische Divisor semiample ist.

Unser erstes Resultat reduziert das Problem der Existenz guter Modelle für nicht-unigeregelte Paare auf den Fall von glatten Varietäten mit effektiver kanonischer Klasse. Etwas präziser formuliert, die Existenz guter Modelle für klt Paare in Dimensionen höchstens $n-1$ vorausge-

setzt, wir zeigen, dass die Existenz guter Modelle für nicht-unigeregelte klt Paare in Dimension n, die Existenz guter Modelle für unigeregelte klt Paare in Dimension n impliziert. Dies ist die Verallgemeinerung der Strategie, die für Varietäten der Dimension drei zum Ziel führte, und stellt den ersten Schritt zum Beweis der Existenz guter Modelle dar.

(b) Die Form verschiedener Kegel im Néron-Severi Raum $N^1(X)_{\mathbb{R}}$ einer Varietät X trägt wichtige Information über die Geometrie von X. Aus Sicht der birationalen Geometrie sind die Kegel der nef Divisoren und der beweglichen Divisoren von besonderem Interesse. Die Kegel-Vermutung von Morrison und Kawamata sagt nun voraus, dass auf einer Calabi-Yau Varietät beide Kegel modulo der Wirkung gewisser natürlicher Gruppen rational polyedrisch sind.

Ein Ergebnis der vorliegenden Arbeit ist der Beweis der Kegel-Vermutung für Calabi-Yau Mannigfaltigkeiten der Picardzahl 2 und unendlicher Gruppe $\mathrm{Bir}(X)$ birationaler Automorphismen. Damit wird die Kegel-Vermutung in großer Allgemeinheit und insbesondere für eine breite Klasse von Dreifaltigkeiten bewiesen.

(c) Es ist eine wichtige und seit langem offene Vermutung, dass die Anzahl minimaler Modelle einer glatten projektiven Varietät bis auf Isomorphie endlich ist. Die Kegel-Vermutung, zusammen mit der Existenz guter Modelle, würde dies nun implizieren. Diese Anwendung kann man als eigentliche Motivation für die Kegel-Vermutung betrachten. Wenn die Endlichkeit nun bereits bekannt ist, ist es naheliegend nach der Anzahl der minimalen Modelle zu fragen und weiter, ob diese eine rein topologische Invariante ist.

Das dritte Resultat dieser Arbeit besagt, dass die Zahl der minimalen Modelle bestimmter log-glatter Paare der Dimension drei nur vom topologischen Typ dieser Paare abhängig ist. Zwei log-glatte Paare (X_1, Δ_1) und (X_2, Δ_2) sind dabei vom selben topologischen Typ, falls ein Homöomorphismus $\varphi\colon X_1 \to X_2$ existiert, der einen Homöomophismus zwischen den Trägern von Δ_1 und Δ_2 induziert.

(d) Es gibt zwei Klassen projektiver Varietäten, deren birationale Geometrie besonders interessant ist. Die erste Klasse enthält Varietäten, für die das klassische Minimal Model Program erfolgreich ausgeführt werden kann. Die zweite Klasse enthält sogenannte 'Mori Dream Spaces'. Es ist bekannt, dass in beiden Fällen die birationale Geometrie komplett durch gewisse endlich erzeugte Ringe bestimmt wird, aber a priori sind die jeweiligen Ringe von ganz unterschiedlicher Gestalt und Herkunft.

In dieser Schrift behandeln wir nun beide Klassen von Varietäten mit dem gleichen Ansatz. Wir identifizieren dabei die maximale Klasse der Varietäten und deren Divisoren, die mit dem MMP behandelt werden können.

Chapter 1

Introduction

The objects of algebraic geometry are varieties, i.e. zeroes of systems of polynomial equations defined over a certain field – in this work, that field of definition is the field of complex numbers $\mathbb{C}$. In this thesis we are interested in *projective varieties*, which are sets given as common zeroes of a system of *homogeneous* polynomials. These are fundamental objects in mathematics, which pop up also in differential geometry, arithmetic geometry, number theory, topology and so on. As in every other corner of mathematics, the principal goal of algebraic geometry is to give a meaningful classification of its main objects. This thesis deals with several questions related to a partly still conjectural programme of classification of varieties: the *Minimal Model Program* (or the *MMP*), as explained below.

One of the main tools to study algebraic varieties is to study behaviour of their subvarieties, and in particular two extreme cases are very important:

(1) the case of *curves*, that is varieties of dimension 1,

(2) the case of *prime divisors*, that is subvarieties of codimension 1.

We concentrate here on the study of $\mathbb{Q}$-Weil divisors on a variety X, i.e. formal $\mathbb{Q}$-linear combinations of prime divisors on X; and on $\mathbb{Q}$-Cartier divisors on X, which are, up to a rational multiple, locally given by the sum of zeroes and poles of a rational function on X. Then we have a good intersection theory of $\mathbb{Q}$-Cartier divisors with curves as explained in [Ful98].

The most important sheaf on a, say, smooth projective variety X of dimension n is its *canonical line bundle*

$$\omega_X = \bigwedge^n (T_X^*)$$

(where T_X is the tangent sheaf of X), as well as the associated *canonical divisor* (or canonical class) K_X, which satisfies

$$\mathcal{O}_X(K_X) \simeq \omega_X.$$

As its name says, it is *canonical*: its definition is intrinsic, and it is naturally defined on every (smooth or normal) variety.

Ever since Riemann's work on curves in the 19th century, the importance of ω_X has been realised: in part because of the Riemann-Roch theorem, and in part because often it is very difficult to find reasonable and useful divisors on X. Of course, in the 20th century it was understood further that this line bundle is important because of Serre duality, Kodaira vanishing and so on. Therefore, it is logical to concentrate on ω_X as the centre point of the MMP, apart from more profound further reasons elaborated on below.

On the other hand, having ample divisors on a projective variety X is extremely important: they give embeddings of X into some projective space, and their cohomological and numerical properties are as nice as one can hope for. The crux of the Minimal Model Program is the study of the question – when can one make the canonical bundle ample.

The Minimal Model Program has seen tremendous progress in the last decade, which is measurable both in scope of the results achieved, as well as in the depth of our understanding of the subject. The seminal paper [BCHM10], building on earlier results of Mori, Reid, Kawamata, Kollár, Shokurov, Siu, Corti, Nakayama and many others, settled many results and advanced hugely our knowledge of the theory. The main result of that paper is that the Minimal Model Program for a smooth projective variety X terminates if either K_X is a big divisor (in other words, the dimensions of the vector spaces $H^0(X, mK_X)$ grow maximally with m – like $m^{\dim X}$) or if it is not pseudoeffective (in other words, K_X is *numerically* not a limit of divisors whose multiples have global sections). In particular, all varieties with big canonical bundle have a birational model Y on which a multiple of K_Y is a big basepoint divisor free. Furthermore, the number of such models Y is finite up to isomorphism.

The main outstanding problem in birational geometry is to prove that models with similar properties exist if X is not necessarily of general type. Progress towards the solution of this problem is the topic of this thesis.

1.1 Classification of curves and surfaces

The classification of curves is classical and was done in the 19th century. The rough classification is according to the genus of a smooth projective curve.

The situation with surfaces is already more complicated. If we start with a smooth projective surface, and want our classification procedure to simplify it in tangible ways, we would therefore want some basic invariants, like the Picard number (i.e. the rank of the group of Cartier divisors modulo numerical equivalence) to be as minimal as possible.

To this end, recall that if $\pi : Y \to X$ is a blow up of a point on a smooth surface X, then the exceptional divisor $E \subseteq Y$ is a (-1)-curve, that is

$$E \simeq \mathbb{P}^1 \quad \text{and} \quad E^2 = -1.$$

The starting point of the classification of surfaces is the following Castelnuovo's theorem [Har77, Theorem V.5.7], which says that if we start with a (-1)-curve on Y, we can invert the blowup construction:

Theorem 1.1. *Let Y be a nonsingular projective surface containing a (-1)-curve E.*

Then there exists a birational morphism $f : Y \to X$ to a smooth projective surface X such that E is contracted to a point, and moreover, f is a blowup of X at $f(E)$.

Now it is easy to see how the classification works in dimension 2. Once we have our smooth surface, we ask whether the surface obtained has a (-1)-curve. If not, we have our *relatively minimal model*. If yes, then we use Castelnuovo contraction to contract a (-1)-curve. We repeat the process for the new surface. The process is finite since after each step, the Picard number drops, as well as the second beti number.

Note however, that the criterion "does X have a (-1)-curve" does not have a meaningful generalisation to higher dimensions. Also, it is not clear that it gives the right notion – in other words, it is not obvious that this is an intrinsic notion of X with special implications on the geometry of X.

However, note that, by the adjunction formula, E is a (-1)-curve on X if and only if

$$E \simeq \mathbb{P}^1 \quad \text{and} \quad K_X \cdot E < 0.$$

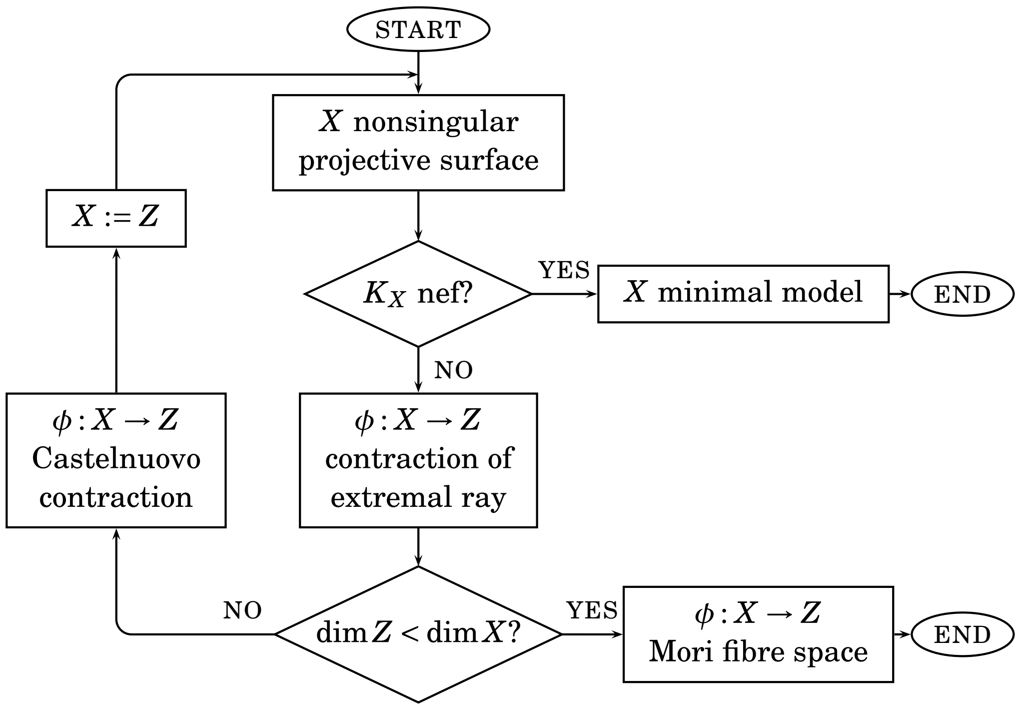
Recall that a divisor D on a variety X is *nef* if $D \cdot C \geq 0$ for every irreducible curve C on X; such divisors are (numerically) limits of ample

$\mathbb{Q}$-divisors. Therefore, if X has a (-1)-curve, then its canonical class cannot be nef.

There are three possibilities for the relatively minimal model X. If K_X is nef, then a further fine classification gives that it is actually *semi-ample*, i.e. some multiple of K_X is basepoint free. Then, by a result of Iitaka, any high multiple of K_X defines a fibration $X \to Z$ to another projective variety Z, and we can further analyse X with the aid of this map. In this case, we also say that X is the *(absolute) minimal model*.

If K_X is not nef, then one can show that either there exists a morphism $\varphi\colon X \to Z$ to a smooth projective curve Z such that X is a $\mathbb{P}^1$-bundle over Z via φ, or $X \simeq \mathbb{P}^2$. In these last two cases, one says that X is a *Mori fibre space*.

This gives the following *hard dichotomy* for surfaces: the end product of the classification is either a minimal model (unique up to isomorphism) if $\kappa(X) \geq 0$ or a Mori fibre space if $\kappa(X) = -\infty$.



FLOWCHART 1.1: Minimal Model Program in dimension 2

1.2 What is the Minimal Model Program?

I sketch briefly what is understood by a good minimal model theory. The presentation differs from the classical one in the sense that it stresses different properties, and it allows to consider the theory for divisors which are not necessarily (close to being) canonical.

One of the ingenious insights of Mori was introducing a new criterion for determining whether a variety X is a minimal model:

Is K_X nef?

There are many reasons why this is a meaningful question to pose. First, it makes sense by analogy with surfaces, as presented above. Second, on a random (smooth, projective) variety X it is usually very hard to find any useful divisors, especially those which carry essential information about the geometry of X – the only obvious candidate is K_X, by its very construction.

Further, in an ideal situation we would have that K_X is ample – indeed, this would mean that some multiple of K_X itself gives an embedding into a projective space, and that it enjoys many nice numerical and cohomological properties.

Therefore, from now on we assume that K_X is pseudoeffective. Then, a reasonable question to pose is:

Question. Is there a birational map $f\colon X \dashrightarrow Y$ such that the divisor f_*K_X is ample?

Here the map f is a *birational contraction* – in other words, f^{-1} should not contract divisors. This is an important condition since the variety Y should be in almost every way simpler than X; in particular, as in the case of surfaces, some of its main invariants, such as the Picard number, should not increase. Likewise, we would like to have the equality

$$K_Y = f_*K_X,$$

and this will almost never happen if f extracts divisors.

What we almost always have to sacrifice is smoothness – in other words, we cannot expect that the variety Y is smooth, even if we start with a smooth variety X. This issue is by now well understood, and it presents more a philosophical (or psychological) than a technical obstacle. The varieties we allow are in some sense pretty close to being smooth, in the sense of *singularities of pairs* which will be explained below.

Further, we impose that f should *preserve* global sections of all positive multiples of K_X. This is also important, since global sections are something we definitely want to keep track of, if we want the divisor $K_Y = f_*K_X$ to bear any connection with K_X. Another way to state this is as follows. Consider the *canonical ring* of X:

$$R(X,K_X) = \bigoplus_{m\in\mathbb{N}} H^0(X,mK_X).$$

Then we require that f induces an isomorphism

$$R(X,K_X) \simeq R(Y,K_Y).$$

We immediately see that the answer to the question above is in general "no" – the condition would imply that K_X is a big divisor. In fact, and perhaps surprisingly, the converse is true by a theorem of Reid [Rei80, Proposition 1.2] – a similar statement is given in Lemma 5.8 below.

We now return to Question above, in order to see if we can modify it to something more probable. We can settle for something weaker, but still sufficient for our purposes: we require that the divisor K_Y is semiample. Then we would have the associated Iitaka fibration $g\colon Y \to Z$ and an ample $\mathbb{Q}$-divisor A such that $K_Y \sim_{\mathbb{Q}} g^*A$.

$$\begin{array}{ccc} X & \stackrel{f}{\dashrightarrow} & Y \\ & \searrow & \downarrow g \\ & & Z \end{array}$$

The composite map $X \dashrightarrow Z$, which is now not necessarily birational, would give

$$R(X,pK_X) \simeq \bigoplus_{n\in\mathbb{N}} H^0(Z,npA)$$

for some positive integer p. In particular, this would imply that the canonical ring $R(X,K_X)$ is finitely generated.

This would clearly be astonishing: we would be able to construct the projective variety

$$Z = \operatorname{Proj} R(X,K_X)$$

just from the geometric data on X. In fact, the wish that the canonical ring is finitely generated predates the modern Minimal Model Program, and goes back to the seminal work of Zariski [Zar62]. This is now a theorem, settled first in [BCHM10, HM10] by the methods of the Minimal Model Program, and then in [CL12] by a self-contained induction.

Theorem 1.2. *Let X be a smooth projective variety over $\mathbb{C}$. Then the canonical ring*

$$R(X,K_X) = \bigoplus_{n\in\mathbb{N}} H^0(X,nK_X)$$

is finitely generated as a $\mathbb{C}$-algebra.

By analogy with surfaces, the search for a map f as above splits into two problems:

(a) find a birational map $f\colon X \dashrightarrow Y$ such that the divisor $K_Y = f_*K_X$ is nef (Y is a *minimal model*),

(b) prove that the nef divisor K_Y is semiample (Y is a *good model*).

Part (b) is the *Abundance conjecture*, and I discuss in Section 1.4 to which extent it is known.

Finally, if we start with a smooth variety X on which the divisor K_X is not pseudoeffective, then one would hope that sort of the opposite to the above holds – that there exists a birational map $f\colon X \dashrightarrow Y$ together with a morphism $g\colon Y \to Z$ such that the a general fibre of g is a Fano variety, i.e. the canonical sheaf of the fibre is anti-ample.

$$\begin{array}{ccc} X & \overset{f}{\dashrightarrow} & Y \\ & \searrow & \downarrow{\scriptstyle g} \\ & & Z \end{array}$$

In this case we call Y a *Mori fibre space*. This is indeed now a theorem [BCHM10].

1.3 Pairs and their singularities

It has become clear in the last several decades that sometimes varieties are not the right objects to look at – often, it is much more convenient to look at pairs (X,Δ), where X is a normal projective variety and Δ is a Weil $\mathbb{Q}$-divisor on X such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier. There are at least two very good reasons why this is the right setup:

(a) we expect the proofs in the field should go by induction on the dimension, and if one wants to use adjunction formula, one has to consider pairs; and

(b) crucially, one cannot consider only the canonical bundle of a variety, if one leaves the category of varieties of general type.

To see (b), consider a good model X and a morphism $\varphi\colon X\to Z$, which is the Iitaka fibration of the semiample divisor K_X. When K_X is not big, it is in general hopeless to expect that $K_X\sim_{\mathbb{Q}}\varphi^*K_Z$. However, it can be shown that there exists an effective $\mathbb{Q}$-divisor Δ on Z such that the pair has nice properties (in the sense explained a bit below) and such that

$$K_X\sim_{\mathbb{Q}}\varphi^*(K_Z+\Delta),$$

cf. [Amb05].

Now assume we are given a pair (X,Δ), and let $f\colon Y\to X$ be a log resolution of the pair, i.e. the variety Y is smooth, the set $\operatorname{Exc} f$ is a divisor, and the support of the divisor $\operatorname{Exc} f\cup f^*\Delta$ has simple normal crossings. Then there exists a unique $\mathbb{Q}$-divisor R on Y such that

$$K_Y\sim_{\mathbb{Q}} f^*(K_X+\Delta)+R,$$

where the divisor R is supported on the proper transform of Δ and on the exceptional divisors of f. For every prime divisor E on Y, we denote the coefficient of E in R by $a(E,X,\Delta)$, called the *discrepancy of E with respect to the pair* (X,Δ).

If we set

$$d(X,\Delta)=\inf\{a(E,X,\Delta)\},$$

where the infimum is over all prime divisors E lying on some birational model $Y\to X$, then it is easy to see that $d(X,\Delta)\le 1$, and there is the following dichotomy:

$$\text{either } d(X,\Delta)\ge -1 \text{ or } d(X,\Delta)=-\infty,$$

cf. [KM98].

This is the first indication that the pairs which satisfy $d(X,\Delta) \geq -1$ behave better than other pairs. The following is an example of a pair with $d(X,\Delta) < -1$ whose canonical ring is not finitely generated, hence no reasonable definition of the Minimal Model Program can work for such (X,Δ).

Example 1.3. Let E be an elliptic curve, let D be a non-torsion divisor of degree 0, and let A be an ample divisor on E of large degree, so that $H^0(E, kD + A) \neq 0$ for all $k \geq 0$. Set

$$Y = \mathbb{P}(\mathscr{O}_E(D) \oplus \mathscr{O}_E(A)) \quad \text{and} \quad M = \mathscr{O}_Y(1).$$

If $R_{i,j} = H^0(E, iD + jA)$, then

$$H^0(Y, M^{\otimes k}) \simeq \bigoplus_{i+j=k} R_{i,j}.$$

This implies that the section ring $R(Y,M)$ is not finitely generated: indeed, since $R_{k,0} = 0$ for all $k > 0$, each $R_{k,1}$ consists of minimal generators of $R(Y,M)$.

Set
$$L = M \otimes \omega_Y^{-1} \otimes \mathscr{O}_Y(1) \quad \text{and} \quad \mathscr{E} = L \oplus \mathscr{O}_Y(1)^{\oplus 3},$$

and let $Z = \mathbb{P}(\mathscr{E})$ with the projection map $\pi\colon Z \to Y$. Thus, Z is a smooth $\mathbb{P}^3$-bundle over Y, and denote $\xi = \mathscr{O}_Z(1)$. Then

$$\omega_Z = \pi^*(\omega_Y \otimes \det \mathscr{E}) \otimes \xi^{\otimes -4} = \pi^*(\omega_Y \otimes L \otimes \mathscr{O}_Y(3)) \otimes \xi^{\otimes -4}.$$

Consider the linear system $|\xi \otimes \pi^*\mathscr{O}_Y(-1)|$. It contains smooth divisors S_1, S_2, S_3 corresponding to the quotients $\mathscr{E} \to L \oplus \mathscr{O}_Y(1)^{\oplus 2}$, and note that $P = S_1 \cap S_2 \cap S_3$ is a codimension 3 cycle corresponding to the quotient $\mathscr{E} \to L$. In particular, the base locus of $|(\xi \otimes \pi^*\mathscr{O}_Y(-1))^{\otimes 4}|$ is contained in P.

$$\begin{array}{ccccc} X & \hookrightarrow & Z & \xrightarrow{\pi} & Y \\ & & & & \downarrow \\ & & & & E \end{array}$$

Let X be a general member of $|(\xi \otimes \pi^*\mathscr{O}_Y(-1))^{\otimes 4}|$. Then X is smooth in codimension 1, and since Z is smooth, we have that X is normal and Gorenstein. The adjunction formula [K$^+$92, Proposition 16.4] gives

$$\omega_X = \omega_Z \otimes \mathscr{O}_Z(X) \otimes \mathscr{O}_X = (\pi_{|X})^*(\omega_Y \otimes L \otimes \mathscr{O}_Y(-1)) = (\pi_{|X})^* M.$$

In particular, the canonical ring

$$R(X,\omega_X) \simeq R(Y,M)$$

is not finitely generated, and it is easy to check that $d(X,0) < -1$.

Hence, we have to restrict ourselves to pairs with $d(X,\Delta) \geq -1$. We need the following definition.

Definition. A pair (X,Δ) has *log canonical singularities* (respectively *klt*, *canonical*, *terminal*) if $d(X,\Delta) \geq -1$ (respectively if > -1, ≥ 0, > 0).

Therefore, according to this definition and the previous example, the class of log canonical pairs is the largest class where the Minimal Model Program can be possibly expected to work. All smooth varieties X, viewed as pairs $(X,0)$, clearly belong to this class – indeed, they have terminal singularities.

Our experience of working in the Minimal Model Program shows that klt pairs behave much better than pairs with $d(X,\Delta) = -1$; moreover, currently we know many more results for klt pairs than for log canonical pairs in general. Also of importance for us is that being klt is an open condition, in the following sense. Say you have at hand a klt pair (X,Δ) with X being $\mathbb{Q}$-factorial, and that you have an effective $\mathbb{Q}$-divisor D on X. Then for all rational $0 \leq \varepsilon \ll 1$, the pair $(X,\Delta+\varepsilon D)$ is again klt. This is easy to see from the definition.

By what is said thus far, divisors of the form $K_X + \Delta$ are of special importance for us, and they are called *adjoint divisors*. We set up the Minimal Model Program in the case of pairs in exactly the same way as before, replacing K_X by $K_X + \Delta$ everywhere. We can now give a precise definition of minimal (or log terminal) models and of good models.

Definition 1.4. Let (X,Δ) be a $\mathbb{Q}$-factorial klt pair, and let $f\colon X \dashrightarrow Y$ be a birational contraction to a $\mathbb{Q}$-factorial variety.

(i) The map f is a *log terminal model for* (X,Δ) if $K_Y + f_*\Delta$ is nef, and if there exists a resolution $(p,q)\colon W \to X \times Y$ of the map f

$$\begin{array}{ccc} & W & \\ {}^{p}\swarrow & & \searrow^{q} \\ X & \overset{f}{\dashrightarrow} & Y \end{array}$$

such that

$$p^*(K_X+\Delta) = q^*(K_Y + f_*\Delta) + E,$$

where $E \geq 0$ is a q-exceptional $\mathbb{Q}$-divisor which contains the whole f-exceptional locus in its support.

(ii) If additionally $K_Y + f_*\Delta$ is semiample, the map f is a *good model* for (X,Δ).

1.4 Existence of good models

The existence of good models for klt pairs (X,Δ) with $K_X+\Delta$ pseudoeffective is the main outstanding conjecture in the Minimal Model Program for projective klt pairs in characteristic zero. It is well known that the existence of good models implies the Abundance conjecture.

If the Minimal Model Program holds, then the previous discussion shows that the study of all pairs (X,Δ) can be split into three main building blocks: when the divisor $K_X+\Delta$ is

(i) ample (this happens when we study the base of the Iitaka fibration on a good model),

(ii) trivial (this happens when we study a general fibre of the Iitaka fibration on a good model), or

(iii) anti-ample (this happens when we study a general fibre of a Mori fibre space).

The existence of good models for surfaces is classical, as explained above. For terminal threefolds, minimal models were constructed in [Mor88, Sho85], whereas minimal models of canonical fourfolds exist by [BCHM10, Fuj05].

In higher dimensions, the existence of minimal models for klt pairs of log general type is proved in [HM10, BCHM10], and by different methods in [CL12, CL13], whereas abundance holds for such pairs by [Sho85, Kaw85a]. Minimal models for effective klt pairs exist assuming the Minimal Model Program in lower dimensions [Bir11]. The abundance conjecture was proved in [Miy87, Miy88b, Miy88a, Kaw92] for terminal threefolds, and extended to log canonical threefold pairs (X,Δ) in [KMM94].

If proved, the existence of good models would imply that if (X,Δ) is a klt pair, then

$$K_X+\Delta \text{ is pseudoeffective if and only if it is effective,}$$

i.e. some multiple of $K_X+\Delta$ has global sections. This is analogous to the hard dichotomy on surfaces mentioned in Section 1.1. This statement, also known as *nonvanishing*, presents a large part of proving the existence of good models.

So far, we know the following result [BDPP13, Corollary 0.3].

Theorem 1.5. *Let X be a projective variety with canonical singularities. Then X is uniruled if and only if K_X is not pseudoeffective.*

Recall that a variety X of dimension n is uniruled if there is a dominant rational map

$$\mathbb{P}^1 \times Y \dashrightarrow X,$$

for some variety Y with $\dim Y = n-1$. This property is preserved in the birational equivalence class of X. We say that a pair (X,Δ) is uniruled if the underlying variety X is so, and similarly for a non-uniruled pair.

Therefore, it is a natural problem to try to prove the existence of good models for non-uniruled and uniruled pairs separately. To a certain extent, this was a strategy employed for threefold pairs in [KMM94]. The proof in [KMM94] proceeds by running a certain K_X-MMP which is $(K_X+\Delta)$-trivial, to end up either with a Mori fibre space, or with a model (Y,Δ_Y) on which $K_Y+(1-\varepsilon)\Delta_Y$ is nef for every $0\leq\varepsilon\ll 1$.

In the Mori fibre space case one is almost immediately done by induction on the dimension (even when one runs a similar strategy in higher dimensions), whereas in the second case one uses Chern classes, the geometry of surfaces and the case by case analysis of the numerical Kodaira dimension – the argument follows closely the proof for terminal threefolds by Miyaoka and Kawamata. A variation of the Mori fibre space case was implemented in higher dimensions in [DHP13], and we recall it in Theorem 2.16 below. However, this does not cover all uniruled pairs, as we explain in Remark 2.17.

In Chapter 2 we take a different approach to reduce to the case of smooth varieties with effective canonical class. We show that it suffices to prove the existence of good models and the abundance conjecture for non-uniruled pairs. More precisely:

Theorem A. *Assume the existence of good models for klt pairs in dimensions at most $n-1$.*

If the abundance conjecture holds for non-uniruled klt pairs in dimension n, then the abundance conjecture holds for uniruled klt pairs in dimension n.

Theorem B. *Assume the existence of good models for klt pairs in dimensions at most $n-1$.*

Then the existence of good models for non-uniruled klt pairs in dimension n implies the existence of good models for uniruled klt pairs in dimension n.

By taking a suitable partial resolution, every klt pair can be transformed into a terminal pair, cf. Theorem 2.4. Then by Theorem 1.5, Theorems A and B show that it suffices to prove the existence of good models and the abundance conjecture for terminal pairs (X,Δ) with K_X pseudoeffective. Therefore, this is a proper generalisation of the strategy employed for threefolds, and is the first reduction step towards the proof of the existence of good models.

In fact, we prove a much stronger result, which implies Theorems A and B.

Theorem 1.6. *Assume the existence of good models for klt pairs in dimensions at most* $n-1$.

If good models exist for log smooth klt pairs (X,Δ) *of dimension* n *such that the linear system* $|K_X|$ *is not empty, then good models exist for uniruled klt pairs in dimension* n.

1.5 The Cone conjecture

Recall again that, conjecturally, the study of algebraic varieties splits into three distinct cases: when K_X is either ample, anti-ample, or a torsion divisor. Much is known about the geometry (at least of moduli) in the first two cases. The third case, which I here call *varieties of Calabi-Yau type*, form a rich and extensively studied class.

If X is a variety, we denote by $N^1(X)_{\mathbb{R}}$ the real vector space of $\mathbb{R}$-Cartier divisors modulo numerical equivalence. Then it is a basic question what the shape of interesting cones in $N^1(X)_{\mathbb{R}}$ is.

From the point of view of birational geometry, the interesting cones are the cone of nef divisors $\mathrm{Nef}(X)$ and the movable cone $\overline{\mathrm{Mov}}(X)$ – this is the closure of the cone spanned by all effective Cartier divisors without divisors in their base loci. The nef cone is interesting as elements on its boundary give all morphisms to other varieties, and elements of the movable cone give all maps to other varieties.

In general, these cones can be very wild. However, it follows from Mori's Cone theorem that the nef cone of a Fano manifold is rational polyhedral, and the Minimal Model Program implies the same for the movable cone of a Fano manifold. We give another proof in Theorem 3.2, which rests on the finite generation of certain rings.

Of course, Calabi-Yau manifolds behave less well than Fano manifolds: for instance, it is not too difficult to construct examples of Calabi-

Yau manifolds for which the nef or the movable cone are not rational polyhedral; one such convenient example is Example 1.7. However, the Cone conjecture – introduced below – gives a description of these cones which is the best that we can ever hope for: it predicts that the nef and the movable cones on a Calabi-Yau manifold are *rational polyhedral up to the action of natural groups acting on them*.

Example 1.7. The following slight generalisation of [Ogu14, Proposition 6.1] is an example of a Calabi-Yau manifold whose movable cone is not rational polyhedral.

Let X be the complete intersection

$$H_1 \cap H_2 \cap \cdots \cap H_{n-1} \cap Q \subseteq \mathbb{P}^n \times \mathbb{P}^n,$$

where $n \geq 3$, where H_i are general hypersurfaces of bidegree $(1,1)$, and where Q is a general hypersurface of bidegree $(2,2)$. Then X is a simply connected Calabi-Yau n-fold with Picard number two. More precisely,

$$\mathrm{Pic}(X) = \mathbb{Z}L_1 \oplus \mathbb{Z}L_2,$$

where L_1 and L_2 are pullbacks of the hyperplane classes of factors $\mathbb{P}^n$.

Consider the two birational involutions ι_1, ι_2 induced by the two natural projections of X to $\mathbb{P}^n$. Then the boundary rays of the pseudoeffective cone (which, in this case, is the same as the movable cone) are both irrational, and $\iota_1\iota_2$ is a birational automorphism of X of infinite order. The last statement can be checked by computing $(\iota_1\iota_2)^* L_i$ as in [Ogu14, Proposition 6.1].

Consider a variety X of Calabi-Yau type, and denote by $\mathrm{Aut}(X)$ the automorphism group and by $\mathrm{Bir}(X)$ the group of birational automorphisms. Note that every element of $\mathrm{Bir}(X)$ is an automorphism in codimension 1, which is an easy consequence of Lemma 1.27 below. We have a natural homomorphism

$$r\colon \mathrm{Bir}(X) \to \mathrm{GL}(N^1(X))$$

given by $g \mapsto g^*$. We set

$$\mathscr{A}(X) = r\big(\mathrm{Aut}(X)\big) \quad \text{and} \quad \mathscr{B}(X) = r\big(\mathrm{Bir}(X)\big).$$

Remark 1.8. In general, on a variety X it is more convenient, in the context of the discussion on Fano manifolds below, to consider the group $\mathrm{PsAut}(X)$ of pseudo-automorphisms acting on $N^1(X)$ instead of the group of birational isomorphisms $\mathrm{Bir}(X)$: here, elements of $\mathrm{PsAut}(X)$ are birational automorphisms which are isomorphisms in codimension 1.

Thus, the question is – interesting on its own – how Aut(X) and Bir(X), or equivalently $\mathscr{A}(X)$ and $\mathscr{B}(X)$, act on certain cones in $N^1(X)_{\mathbb{R}}$. The first thing to notice is that $\mathscr{B}(X)$ preserves the effective cone Eff(X) (this is the cone in $N^1(X)_{\mathbb{R}}$ spanned by the numerical classes of effective Cartier divisors on X) and the movable cone $\overline{\mathrm{Mov}}(X)$, and that $\mathscr{A}(X)$ preserves the nef cone Nef(X). A precise answer to the question above is suggested by the following *Cone conjecture*.

But first we need a definition.

Definition 1.9. Let V be a real vector space equipped with a rational structure, and let $\mathscr{C}$ be a cone in V. Let Γ be a subgroup of GL(V) which preserves $\mathscr{C}$. A rational polyhedral cone $\Pi \subseteq \mathscr{C}$ is a *fundamental domain* for the action of Γ on $\mathscr{C}$ if the following holds:

(1) $\mathscr{C} = \bigcup_{g\in\Gamma} g\Pi$,

(2) $\operatorname{int}\Pi \cap \operatorname{int} g\Pi = \emptyset$ if $g \neq \mathrm{id}$.

Conjecture I. *Let X be a variety of Calabi-Yau type.*

(1) There exists a rational polyhedral cone Π which is a fundamental domain for the action of $\mathscr{A}(X)$ on $\mathrm{Nef}(X) \cap \mathrm{Eff}(X)$.

(2) There exists a rational polyhedral cone Π' which is a fundamental domain for the action of $\mathscr{B}(X)$ on $\overline{\mathrm{Mov}}(X) \cap \mathrm{Eff}(X)$.

A version of the first part of the conjecture was formulated by Morrison [Mor93] and was inspired by developments in mirror symmetry. Later it was extended to a version of the second part of the conjecture in [Mor96]. It was presented in the form as above in [Kaw97], and there is a formulation which involves klt pairs and pseudo-automorphisms in [Tot10]. More discussion about these versions of Conjecture I and their consequences is in Section 3.2 below.

The conjecture in its general form seems very difficult, and very little is known. The starting point is the proof of the conjecture on Calabi-Yau surfaces [Ste85, Nam85, Kaw97]. This was generalised by Totaro [Tot10] to klt Calabi-Yau pairs – the proof reinterprets the problem by using hyperbolic geometry. For abelian varieties, the proof is in [PS12].

A version for the movable cone on projective hyperkähler manifolds is in [Mar11], and a version for the nef cone on projective hyperkähler manifolds is in [AV14]. The proof of Conjecture I for the nef cone on projective hyperkähler manifolds of $K3^{[n]}$-type is in [MY14]. Oguiso

[Ogu11] gave a proof of the conjecture for the movable cone of generic hypersurfaces of multi-degree $(2,\dots,2)$ in $(\mathbb{P}^1)^n$ for $n \geq 4$.

In Chapter 3 we present the proof of the Cone conjecture for Calabi-Yau n-folds with Picard number 2 and infinite group $\mathrm{Bir}(X)$ from [LP13].

Theorem C. *Let X be a Calabi-Yau manifold with Picard number 2. If the group $\mathrm{Bir}(X)$ is infinite, then the Cone conjecture holds on X.*

The proof rests on previous work of Oguiso [Ogu14] on the birational automorphism group of Calabi-Yau manifolds with Picard number 2. This is one of the first results to treat the Cone conjecture in such a generality, and the first result to confirm it for a wide class of threefolds.

In fact, in Section 3.4 we explicitly calculate the groups $\mathscr{A}(X)$ and $\mathscr{B}(X)$ on a Calabi-Yau manifold with Picard number 2. A flavour of it is given in the following result.

Theorem 1.10. *Let X be a Calabi-Yau manifold of Picard number 2. Then*

$$\text{either } |\mathscr{A}(X)| \leq 2 \text{ or } \mathscr{A}(X) \text{ is infinite,}$$

and

$$\text{either } |\mathscr{B}(X)| \leq 2 \text{ or } \mathscr{B}(X) \text{ is infinite.}$$

Further discussion. Let us return again to Fano manifolds. As mentioned above, in Theorem 3.2 we show that the nef and movable cones on a Fano manifold X are rational polyhedral. Then the following result from convex geometry, applied to the vector space $V = N^1(X)_{\mathbb{R}}$ with the standard lattice L given by the Néron-Severi group $N^1(X)$ and the induced rational structure, gives that "on a Fano manifold the Cone conjecture holds", when either:

(a) the group $\mathrm{Aut}(X)$ is acting on the nef cone of X,

(b) the group $\mathrm{PsAut}(X)$ is acting on the movable cone of X.

Proposition 1.11. *Let V be a finite dimensional real vector space equipped with a rational structure, and let L be a lattice in V. Let $\mathscr{C}$ be a rational polyhedral cone in V of dimension $\dim V$. Let Γ be a subgroup of $\mathrm{GL}(V)$ which preserves L and $\mathscr{C}$.*

Then Γ is a finite group, and there exists a rational polyhedral fundamental domain for the action of Γ on $\mathscr{C}$.

Proof. Let $\delta_1, \dots, \delta_r$ be *primitive classes* on the extremal rays of the cone $\mathscr{C}$ (in the sense that they are integral classes not divisible in L). Then any element $g \in \Gamma$ permutes these δ_i: this follows since g preserves $\mathscr{C}$, and it sends a primitive class to a primitive class. Therefore, Γ is finite.

The proof of existence of a rational polyhedral fundamental domain is a bit more involved. For every point $x \in V$, let Σ_x denote the stabiliser of x in Γ. Pick a point $x_0 \in \mathscr{C}$ such that

$$\text{for every } z \in \mathscr{C} \text{ we have } |\Sigma_{x_0}| \leq |\Sigma_z|.$$

Then Σ_{x_0} is actually trivial. Indeed, there exists $0 < \varepsilon \ll 1$ such that if $B(x_0, \varepsilon)$ is the ε-ball around x_0 (in the standard norm), then the sets $g\big(B(x_0, \varepsilon) \cap \mathscr{C}\big)$ are pairwise disjoint for $g \notin \Sigma_{x_0}$. By the choice of x_0, this implies that

$$|\Sigma_{x_0}| = |\Sigma_z| \text{ for every } z \in B(x_0, \varepsilon) \cap \mathscr{C}.$$

Hence, for every $g \in \Sigma_{x_0}$ we have that g stabilises $B(x_0, \varepsilon) \cap \mathscr{C}$, and thus $g = \mathrm{id}$ since there exists a basis of V which belongs to $B(x_0, \varepsilon) \cap \mathscr{C}$.

If $\langle\ ,\ \rangle$ denotes the standard scalar product on $V \simeq \mathbb{R}^N$, for every $x, y \in V$ set

$$d(x, y) = \sum_{g \in \Gamma} \langle gx, gy \rangle.$$

Then it is easy to check that $d \colon V \times V \to \mathbb{R}$ is a scalar product, and that $d(x, y) = d(gx, gy)$ for every $x, y \in V$ and every $g \in \Gamma$. Let

$$\Pi = \{x \in \mathscr{C} \mid d(x, x_0) \leq d(x, gx_0) \text{ for every } g \in \Gamma\}.$$

Then Π is cut out from $\mathscr{C}$ by rational half-spaces, and hence Π is a rational polyhedral cone. I claim that Π is a fundamental domain for the action of Γ on $\mathscr{C}$. Indeed, take any $w \in \mathscr{C}$. Then there exists $h \in \Gamma$ such that $d(w, hx_0) \leq d(w, gx_0)$ for every $g \in \Gamma$. This is equivalent to

$$d(h^{-1}w, x_0) \leq d(h^{-1}w, h^{-1}gx_0)$$

for every $g \in \Gamma$, and hence $h^{-1}w \in \Pi$. Therefore,

$$\mathscr{C} = \bigcup_{g \in \Gamma} g\Pi.$$

Since $\Sigma_{x_0} = \{\mathrm{id}\}$, we have $\operatorname{int}\Pi \cap \operatorname{int} g\Pi = \emptyset$ unless $g = \mathrm{id}$ by definition of Π. This completes the proof. □

Finally, we discuss some of the formulations of the Cone conjecture, and which consequences it has for the geometry of a Calabi-Yau. The following result shows that the existence of good models and the Cone conjecture are, in some sense, consistent.

Proposition 1.12. *Let X be an n-dimensional variety of Calabi-Yau type. Assume either the existence of good models in dimension n, or the Cone conjecture in dimension n.*

Then the cones $\mathrm{Nef}(X) \cap \mathrm{Eff}(X)$ *and* $\overline{\mathrm{Mov}}(X) \cap \mathrm{Eff}(X)$ *are spanned by rational divisors.*

Proof. I only show the statements for $\mathrm{Nef}(X) \cap \mathrm{Eff}(X)$, the rest is analogous.

Assume the existence of good models in dimension n. Let D be an $\mathbb{R}$-divisor whose class is in $\mathrm{Nef}(X) \cap \mathrm{Eff}(X)$. Then we can write $D \equiv \sum_{i=1}^{r} \delta_i D_i$ for prime divisors D_i and positive real numbers δ_i. Fix an ample $\mathbb{Q}$-divisor A on X. By Theorem 3.8, the ring

$$R(X; D_1, \ldots, D_r, A)$$

is finitely generated, and hence, the cone

$$\mathcal{N} = \pi^{-1}\big(\mathrm{Nef}(X)\big) \cap \sum \mathbb{R}_+ D_i$$

is rational polyhedral by Proposition 3.5, where $\pi\colon \mathrm{Div}_{\mathbb{R}}(X) \to N^1(X)_{\mathbb{R}}$ is the natural map. Since $\pi(D) \in \mathcal{N}$, the result follows.

Now assume the Cone conjecture in dimension n. Let D be an $\mathbb{R}$-divisor whose class is in $\mathrm{Nef}(X) \cap \mathrm{Eff}(X)$, and let Π be the fundamental domain for the action of $\mathscr{A}(X)$ on $\mathrm{Nef}(X) \cap \mathrm{Eff}(X)$. Then there exists $g \in \mathscr{A}(X)$ such that $D \in g\Pi$, and the conclusion follows since $g\Pi$ is a rational polyhedral cone. □

We end this discussion with a recent result of Looijenga [Loo14, Theorem 4.1, Application 4.15]. The result belongs completely to the realm of convex geometry; however, we will see that it has far-reaching consequences in our situation.

Theorem 1.13. *Let V be a real vector space equipped with a rational structure $V(\mathbb{Q})$, and let L be a lattice in V. Let $\mathscr{C}$ be an open cone in V. Let Γ be a subgroup of* $\mathrm{GL}(V)$ *which preserves L and $\mathscr{C}$. Let $\mathscr{C}_+$ denote the convex hull of the set $\overline{\mathscr{C}} \cap V(\mathbb{Q})$. Assume that there exists a polyhedral cone Π in $\mathscr{C}_+$ with $\mathscr{C} \subseteq \Gamma \cdot \Pi$.*

Then $\Gamma \cdot \Pi = \mathscr{C}_+$, and there exists a rational polyhedral fundamental domain for the action of Γ on $\mathscr{C}_+$.

This is a remarkable result: it shows that as long as we find a *covering* rational polyhedral cone, then the existence of the fundamental domain is automatic.

Note that Theorem 1.13 in particular implies the following.

Corollary 1.14. *Assume Conjecture II in dimension n. Let X be an n-dimensional variety with terminal singularities and of Calabi-Yau type. Then every nef line bundle on X is semiample.*

Note that this result implies something much stronger than the Abundance conjecture: indeed, the Abundance conjecture implies semi-ampleness of every *effective* nef line bundle on a terminal variety of Calabi-Yau type.

This seems to be a believed conjecture, although it is not clear what the evidence for it is. It is worth noting that the original form of the Cone conjecture in [Mor93] did not involve the cone $\mathrm{Nef}(X) \cap \mathrm{Eff}(X)$, but the cone $\mathrm{Nef}(X)_+$ (i.e. the convex hull of the cone $\mathrm{Nef}(X) \cap N^1(X)_{\mathbb{Q}}$), which is consistent with the convex-geometric Theorem 1.13. The cone $\mathrm{Eff}(X)$ entered the formulation in [Kaw97].

We prove in Theorem 3.13 below that

$$\mathrm{Nef}(X) \cap \mathrm{Eff}(X) \subseteq \mathrm{Nef}(X)_+.$$

The Cone conjecture would imply that this inclusion is actually an equality. In all known cases this is, of course, true.

1.6 Number of good models

In [Kaw97], Kawamata formulated the following generalisation of the Cone conjecture in the relative setting.

Conjecture II. *Let X be a normal projective variety of relative Calabi-Yau type, i.e. assume there exists a fibration $X \to S$ such that $K_X \equiv_S 0$.*

(1) There exists a rational polyhedral cone Π which is a fundamental domain for the action of $\mathscr{A}(X/S)$ on $\mathrm{Nef}(X/S) \cap \mathrm{Eff}(X/S)$.

(2) There exists a rational polyhedral cone Π' which is a fundamental domain for the action of $\mathscr{B}(X/S)$ on $\overline{\mathrm{Mov}}(X/S) \cap \mathrm{Eff}(X/S)$.

Here, of course, all groups and cones are the relative analogues of the absolute setting from before, where S was a point.

It is an important and long-standing conjecture that the number of minimal models of a smooth projective variety is finite up to isomorphism. This is known for projective varieties of general type [BCHM10]. A positive answer to Conjecture II together with the existence of good models would imply that the number of minimal models of a terminal variety is finite up to isomorphism; we show this in Theorem 3.11. This gives the main motivation for the Cone conjecture in the realm of birational geometry.

Kawamata [Kaw97] gave a proof of (a weaker form of) Conjecture II when $X \to S$ is a 3-fold over a positive-dimensional base. This, in particular, showed that if X is a 3-fold with positive Kodaira dimension, then the number of its minimal models is finite up to isomorphisms.

One might wonder how much of a birational geometry of a projective variety is captured in its topology. One way to quantify this is to speculate that *the number of minimal models of a smooth projective variety is bounded with respect to its underlying topology as a complex manifold*.

This belief also has roots in other results in the field. According to philosophy starting with [Kol86], vanishing and injectivity theorems in cohomology hold due to topological reasons, and Kollár's effective basepoint freeness [Kol93] gives bounds that depend only on the dimension of a variety. The finite generation of adjoint rings can be proved as a consequence of the Kawamata-Viehweg vanishing theorem [CL12], and this in turns implies finiteness of minimal models for a given pair of log general type [CL13, KKL12]. More precisely, the number of minimal models of a pair (X,Δ) is related to the number of generators of a suitable adjoint ring.

The results of Chapter 4 represent the first attempt to bound the number of minimal models of a given log smooth pair of dimension 3 with respect to the underlying topology as a complex manifold. Two log smooth pairs (X_1,Δ_1) and (X_2,Δ_2) are said to be of the same *topological type* if there is a homeomorphism $\varphi\colon X_1 \to X_2$ which is a homeomophism between $\operatorname{Supp}\Delta_1$ and $\operatorname{Supp}\Delta_2$. The main result of Chapter 4 is the following.

Theorem D. *Let ε be a positive number. Let $\mathfrak{X}$ be the collection of all log smooth 3-fold terminal pairs $(X,\Delta = \sum_{i=1}^{p} \delta_i S_i)$ such that:*

(1) X is not uniruled,

(2) $\varepsilon \le \delta_i \le 1-\varepsilon$ for all i,

(3) $S_1,\dots,S_p$ are distinct prime divisor not contained in

$$\mathbf{B}(K_X + \sum_{i=1}^{p} a_i S_i)$$

for all $0 \le a_i \le 1$*, and*

(4) S_i *span* $\mathrm{Div}_{\mathbb{R}}(X)$ *up to numerical equivalence.*

Then for every $(X_0,\Delta_0) \in \mathfrak{X}$ *there exists a constant* N *such that for every* $(X,\Delta) \in \mathfrak{X}$ *of the topological type as* (X_0,Δ_0)*, the number of log terminal models of* (X,Δ) *is bounded by* N*.*

Here we refer to the definition of the stable base locus in Section 1.8. Theorem D combined with the Cone conjecture suggests that the number of faces of the fundamental domain of the action of the group of birational automorphisms on the movable cone of a Calabi-Yau manifold X is determined by the topological type of X.

1.7 Finding a right general setup

As Example 1.3 shows, there are indeed situations where the classical Minimal Model Program cannot work for the canonical class. On the other hand, there is a special class of varieties, called Mori Dream Spaces, where we can do a version of the MMP for *every* effective divisor.

Definition 1.15. A projective $\mathbb{Q}$-factorial variety X is a Mori Dream Space if

(1) $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Q}}$,

(2) $\mathrm{Nef}(X)$ is the affine hull of finitely many semiample line bundles, and

(3) there are finitely many birational maps $f_i\colon X \dashrightarrow X_i$ to projective $\mathbb{Q}$-factorial varieties X_i such that each f_i is an isomorphism in codimension 1, each X_i satisfies (2), and

$$\overline{\mathrm{Mov}}(X) = \bigcup f_i^*\big(\mathrm{Nef}(X_i)\big).$$

If $D_1,\dots,D_r$ is a basis of $\mathrm{Pic}(X)_{\mathbb{Q}}$ such that $\overline{\mathrm{Eff}}(X) \subseteq \sum \mathbb{R}_+ D_i$, then

$$R(X;D_1,\dots,D_r) = \bigoplus_{(n_1,\dots,n_r)\in\mathbb{N}^r} H^0(X, n_1D_1+\dots+n_rD_r)$$

is a *Cox ring* of X. The finite generation of this ring is independent of the choice of $D_1,\dots,D_r$.

The class of Mori Dream Spaces was introduced in [HK00]. It contains, for instance, all toric varieties [HK00] or Fano varieties, see Corollary 5.22. Of course, this is an exceptionally nice example, and we would like to find, in some sense the *maximal* class of varieties where a version of the Minimal Model Program can be performed.

Maybe it is too much to hope that there exists such a class which contains both the setup of the classical MMP as well as Mori Dream Spaces, since they can be, in some sense, unrelated or only loosely related. However, we will see in Chapter 5 that we can indeed build a theory which contains both of these *pictures* as special instances.

Say we have a $\mathbb{Q}$-factorial projective variety X and a $\mathbb{Q}$-divisor D on X; note that here we allow X to be *arbitrarily* singular. Then the group of global sections of D is

$$H^0(X,D) = \{f \in k(X) \mid \operatorname{div} f + D \geq 0\},$$

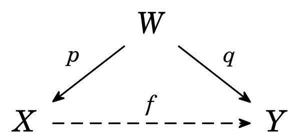
and the associated *section ring* is defined as

$$R(X,D) = \bigoplus_{m\in\mathbb{N}} H^0(X,mD).$$

Analogously to the case of adjoint divisors, we can give a good definition of a good model for D.

Definition 1.16. Let $D \in \mathrm{Div}_{\mathbb{R}}(X)$ and let $\varphi\colon X \dashrightarrow Y$ be a contraction map to a normal projective variety Y such that $D' = \varphi_* D$ is $\mathbb{R}$-Cartier.

(1) The map φ is *D-nonpositive* (respectively *D-negative*) if it is birational, and for a common resolution $(p,q)\colon W \to X \times Y$



we can write $p^*D = q^*D' + E$, where $E \geq 0$ is q-exceptional (respectively $E \geq 0$ is q-exceptional and $\operatorname{Supp} E$ contains the strict transform of the φ-exceptional divisors). In particular, $H^0(X,D) \simeq H^0(Y,\varphi_* D)$ and

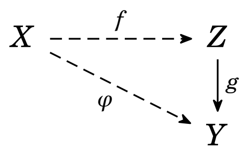
$$R(X,D) \simeq R(Y, f_* D).$$

(2) The map φ is an *optimal model* of D if φ is D-negative, Y is $\mathbb{Q}$-factorial and D' is nef.

(3) The map φ is a *semiample model* of D if φ is D-nonpositive and D' is semiample.

(4) The map φ is a *good model* of D if φ is an optimal model such that D' is semiample.

(5) The map φ is the *ample model* of D if there exist a birational contraction $f\colon X \dashrightarrow Z$ which is a semiample model of D, and a morphism with connected fibres $g\colon Z \to Y$ such that $\varphi = g \circ f$ and $f_*D \sim_{\mathbb{Q}} g^*A$, where A is an ample $\mathbb{R}$-divisor on Y.



Remark 1.17. The ample model is unique up to isomorphism. Indeed, with the notation from the definition, we have $R(X,pD) \simeq R(Z,pf_*D)$ for some large positive integer p. This last ring is isomorphic to $R(Y,pA)$, and therefore $Y \simeq \operatorname{Proj} R(X,D)$.

We first notice that, if an MMP can be performed for our $\mathbb{Q}$-divisor D (in other words, if a good model for D exists), then D cannot be *isolated* in the Néron-Severi space $N^1(X)_{\mathbb{R}}$. The following lemma makes this more precise, but first we need a definition.

Definition 1.18. If X is a normal projective variety, and if $\mathscr{S} \subseteq \operatorname{Div}_{\mathbb{Q}}(X)$ is a finitely generated monoid, then

$$R(X,\mathscr{S}) = \bigoplus_{D\in\mathscr{S}} H^0\big(X,D\big)$$

is a *divisorial $\mathscr{S}$-graded ring*. If $\mathscr{C} \subseteq \operatorname{Div}_{\mathbb{R}}(X)$ is a rational polyhedral cone, then $\mathscr{S} = \mathscr{C} \cap \operatorname{Div}(X)$ is a finitely generated monoid by Gordan's lemma, and we define

$$R(X,\mathscr{C}) := R(X,\mathscr{S}).$$

If $D_1,\dots,D_r$ be $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors on X, then we have the associated *divisorial ring*

$$\mathfrak{R} = R(X;D_1,\dots,D_r) = \bigoplus_{(n_1,\dots,n_r)\in\mathbb{N}^r} H^0(X,n_1D_1+\cdots+n_rD_r).$$

When all D_i are adjoint divisors, then the ring $\mathfrak{R}$ is an *adjoint ring*. The *support* of $\mathfrak{R}$ is the cone

$$\operatorname{Supp}\mathfrak{R} = \{D \in \sum \mathbb{R}_+ D_i \mid |D|_{\mathbb{R}} \neq \emptyset\} \subseteq \operatorname{Div}_{\mathbb{R}}(X),$$

and similarly for rings of the form $R(X,\mathscr{C})$.

Lemma 1.19. *Let X be a $\mathbb{Q}$-factorial projective variety, and let D be a $\mathbb{Q}$-divisor on X. Assume that there exists a good model for D, and let $\pi\colon \operatorname{Div}_{\mathbb{R}}(X) \to N^1(X)_{\mathbb{R}}$ be the natural projection.*

Then there exist $\mathbb{Q}$-divisors $D_1,\dots,D_r$ such that

(1) $D \in \sum \mathbb{R}_+ D_i \subseteq \operatorname{Div}_{\mathbb{R}}(X)$,

(2) $\dim \pi(\sum \mathbb{R}_+ D_i) = \dim N^1(X)_{\mathbb{R}}$,

(3) the ring $R(X;D_1,\dots,D_r)$ is finitely generated.

Proof. We assume the notation as above. In particular, let $f\colon X \dashrightarrow Y$ be a good model for D. Since f_*D is semiample, there exist semiample $\mathbb{Q}$-divisors $G_1,\dots,G_m$ on Y such that:

(i) $f_*D \in \sum \mathbb{R}_+ G_i \subseteq \operatorname{Div}_{\mathbb{R}}(Y)$,

(ii) the dimension of the image of the cone $\sum \mathbb{R}_+ G_i$ in $N^1(Y)_{\mathbb{R}}$ is maximal, and

(iii) the ring $R(Y;G_1,\dots,G_m)$ is finitely generated.

Indeed, we take $G_1 = f_*D$, and we can pick $G_2,\dots,G_m$ to be ample.

If $E_1,\dots,E_\ell$ are the prime divisors contracted by f, then we have

$$D = f^*f_*D + \sum r_i E_i$$

for some $r_i \geq 0$. Now we define $D_1,\dots,D_r$, with $r = m+\ell$, as follows. Set

$$D_i = f^*G_i$$

for $i = 1,\dots,m$, and set

$$D_{m+i} = f^*G_1 + \lambda_i E_i$$

for $i = 1,\dots,\ell$, where $\lambda_i = \ell r_i$. Then it is easy to see that (1) and (2) hold. It remains to show that the ring $R(X;D_1,\dots,D_r)$ is finitely generated.

For non-negative integers $k_1,\ldots,k_r$, denote $D_{k_1,\ldots,k_r} = \sum k_i D_i$, and note that

$$D_{k_1,\ldots,k_r} = \sum_{i=1}^m f^*(k_i G_i) + \left(\sum_{i=m+1}^r k_i\right) f^* G_1 + \sum_{i=m+1}^r k_i \lambda_i E_i.$$

This implies

$$H^0(X, D_{k_1,\ldots,k_r}) = H^0\Big(X, \sum_{i=1}^m k_i D_i + \left(\sum_{i=m+1}^r k_i\right) D_1\Big),$$

and thus

$$R(X; D_1,\ldots,D_r) \simeq R(X; D_1,\ldots,D_m, D_1,\ldots,D_1).$$

Now, this last ring is finitely generated by Lemma 1.21 below, as the ring

$$R(X; D_1,\ldots,D_m) \simeq R(Y; G_1,\ldots,G_m)$$

is finitely generated. □

Therefore, Lemma 1.19 says that unless we have a finitely generated divisorial ring $\mathfrak{R}$ such that $D \in \operatorname{Supp}\mathfrak{R}$ which is *full* (in the sense that the image of $\operatorname{Supp}\mathfrak{R}$ in $N^1(X)_{\mathbb{R}}$ is maximal dimensional), then we stand no chance of ever performing the Minimal Model Program for this D.

With notation from Lemma 1.19, we have the graded ring

$$\mathfrak{R} = R(X; D_1,\ldots,D_r),$$

and we want to determine *sufficient* conditions to allow us to perform a Minimal Model Program for every divisor in $\operatorname{Supp}\mathfrak{R}$. We will see in Theorem 5.9 that there exist finitely many natural maps

$$\varphi_i \colon X \dashrightarrow X_i$$

associated to a certain decomposition of $\operatorname{Supp}\mathfrak{R}$. A fundamental requirement is that all X_i are $\mathbb{Q}$-factorial varieties. The varieties X_i are isomorphic to $\operatorname{Proj} R(X, G_i)$ for some $\mathbb{Q}$-divisors G_i in the interior of $\operatorname{Supp}\mathfrak{R}$.

Let G_i' be any $\mathbb{Q}$-divisor such that $G_i \equiv G_i'$. If X_i is $\mathbb{Q}$-factorial, then in particular, the divisor $(\varphi_i)_* G_i'$ is $\mathbb{Q}$-Cartier. It is easy to show, see Lemma 5.13, that in that case, the section ring $R(X, G_i')$ is also finitely generated. Therefore, the divisors in the interior of $\operatorname{Supp}\mathfrak{R}$ must be pretty special – it is not in general true that finite generation of section rings is a numerical property, see Example 5.14. These divisors deserve a special name.

Definition 1.20. Let X be a $\mathbb{Q}$-factorial projective variety. A $\mathbb{Q}$-divisor D is *gen* if for every $\mathbb{Q}$-divisor $D' \equiv D$, the section ring $R(X,D')$ is finitely generated.

Therefore, our last requirement must be that all the divisors in the interior of $\operatorname{Supp}\mathfrak{R}$ are gen. The main result of Chapter 5 is the following.

Theorem E. *Let X be a projective $\mathbb{Q}$-factorial variety, let $D_1,\dots,D_r$ be effective $\mathbb{Q}$-divisors on X, and assume that the numerical classes of D_i span $N^1(X)_{\mathbb{R}}$. Assume that the ring*

$$R(X;D_1,\dots,D_r)$$

is finitely generated, that the cone $\sum \mathbb{R}_+ D_i$ contains an ample divisor, and that every divisor in the interior of this cone is gen.

Then there is a finite decomposition

$$\sum \mathbb{R}_+ D_i = \coprod \mathcal{N}_i$$

into cones having the following properties:

(1) each $\overline{\mathcal{N}_i}$ is a rational polyhedral cone,

(2) for each i, there exists a $\mathbb{Q}$-factorial projective variety X_i and a birational contraction $\varphi_i\colon X \dashrightarrow X_i$ such that φ_i is a good model for every divisor in $\mathcal{N}_i$.

In fact, we prove a stronger result: we show that for any $\mathbb{Q}$-divisor $D \in \sum \mathbb{R}_+ D_i$, we can run a D-MMP which terminates, see Theorem 5.19 for the precise statement. The decomposition in Theorem E determines a *geography of optimal models* associated to $R(X;D_1,\dots,D_r)$. This also allows us to recover some of the main results of [BCHM10] and [HK00], see Corollaries 5.21 and 5.22.

1.8 Notation and conventions

Throughout the manuscript, unless otherwise stated all varieties are normal and projective, and everything happens over the complex numbers. We denote by $\mathbb{R}_+$ and $\mathbb{Q}_+$ the sets of non-negative real and rational numbers. A pair (X,Δ) is *log smooth* if X is smooth and if the support of Δ has simple normal crossings.

I follow notation and conventions from [Laz04], and anything which is not explicitly defined here, can be found there.

Apart from the notation introduced thusfar, we need several more concepts. Additional notation will be introduced in each chapter if necessary.

Divisors and line bundles. Let X be a normal projective variety and let $\mathbf{k} \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$. We denote by $\mathrm{Div}_{\mathbf{k}}(X)$ the group of $\mathbf{k}$-Cartier $\mathbf{k}$-divisors on X, and $\sim_{\mathbf{k}}$ and $\equiv$ denote the $\mathbf{k}$-linear and numerical equivalence of $\mathbb{R}$-divisors. If there is a morphism $X \to Y$ to another normal projective variety, numerical equivalence over Y is denoted by $\equiv_Y$. We denote

$$\mathrm{Pic}(X)_{\mathbf{k}} = \mathrm{Div}_{\mathbf{k}}(X)/\sim_{\mathbf{k}} \quad \text{and} \quad N^1(X)_{\mathbf{k}} = \mathrm{Div}_{\mathbf{k}}(X)/\equiv.$$

As mentioned above, $\mathrm{Nef}(X) \subseteq N^1(X)_{\mathbb{R}}$ denotes the closed cone of nef divisors, $\mathrm{Big}(X)$ stands for the open cone of big divisors, $\overline{\mathrm{Mov}}(X)$ is the closure of the cone generated by mobile divisors (that is, effective divisors whose base locus does not contain divisors), and $\mathrm{Mov}(X)$ is its interior. Finally, $\mathrm{Eff}(X)$ is the effective cone, and $\overline{\mathrm{Eff}}(X)$ is the pseudo-effective cone (the closure of the effective cone, or equivalently, the closure of the big cone).

If X is a normal projective variety and if D is an integral divisor on X, we denote by $\mathrm{Bs}\,|D|$ the base locus of D, whereas $\mathrm{Fix}\,|D|$ and $\mathrm{Mob}(D)$ denote the fixed and mobile parts of D. If S is a prime divisor on X such that $S \not\subseteq \mathrm{Bs}\,|D|$, then $|D|_S$ denotes the image of the linear system $|D|$ under restriction to S. If D is an $\mathbb{R}$-divisor on X, we denote

$$|D|_{\mathbb{R}} = \{D' \in \mathrm{Div}_{\mathbb{R}}(X) \mid D \sim_{\mathbb{R}} D' \geq 0\} \quad \text{and} \quad \mathbf{B}(D) = \bigcap_{D' \in |D|_{\mathbb{R}}} \mathrm{Supp}\, D',$$

and we call $\mathbf{B}(D)$ the *stable base locus* of D. If A is any ample divisor on X, then

$$\mathbf{B}_+(D) = \bigcap_{\varepsilon > 0} \mathbf{B}(D - \varepsilon A)$$

is the *augmented base locus* of D, and we clearly have

$$\mathbf{B}(D) \subseteq \mathbf{B}_+(D).$$

Divisorial rings. In the manuscript, we use several properties of finitely generated divisorial rings without explicit mention, see [CL12, §2.4]. The one we use most is recalled in the following lemma.

Lemma 1.21. *Let X be a normal projective variety, let $D_1, \dots, D_r$ be divisors in $\mathrm{Div}_{\mathbb{Q}}(X)$, and let $p_1, \dots, p_r$ be positive rational numbers.*

Then the ring $R(X; D_1, \dots, D_r)$ is finitely generated if and only if the ring $R(X; p_1 D_1, \dots, p_r D_r)$ is finitely generated.

Asymptotic valuations. A geometric valuation Γ on a normal variety X is a valuation on the function field $k(X)$ given by the order of vanishing at the generic point of a prime divisor on some proper birational model $f\colon Y \to X$; by abusing notation, we identify Γ with the corresponding prime divisor. If D is an $\mathbb{R}$-Cartier divisor on X, we use $\operatorname{mult}_\Gamma D$ to denote $\operatorname{mult}_\Gamma f^*D$. The set $f(\Gamma)$ is the *centre of* Γ *on* X and is denoted by $c_X(\Gamma)$.

Definition 1.22. Let X be a normal projective variety, let D be an $\mathbb{R}$-Cartier divisor such that $|D|_{\mathbb{R}} \neq \emptyset$, and let Γ be a geometric valuation over X. The *asymptotic order of vanishing* of D along Γ is

$$o_\Gamma(D) = \inf\{\operatorname{mult}_\Gamma D' \mid D' \in |D|_{\mathbb{R}}\}.$$

Finite generation of a divisorial ring $\mathfrak{R}$ has important consequences on the behavior of the asymptotic order functions, and therefore on the convex geometry of its support $\operatorname{Supp}\mathfrak{R}$, as observed in [ELM$^+$06].

Theorem 1.23. *Let X be a projective $\mathbb{Q}$-factorial variety, and let $\mathscr{C} \subseteq \operatorname{Div}_{\mathbb{R}}(X)$ be a rational polyhedral cone. Assume that the ring $\mathfrak{R} = R(X, \mathscr{C})$ is finitely generated. Then:*

(1) $\operatorname{Supp}\mathfrak{R}$ *is a rational polyhedral cone,*

(2) if $\operatorname{Supp}\mathfrak{R}$ *contains a big divisor, then all pseudo-effective divisors in* $\operatorname{Supp}\mathfrak{R}$ *are in fact effective,*

(3) there is a finite rational polyhedral subdivision $\operatorname{Supp}\mathfrak{R} = \bigcup \mathscr{C}_i$ *such that o_Γ is linear on $\mathscr{C}_i$ for every geometric valuation Γ over X, and the cones $\mathscr{C}_i$ form a fan,*

(4) there is a positive integer d and a resolution $f\colon \widetilde{X} \to X$ such that $\operatorname{Mob} f^*(dD)$ *is basepoint free for every* $D \in \operatorname{Supp}\mathfrak{R} \cap \operatorname{Div}(X)$*, and*

$$\operatorname{Mob} f^*(kdD) = k \operatorname{Mob} f^*(dD)$$

for every positive integer k.

Proof. This is essentially [ELM$^+$06, Theorem 4.1], see [CL13, Theorem 3.6]. □

Convex geometry. Let $\mathscr{C} \subseteq \mathbb{R}^N$ be a convex set. A subset $F \subseteq \mathscr{C}$ is a *face* of $\mathscr{C}$ if it is convex, and whenever $tu + (1-t)v \in F$ for some $u, v \in \mathscr{C}$ and $0 < t < 1$, then $u, v \in F$. Note that $\mathscr{C}$ is itself a face of $\mathscr{C}$. We say that $x \in \mathscr{C}$ is an *extreme point* of $\mathscr{C}$ if $\{x\}$ is a face of $\mathscr{C}$.

The topological closure of a set $\mathscr{S} \subseteq \mathbb{R}^N$ is denoted by $\overline{\mathscr{S}}$. The boundary of a closed set $\mathscr{C} \subseteq \mathbb{R}^N$ is denoted by $\partial\mathscr{C}$.

A *rational polytope* in $\mathbb{R}^N$ is a compact set which is the convex hull of finitely many rational points in $\mathbb{R}^N$. A *rational polyhedral cone* in $\mathbb{R}^N$ is a convex cone spanned by finitely many rational vectors. The dimension of a cone in $\mathbb{R}^N$ is the dimension of the minimal $\mathbb{R}$-vector space containing it.

A finite rational polyhedral subdivision $\mathscr{C} = \bigcup \mathscr{C}_i$ of a rational polyhedral cone $\mathscr{C}$ is a *fan* if each face of $\mathscr{C}_i$ is also a cone in the decomposition, and the intersection of two cones in the decomposition is a face of each.

We need some naturally defined convex sets on the space of divisors.

Definition 1.24. Let X be a projective $\mathbb{Q}$-factorial variety, let $S_1,\dots, S_p$ be distinct prime divisors on X, denote $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$, and let A be an ample $\mathbb{Q}$-divisor on X. We define

$$\mathscr{L}(V) = \{\Delta \in V \mid (X,\Delta) \text{ is log canonical}\},$$
$$\mathscr{E}_A(V) = \{\Delta \in \mathscr{L}(V) \mid |K_X + A + \Delta|_{\mathbb{R}} \neq \emptyset\}.$$

It is easy to check that $\mathscr{L}(V)$ is a rational polytope, cf. [BCHM10, Lemma 3.7.2]. On the other hand, the fact that $\mathscr{E}_A(V)$ is a rational polytope is much harder, see Corollary 1.26.

Finitely generated adjoint rings. Lemma 1.19 shows that the existence of good models implies finite generation of certain multi-graded rings. In the case of adjoint divisors, this is indeed now a theorem, proved in [BCHM10, HM10], and also in [CL12] by different methods.

Theorem 1.25. *Let X be a $\mathbb{Q}$-factorial projective variety, and let $\Delta_1,\dots,\Delta_r$ be $\mathbb{Q}$-divisors such that all pairs (X,Δ_i) are klt.*

(1) If $A_1,\dots,A_r$ are ample $\mathbb{Q}$-divisors, then the adjoint ring

$$R(X;K_X+\Delta_1+A_1,\dots,K_X+\Delta_r+A_r)$$

is finitely generated.

(2) If Δ_i are big, then the adjoint ring

$$R(X;K_X+\Delta_1,\dots,K_X+\Delta_r)$$

is finitely generated.

We then have the following easy corollary.

Corollary 1.26. *Let X be a projective $\mathbb{Q}$-factorial variety, let $S_1,\dots, S_p$ be distinct prime divisors on X, denote $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$, and let A be an ample $\mathbb{Q}$-divisor on X. Let $\mathscr{C} \subseteq \mathscr{L}(V)$ be a rational polytope such that for every $\Delta \in \mathscr{C}$, the pair (X,Δ) is klt.*

Then the set $\mathscr{C} \cap \mathscr{E}_A(V)$ is a rational polytope, and the ring

$$R(X, \mathbb{R}_+(K_X + A + \mathscr{C} \cap \mathscr{E}_A(V)))$$

is finitely generated.

Proof. Let $B_1,\dots,B_r$ be the vertices of $\mathscr{C}$. Then the ring

$$\mathfrak{R} = R(X; K_X + B_1 + A, \dots, K_X + B_r + A)$$

is finitely generated by Theorem 1.25, which implies the second claim since there is a natural surjection from $\mathfrak{R}$ to $R(X, \mathbb{R}_+(K_X + A + \mathscr{C} \cap \mathscr{E}_A(V)))$. Since

$$\operatorname{Supp} \mathfrak{R} = \mathbb{R}_+(K_X + A + \mathscr{C} \cap \mathscr{E}_A(V)).$$

the first claim follows from Theorem 1.23(i). □

Negativity Lemma. We recall the following important result known as the Negativity lemma, see [K+92, Lemma 2.19].

Lemma 1.27. *Let $f: X \to Y$ be a proper birational morphism, where X is normal, and let E be an f-exceptional divisor on X. Assume that*

$$E \equiv_Y H + D,$$

where H is f-nef and $D \geq 0$ has no common components with E. Then $E \leq 0$.

The following corollary, cf. [BCHM10, Lemma 3.6.4], will be used in Chapter 5.

Corollary 1.28. *Let $X \to Z$ and $Y \to Z$ be projective morphisms of normal projective varieties. Let $f: X \dashrightarrow Y$ be a birational contraction over Z, and let $(p,q): W \to X \times Y$ be a resolution of f. Let D and D' be $\mathbb{R}$-Cartier divisors on X such that f_*D and f_*D' are $\mathbb{R}$-Cartier on Y, and assume that $D \equiv_Z D'$. Then*

$$p^*D - q^*f_*D = p^*D' - q^*f_*D'.$$

In particular, f is D-nonpositive (respectively D-negative) if and only if f is D'-nonpositive (respectively D'-negative).

Proof. The divisor $E = p^*(D - D') - q^*f_*(D - D')$ is q-exceptional since f is a contraction, and we have $E \equiv_Y 0$. We conclude by Lemma 1.27. □

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Chapter 2

The existence of good models

2.1 Introduction

The results of this chapter are taken from [DL14]. We prove the following results announced in Chapter 1.

Theorem A. *Assume the existence of good models for klt pairs in dimensions at most* $n-1$.

If the abundance conjecture holds for non-uniruled klt pairs in dimension n, then the abundance conjecture holds for uniruled klt pairs in dimension n.

Theorem B. *Assume the existence of good models for klt pairs in dimensions at most* $n-1$.

Then the existence of good models for non-uniruled klt pairs in dimension n implies the existence of good models for uniruled klt pairs in dimension n.

We briefly explain the strategy of the proof. If (X,Δ) is a uniruled klt pair, then by [DHP13, Proposition 8.7] we may assume that the adjoint divisor $K_X+\Delta$ is effective; we reprove this result below in Theorem 2.16. We first show that we may furthermore assume that X is smooth and Δ is a reduced simple normal crossings divisor, and that there exists an effective $\mathbb{Q}$-divisor D such that $K_X+\Delta\sim_{\mathbb{Q}}D$ and the supports of Δ and D are the same. Then we use ramified covers, dlt models and log resolutions to construct a log smooth pair (W,Δ_W) and a generically finite morphism $w\colon W\to X$ such that K_W is an effective divisor – we do this by carefully analysing the behaviour of valuations under finite morphisms. We conclude by the construction of w and since the Kodaira dimension and the numerical Kodaira dimension are preserved under proper morphisms, cf. Lemma 2.7.

In fact, our techniques lead to the following main technical result of the chapter, which implies Theorems A and B.

Theorem 2.1. *Assume the existence of good models for klt pairs in dimensions at most* $n-1$.

If good models exist for log smooth klt pairs (X,Δ) *of dimension* n *such that the linear system* $|K_X|$ *is not empty, then good models exist for uniruled klt pairs in dimension* n.

As a by-product, we obtain in Lemma 2.22 a result which can be viewed as a global version of the index one cover [Rei80, Corollary 1.9], and might be of independent interest.

2.2 Previous results

In this section we gather previous results which will be used in Section 2.3. We pay special attention to the behaviour of discrepancies under finite morphisms – this is also known, but we provide the details for the benefit of the reader.

2.2.1 Terminal and dlt models

Terminal and dlt models allow us to make the singularities of pairs simpler, in the first case by replacing klt by terminal singularities, and in the second case by replacing log canonical by dlt singularities. For us, particularly the dlt models and their precise definition will be useful.

Definition 2.2. Let (X,Δ) be a klt pair. A pair (Y,Γ) together with a proper birational morphism $f\colon Y\to X$ is a *terminal model of* (X,Δ) if the following holds:

(i) the pair (Y,Γ) is terminal,

(ii) Y is $\mathbb{Q}$-factorial,

(iii) $K_Y+\Gamma\sim_{\mathbb{Q}} f^*(K_X+\Delta)$.

Definition 2.3. Let (X,Δ) be a log canonical pair. A pair (Y,Γ) together with a proper birational morphism $f\colon Y\to X$ is a *dlt model of* (X,Δ) if the following holds:

(i) the pair (Y,Γ) is dlt,

(ii) the divisor Γ is the sum of $f_*^{-1}\Delta$ and all exceptional prime divisors with discrepancy -1,

(iii) Y is $\mathbb{Q}$-factorial,

(iv) $K_Y+\Gamma\sim_{\mathbb{Q}} f^*(K_X+\Delta)$.

The starting point is the following existence result.

Theorem 2.4. *Let* (X,Δ) *be a pair.*

(a) If (X,Δ) *is klt, then a terminal model of* (X,Δ) *exists.*

(b) If (X,Δ) *is log canonical, then a dlt model of* (X,Δ) *exists.*

Proof. For part (a), see [BCHM10, Corollary 1.4.3] and the paragraph after that result. Part (b) is [KK10, Theorem 3.1]. □

2.2.2 Good models

Note that if (X,Δ) is a klt pair, then it has a good model if and only if there exists a Minimal Model Program with scaling of an ample divisor which terminates with a good model of (X,Δ), cf. [Lai11, Propositions 2.4 and 2.5].

Theorem 2.5. *Assume the existence of good models for klt pairs in dimensions at most* $n-1$.

Let (X,Δ) *be a klt pair of dimension* n *which is projective over a projective variety* Z *such that* $K_X+\Delta$ *is effective over* Z*. Then* (X,Δ) *has a log terminal model over* Z*.*

Proof. By [Bir11, Corollary 1.7 and the paragraph after Definition 2.2], it is enough to show that every $\mathbb{Q}$-factorial dlt pair (Y,Γ) of dimension at most $n-1$ such that $K_Y+\Gamma$ is pseudoeffective has a minimal model in the sense of Birkar and Shokurov, cf. [Bir11, Definition 2.1]. To this end, note first that $\kappa(Y,K_Y+\Gamma)\geq 0$ by our assumption and by [Gon12, Theorem 1.5] and [FG14, Theorem 5.5]. Then we conclude by induction and by [Bir11, Corollary 1.7] again. □

Kawamata [Kaw85b] was the first to realise that the numerical Kodaira dimension, in the case of nef divisors, plays a crucial role in the abundance conjecture. The concept was generalised in [Nak04] to the case of pseudoeffective divisors.

Definition 2.6. Let X be a smooth projective variety and let D be a pseudoeffective $\mathbb{Q}$-divisor on X. If we denote

$$\sigma(D,A) = \sup\{k \in \mathbb{N} \mid \liminf_{m\to\infty} h^0(X, \lfloor mD \rfloor + A)/m^k > 0\}$$

for a Cartier divisor A on X, then the *numerical Kodaira dimension* of D is

$$\kappa_\sigma(X,D) = \sup\{\sigma(D,A) \mid A \text{ is ample}\}.$$

If X is a projective variety and if D is a pseudoeffective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on X, then we set $\kappa_\sigma(X,D) = \kappa_\sigma(Y, f^*D)$ for any birational morphism $f\colon Y \to X$ from a smooth projective variety Y.

The function κ_σ behaves similarly to the Kodaira dimension under proper pullbacks:

Lemma 2.7. *Let D be a $\mathbb{Q}$-divisor on a $\mathbb{Q}$-factorial variety X, and let $f\colon Y \to X$ be a proper surjective morphism. Then*

$$\kappa(X,D) = \kappa(Y, f^*D) \quad \textit{and} \quad \kappa_\sigma(X,D) = \kappa_\sigma(Y, f^*D).$$

If f is birational and E is an effective f-exceptional divisor on Y, then

$$\kappa(X,D) = \kappa(Y, f^*D + E) \quad \textit{and} \quad \kappa_\sigma(X,D) = \kappa_\sigma(Y, f^*D + E).$$

Proof. The first three relations are [Nak04, Lemma II.3.11, Proposition V.2.7(4)]. For the last one, we have $P_\sigma(f^*D + E) = P_\sigma(f^*D)$ by [GL13, Lemma 2.16], hence $\kappa_\sigma(Y, f^*D + E) = \kappa_\sigma(Y, f^*D)$ by [Leh13, Theorem 6.7]. □

The following result generalises [Kaw85b, Theorem 6.1], and it will be crucial in the proofs in the following section.

Lemma 2.8. *Let (X,Δ) be a klt pair. Then (X,Δ) has a good model if and only if $\kappa(X, K_X + \Delta) = \kappa_\sigma(X, K_X + \Delta)$.*

Proof. This is [GL13, Theorem 4.3]. □

Lemma 2.9. *Let (X,Δ) and (X,Δ') be pairs, and assume that there exist $\mathbb{Q}$-divisors $D \geq 0$ and $D' \geq 0$ such that*

$$K_X + \Delta \sim_{\mathbb{Q}} D \geq 0, \quad K_X + \Delta' \sim_{\mathbb{Q}} D' \geq 0 \quad \textit{and} \quad \operatorname{Supp} D' = \operatorname{Supp} D.$$

Then

$$\kappa(X, K_X + \Delta) = \kappa(X', K_{X'} + \Delta') \quad \textit{and} \quad \kappa_\sigma(X, K_X + \Delta) = \kappa_\sigma(X', K_{X'} + \Delta')$$

Proof. There exist positive rational numbers t_1 and t_2 such that $t_1 D \leq D' \leq t_2 D$, hence $\kappa(X, t_1 D) \leq \kappa(X, D') \leq \kappa(X, t_2 D)$. This implies the first equality, and the second is analogous. □

2.2.3 Valuations under finite morphisms

We first prove an easy algebraic result that we use in the proof of Proposition 2.13.

Lemma 2.10. *Let $k \subseteq K$ be an algebraic extension of fields. Let (B, m_B) be a discrete valuation ring with the quotient field K, and let $A = B \cap k$ and $m_A = m_B \cap k$. Then (A, m_A) is a discrete valuation ring with the quotient field k such that the field extension $A/m_A \subseteq B/m_B$ is algebraic.*

Proof. Let $\nu\colon K \to \mathbb{Z} \cup \{\infty\}$ be the valuation function corresponding to (B, m_B). Then $A = \{a \in k \mid \nu(a) \geq 0\}$ and $m_A = \{a \in k \mid \nu(a) > 0\}$, and it is immediate that k is the quotient field of A. Let $b \in B$ and denote $\overline{b} = b + m_B \in B/m_B$. Then there is a polynomial

$$p = T^n + r_{n-1}T^{n-1} + \cdots + r_0 \in k[T]$$

such that $p(b) = 0$, and fix $j \in \{0, \ldots, n-1\}$ such that $\nu(r_j) \leq \nu(r_i)$ for all i. If $\nu(r_j) \geq 0$, then $p \in A[T]$ and $\overline{b}$ is algebraic over A/m_A. If $\nu(r_j) < 0$, then $r_j^{-1} \in m_A$ and $\nu(r_j^{-1} r_i) \geq 0$ for all i. Therefore,

$$\overline{p} = r_j^{-1} p \mod m_A \in (A/m_A)[T]$$

is a non-zero polynomial such that $\overline{p}(\overline{b}) = 0$, which proves the last claim. It remains to show that $m_A \neq \{0\}$. Fix $b \in B$ with $\nu(b) > 0$ and let

$$p = a_n T^n + a_{n-1}T^{n-1} + \cdots + a_0 \in A[T]$$

be a polynomial of minimal degree such that $p(b) = 0$, so that, in particular, $a_0 \neq 0$. Then we have

$$0 < \nu(b) \leq \nu\big(b(a_n b^{n-1} + a_{n-1}b^{n-2} + \cdots + a_1)\big) = \nu(-a_0),$$

hence $a_0 \in m_A$. □

We make a couple of remarks on geometric valuations.

Remark 2.11. Let X be a variety and let Γ be a geometric valuation over X which is a divisor on a birational model $Y \to X$. Let R be a discrete valuation ring with quotient field $k(X)$ which dominates the local ring $\mathscr{O}_{X, c_X(\Gamma)} \subseteq k(X)$. Then there exists a morphism $\operatorname{Spec} R \to X$ which sends the generic point of $\operatorname{Spec} R$ to the generic point of X, and the closed point of $\operatorname{Spec} R$ to the generic point of $c_X(\Gamma)$, cf. [Har77, Lemma II.4.4]. In particular, this holds if $R = \mathscr{O}_{Y,\Gamma}$.

Remark 2.12. Let X be a normal variety and let (R,m) be a discrete valuation ring such that the quotient field of R is $k(X)$. Assume that there is a morphism $\operatorname{Spec} R \to X$ which sends the generic point of $\operatorname{Spec} R$ to the generic point of X. Assume that $\operatorname{trdeg}_{\mathbb{C}}(R/m) = \dim X - 1$. Then by a lemma of Zariski [KM98, Lemma 2.45], the corresponding valuation is a geometric valuation on X.

Proposition 2.13. *Let $\pi\colon X' \to X$ be a finite morphism of degree m between normal varieties, let Δ be a $\mathbb{Q}$-divisor on X such that (X,Δ) is a pair, and let Δ' be a $\mathbb{Q}$-divisor on X' such that $K_{X'} + \Delta' = \pi^*(K_X + \Delta)$.*

(i) For every geometric valuation E' over X' there exists a geometric valuation E over X and an integer $1 \le r \le m$ such that $\pi(c_{X'}(E')) = c_X(E)$ and

$$a(E',X',\Delta') + 1 = r(a(E,X,\Delta) + 1).$$

(ii) For every geometric valuation E over X there exists a geometric valuation E' over X' and an integer $1 \le r \le m$ such that $\pi(c_{X'}(E')) = c_X(E)$ and

$$a(E',X',\Delta') + 1 = r(a(E,X,\Delta) + 1).$$

In particular, the pair (X,Δ) is log canonical (respectively klt) if and only if the pair (X',Δ') is log canonical (respectively klt).

Proof. This is [KM98, Proposition 5.20], and in the following we reproduce the proof with more details.

We claim that both in (i) and (ii) there is a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\pi'} & Y \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow{\pi} & X \end{array} \tag{2.1}$$

where f and f' are birational morphisms, π' is finite and there are prime divisors $E \subseteq Y$ and $E' \subseteq Y'$ such that $\pi'(E') = E$. The claim immediately implies the proposition: indeed, let $r = \operatorname{mult}_{E'}(\pi')^*E$. Then locally around the generic point of E' we have

$$\begin{aligned} K_{Y'} - (r-1)E' &= (\pi')^*K_Y \sim_{\mathbb{Q}} (\pi')^*(f^*(K_X + \Delta) + a(E,X,\Delta)\cdot E) \\ &= (f')^*(K'_X + \Delta') + r\cdot a(E,X,\Delta)\cdot E' \\ &\sim_{\mathbb{Q}} K_{Y'} - a(E',X',\Delta')\cdot E' + r\cdot a(E,X,\Delta)\cdot E', \end{aligned}$$

hence (i) and (ii) follow.

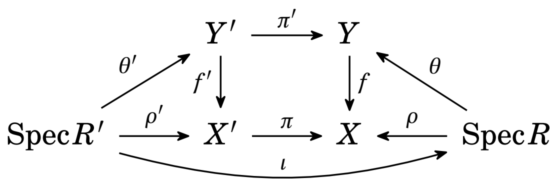
To see the claim in the case (ii), let $f\colon Y \to X$ be a birational morphism such that $E \subseteq Y$ is a prime divisor, and let Y' be a component of the normalisation of the fibre product $X' \times_X Y$ that maps onto Y. Then we obtain the diagram (2.1), and since π' is surjective, there is a prime divisor $E' \subseteq Y'$ with $\pi'(E') = E$.

In the case (i), let $(R', m_{R'})$ be the discrete valuation ring corresponding to the valuation E', and let $R = R' \cap k(X)$ and $m_R = m_{R'} \cap k(X)$. Since $k(X) \subseteq k(X')$ is an algebraic extension of fields, R is a discrete valuation ring with quotient field $k(X)$ such that $\operatorname{trdeg}_{\mathbb{C}}(R/m_R) = \dim X - 1$ by Lemma 2.10. If E is the corresponding discrete valuation, then E is a divisorial valuation by Remark 2.12. By Remark 2.11, there is a morphism $\rho'\colon \operatorname{Spec} R' \to X'$ which sends the generic point of $\operatorname{Spec} R'$ to the generic point of X', and the closed point of $\operatorname{Spec} R'$ to the generic point η' of $c_{X'}(E')$. If $\eta = \pi(\eta')$, then

$$\mathscr{O}_{X,\eta} \subseteq \mathscr{O}_{X',\eta'} \cap k(X) \subseteq R' \cap k(X) = R,$$

hence by Remark 2.11 there is a morphism $\rho\colon \operatorname{Spec} R \to X$ which sends the generic point of $\operatorname{Spec} R$ to the generic point of X, and the closed point of $\operatorname{Spec} R$ to η.

Let $f\colon Y \to X$ be a birational morphism such that E is a divisor on Y, and denote by X' a component of the normalization of the fibre product $X' \times_X Y$ that maps onto Y, so that we have the diagram (2.1). By the valuative criterion of properness, we have the diagram



where $\iota\colon \operatorname{Spec} R' \to \operatorname{Spec} R$ is the morphism induced by the inclusion $R \subseteq R'$. Since f is separated, we have $\pi' \circ \theta' = \theta \circ \iota$, and this just says that E' is a prime divisor on Y' such that $\pi'(c_{Y'}(E')) = c_Y(E)$. $\square$

2.3 Good models for uniruled pairs

Lemma 2.14. *Let (X,Δ) be a pair, and let $f\colon X \dashrightarrow Y$ be a birational contraction to a normal projective variety such that $K_Y + f_*\Delta$ is $\mathbb{Q}$-Cartier. Then*

$$\kappa_\sigma(X, K_X + \Delta) \le \kappa_\sigma(Y, K_Y + f_*\Delta).$$

Proof. Let $(p,q)\colon W \to X \times Y$ be a resolution of the map f. Write

$$K_W + \Delta_W \sim_{\mathbb{Q}} p^*(K_X + \Delta) + E \quad \text{and} \quad K_W + \Delta'_W \sim_{\mathbb{Q}} q^*(K_Y + f_*\Delta) + E',$$

where $\Delta_W \geq 0$ and $E \geq 0$ have no common components, and $\Delta'_W \geq 0$ and $E' \geq 0$ have no common components. Since f is a contraction, the divisor $\Delta_W - \Delta'_W$ is q-exceptional, and there are effective q-exceptional $\mathbb{Q}$-divisors E^+ and E^- such that $\Delta_W - \Delta'_W = E^+ - E^-$. Therefore,

$$K_W + \Delta_W + E^- = K_W + \Delta'_W + E^+ \sim_{\mathbb{Q}} q^*(K_Y + f_*\Delta) + E' + E^+,$$

hence $\kappa_\sigma(W, K_W + \Delta_W + E^-) = \kappa_\sigma(Y, K_Y + f_*\Delta)$ by Lemma 2.7. We conclude since $\kappa_\sigma(X, K_X + \Delta) = \kappa_\sigma(W, K_W + \Delta_W) \leq \kappa_\sigma(W, K_W + \Delta_W + E^-)$ by Lemma 2.7. □

Definition 2.15. Let (X,Δ) be a klt pair. Let G be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $K_X + \Delta + G$ is pseudoeffective. Then the *pseudoeffective threshold* $\tau(X,\Delta;G)$ is defined as

$$\tau(X,\Delta;G) = \min\{t \in \mathbb{R} \mid K_X + \Delta + tG \text{ is pseudoeffective}\}.$$

Theorem 2.16. *Assume the existence of good models for klt pairs in dimensions at most $n-1$.*

Let (X,Δ) be a klt pair of dimension n. Let G be an effective $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor such that $(X,\Delta+G)$ is klt and $K_X + \Delta + G$ is pseudoeffective. Assume that $K_X + \Delta$ is not pseudoeffective, i.e. that $\tau = \tau(X,\Delta;G) > 0$.

Then $\tau \in \mathbb{Q}$, and there exists a good model of $(X,\Delta+\tau G)$. In particular,

$$\kappa(X, K_X + \Delta + \tau G) \geq 0.$$

Proof. We follow closely the proof of [DHP13, Proposition 8.7, Theorem 8.8]. Fix an ample divisor A on X. For any rational number $0 \leq x \leq \tau$ let $y_x = \tau(X,\Delta + xG; A)$. Note that $y_\tau = 0$ and that y_x is a positive rational number for $0 \leq x < \tau$ – rationality follows from [BCHM10, Corollary 1.1.7], and positivity from the fact that $K_X + \Delta + xG$ is not pseudoeffective when $x < \tau$.

Let (x_i) be an increasing sequence of non-negative rational numbers such that $\lim\limits_{i\to\infty} x_i = \tau$, and denote $y_i = y_{x_i}$. Fix i, let $f_i\colon X \dashrightarrow Y_i$ be the $(K_X + \Delta + x_i G)$-MMP with scaling of A, and denote by Δ_i, G_i and A_i the proper transforms of Δ, G and A on Y_i. By [BCHM10, Corollary 1.3.3], there is an extremal contraction $g_i\colon Y_i \to Z_i$ of fibre type such that

$$K_{Y_i} + \Delta_i + x_i G_i + y_i A_i \equiv_{g_i} 0.$$

Let E_j be effective divisors on Y_i whose classes converge to the class of $K_{Y_i}+\Delta_i+\tau G_i$ in $N^1(Y_i)_{\mathbb{R}}$, and let C be a curve on Y_i which does not belong to $\bigcup \operatorname{Supp} E_j$ and is contracted by g_i. Then

$$(K_{Y_i}+\Delta_i+\tau G_i)\cdot C \geq 0 \quad \text{and} \quad (K_{Y_i}+\Delta_i+x_iG_i+y_iA_i)\cdot C = 0.$$

Therefore, there exists a rational number $\eta_i \in (x_i,\tau]$ such that $(K_{Y_i}+\Delta_i+\eta_iG_i)\cdot C=0$, hence

$$K_{Y_i}+\Delta_i+\eta_iG_i \equiv_{g_i} 0$$

since all contracted curves are numerically proportional. In particular, if F_i is a general fibre of g_i, and $\Delta_{F_i}=\Delta_i|_{F_i}$ and $G_{F_i}=G_i|_{F_i}$, then

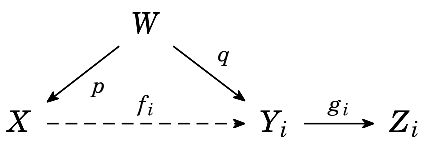
$$K_{F_i}+\Delta_{F_i}+\eta_iG_{F_i} \equiv 0. \tag{2.2}$$

Denoting

$$\tau_i=\max\{t\in\mathbb{R} \mid K_{F_i}+\Delta_{F_i}+tG_{F_i} \text{ is log canonical}\},$$

we have $x_i \leq \tau_i$ since $K_{F_i}+\Delta_{F_i}+x_iG_{F_i}$ is log canonical for every i. If $K_{F_i}+\Delta_{F_i}+\tau G_{F_i}$ is not log canonical for infinitely many i, then after passing to a subsequence we can assume that $\tau_i<\tau$ for all i, and since $x_i \leq \tau_i$ and $\lim x_i=\tau$, we can assume that the sequence (τ_i) is strictly increasing, which contradicts [HMX12, Theorem 1.1]. Therefore, $K_{F_i}+\Delta_{F_i}+\tau G_{F_i}$ is log canonical for $i \gg 0$, and then [HMX12, Theorem 1.5] implies that the sequence (η_i) is eventually constant, hence $\eta_i=\tau$ for $i\gg 0$. In particular, $\tau\in\mathbb{Q}$.

Now, for the rest of the proof fix any such $i \gg 0$ for which $\eta_i=\tau$, and let $(p,q)\colon W\to X\times Y_i$ be a resolution of the map f_i.



We may write

$$K_W+\Delta_W \sim_{\mathbb{Q}} p^*(K_X+\Delta+\tau G)+E,$$

where Δ_W and E are effective $\mathbb{Q}$-divisors without common components. We want to prove that $(X,\Delta+\tau G)$ has a good minimal model, hence by Lemmas 2.7 and 2.8, it is enough to show that

$$\kappa(W,K_W+\Delta_W)=\kappa_\sigma(W,K_W+\Delta_W). \tag{2.3}$$

If we denote $F_W = q^{-1}(F_i) \subseteq W$, then $q_*(K_{F_W} + \Delta_W|_{F_W}) = K_{F_i} + \Delta_{F_i} + \tau G_{F_i}$, hence by Lemma 2.14 and by (2.2),

$$\kappa_\sigma(F_W, K_{F_W} + \Delta_W|_{F_W}) \leq \kappa_\sigma(F_i, K_{F_i} + \Delta_{F_i} + \tau G_{F_i}) = 0. \tag{2.4}$$

When $\dim Z_i = 0$, then $F_W = W$ and (2.4) implies (2.3) by [Nak04, Corollary V.4.9].

When $\dim Z_i > 0$, then $K_W + \Delta_W$ is effective over Z_i by induction on the dimension and by [BCHM10, Lemma 3.2.1]. By Theorem 2.5 and by [Fuj11, Theorem 1.1] there exists a good model $(W, \Delta_W) \dashrightarrow (W_{\min}, \Delta_{\min})$ of (W, Δ_W) over Z_i. Let $\varphi\colon W_{\min} \to W_{\text{can}}$ be the corresponding fibration to the canonical model of (W, Δ_W) over Z_i. Since $K_W + \Delta_W$ is not big over Z_i by (2.4), we have $\dim W_{\text{can}} < \dim X$. By [Amb05, Theorem 0.2], there exists a divisor Δ_{can} on W_{can} such that the pair $(W_{\text{can}}, \Delta_{\text{can}})$ is klt and

$$K_{W_{\min}} + \Delta_{\min} \sim_{\mathbb{Q}} \varphi^*(K_{W_{\text{can}}} + \Delta_{\text{can}}).$$

Since we assume the existence of good models for klt pairs in dimensions at most $n-1$, we have $\kappa(W_{\text{can}}, K_{W_{\text{can}}} + \Delta_{\text{can}}) = \kappa_\sigma(W_{\text{can}}, K_{W_{\text{can}}} + \Delta_{\text{can}})$ by Lemma 2.8, and hence (2.3) holds by Lemma 2.7, which concludes the proof. □

Remark 2.17. Let (X, Δ) be a uniruled klt pair such that K_X is not pseudoeffective and $K_X + \Delta$ is pseudoeffective. A natural strategy to construct a good model of (X, Δ) is to run a $(K_X + \tau\Delta)$-MMP, where $\tau = \tau(X, 0; \Delta)$, and which we know terminates with a good model (Y, Δ_Y) by Theorem 2.16. The main problem is that this MMP does not preserve sections of $K_X + \Delta$. An instructive example is when $K_X \sim_{\mathbb{Q}} -\tau\Delta$, where Δ is nef and not big, and for instance $\rho(X) = 2$. Then one might want to run the $(K_X + (\tau - \varepsilon)\Delta)$-MMP with scaling of an ample divisor A, where $0 < \varepsilon \ll 1$. If $\mathrm{Nef}(X) \neq \overline{\mathrm{Eff}}(X)$, then this MMP ends up with a model on which the proper transform of $K_X + \Delta$ is ample, regardless of the Kodaira dimension of $K_X + \Delta$.

Theorem 2.18. *Assume the existence of good models for klt pairs in dimensions at most $n-1$, and the existence of good models for log smooth klt pairs (X, Δ) in dimension n such that $|K_X| \neq \emptyset$.*

Let (X, Δ) be a log smooth log canonical pair of dimension n and assume that there exists a $\mathbb{Q}$-divisor $D \geq 0$ such that $K_X + \Delta \sim_{\mathbb{Q}} D$ and $\operatorname{Supp}\Delta = \operatorname{Supp} D$. *Then*

$$\kappa(X, K_X + \Delta) = \kappa_\sigma(X, K_X + \Delta).$$

Proof. Replacing Δ by $\lceil\Delta\rceil$, by Lemma 2.9 we may assume that the divisor Δ is reduced. In the course of the proof, we construct a tower of proper maps

$$(T,\Delta_T)\xrightarrow{\mu}(W,\Delta_W)\xrightarrow{g}(Z,\Delta_Z)\xrightarrow{f}(X',\Delta_{X'})\xrightarrow{\pi}(X,\Delta),$$

where π and μ are finite, and f and g are birational, such that for each $\mathscr{X}\in\{T,W,Z,X'\}$ we have

$$\kappa(\mathscr{X},K_{\mathscr{X}}+\Delta_{\mathscr{X}})=\kappa(X,K_X+\Delta)\quad\text{and}\quad\kappa_\sigma(\mathscr{X},K_{\mathscr{X}}+\Delta_{\mathscr{X}})=\kappa_\sigma(X,K_X+\Delta).$$

The pair (T,Δ_T) will be log smooth with $|K_T|\neq\emptyset$ which allows us to conclude.

Let m be the smallest positive integer such that $m(K_X+\Delta)\sim mD$, and denote $G=mD$. Let $\pi\colon X'\to X$ be the normalisation of the corresponding m-fold cyclic covering ramified along G. Note that X' is irreducible by [EV92, Lemma 3.15(a)] since m is minimal. Then there exists an effective Cartier divisor G' on X' such that

$$\pi^*G=mG'\quad\text{and}\quad\pi^*(K_X+\Delta)\sim G',$$

and let $\Delta'=(G')_{\mathrm{red}}$. By the Hurwitz formula, we have

$$K_{X'}+\Delta'=\pi^*(K_X+\Delta),$$

and the pair (X',Δ') is log canonical by Proposition 2.13. By Theorem 2.4, there exists a dlt model $f\colon(Z,\Delta_Z)\to X'$ of (X',Δ'), and we have

$$\kappa(X,K_X+\Delta)=\kappa(Z,K_Z+\Delta_Z)\quad\text{and}\quad\kappa_\sigma(X,K_X+\Delta)=\kappa_\sigma(Z,K_Z+\Delta_Z)$$

by Lemma 2.7. Denote $G_Z=f^*G'$. We claim that for every geometric valuation E' over Z we have $a(E',Z,\Delta_Z)\in\mathbb{Z}$. To prove the claim, let E' be a geometric valuation over Z. Then by Proposition 2.13, there exists a geometric valuation E over X and an integer $1\le r\le m$ such that

$$a(E',Z,\Delta_Z)+1=a(E',X',\Delta')+1=r(a(E,X,\Delta)+1),\tag{2.5}$$

where the first equality holds because $K_Z+\Delta_Z\sim_{\mathbb{Q}}f^*(K_{X'}+\Delta')$. Since (X,Δ) is log smooth and Δ is reduced, we have $a(E,X,\Delta)\in\mathbb{Z}$, which together with (2.5) implies the claim.

Now, if $g\colon W\to Z$ is a log resolution of the pair (Z,Δ_Z), by the claim we may write

$$K_W+\Delta_W\sim_{\mathbb{Q}}g^*(K_Z+\Delta_Z)+E_W\sim_{\mathbb{Q}}g^*G_Z+E_W,$$

where Δ_W and E_W are effective integral divisors with no common components. Then

$$\kappa(X,K_X+\Delta)=\kappa(W,K_W+\Delta_W) \quad \text{and} \quad \kappa_\sigma(X,K_X+\Delta)=\kappa_\sigma(W,K_W+\Delta_W)$$

by Lemma 2.7, and the divisor $G_W = g^*G_Z + E_W - \Delta_W$ is Cartier. We have

$$K_W \sim_{\mathbb{Q}} G_W,$$

and we claim that $G_W \geq 0$. Indeed, if S is a component of Δ_W, then $a(S,Z,\Delta_Z) = a(S',X',\Delta') = -1$. By Proposition 2.13, there exists a geometric valuation S over X and an integer $1 \leq r \leq m$ such that $\pi(c_{X'}(S')) = c_X(S)$ and

$$a(S',X',\Delta')+1 = r(a(S,X,\Delta)+1).$$

This implies $a(S,X,\Delta) = -1$, thus $c_X(S) \subseteq \operatorname{Supp}\Delta$ because (X,Δ) is log smooth. From here we obtain $c_{X'}(S') \subseteq \pi^{-1}(\operatorname{Supp}\Delta) = \operatorname{Supp}G'$, and in particular $S' \subseteq \operatorname{Supp}G_Z$. Therefore $\operatorname{mult}_S g^*G_Z \geq 1 = \operatorname{mult}_S \Delta_W$, and the claim follows.

Now, consider the klt pair $(K_W, \frac{1}{2}\Delta_W)$. Since $K_W + \frac{1}{2}\Delta_W \sim_{\mathbb{Q}} G_W + \frac{1}{2}\Delta_W$, $K_W + \Delta_W \sim_{\mathbb{Q}} G_W + \Delta_W$ and $\operatorname{Supp}(G_W + \frac{1}{2}\Delta_W) = \operatorname{Supp}(G_W + \Delta_W)$, by Lemma 2.9 we have

$$\kappa(X,K_X+\Delta)=\kappa(W,K_W+\tfrac{1}{2}\Delta_W) \quad \text{and} \quad \kappa_\sigma(X,K_X+\Delta)=\kappa_\sigma(W,K_W+\tfrac{1}{2}\Delta_W).$$

Let k be the smallest positive integer such that $k(K_W - G_W) \sim 0$, and let $\mu\colon T \to W$ be the corresponding k-fold étale covering. Then

$$K_T = \mu^*K_W \sim \mu^*G_W,$$

and setting $\Delta_T = \mu^*(\frac{1}{2}\Delta_W)$, the pair (K_T, Δ_T) is klt by Proposition 2.13. We have

$$\kappa(X,K_X+\Delta)=\kappa(T,K_T+\Delta_T) \quad \text{and} \quad \kappa_\sigma(X,K_X+\Delta)=\kappa_\sigma(T,K_T+\Delta_T)$$

by Lemma 2.9, hence $\kappa(X,K_X+\Delta) = \kappa_\sigma(X,K_X+\Delta)$ by our assumptions and by Lemma 2.8. □

Remark 2.19. With the notation from the proof of Theorem 2.18, one can show that the variety Z has canonical singularities, so that Z is not uniruled by Theorem 1.5, without passing to a log resolution.

Remark 2.20. In the proof of Theorem 2.18, $X' \setminus \Delta' \subseteq X'$ is a toroidal embedding since the pair (X,Δ) is log smooth [Ara14, Lemma 1.1], i.e. it is locally analytically on X' isomorphic to an embedding of a torus into a toric variety. By [AW97, Theorem 0.2], there exists a toroidal resolution $h\colon (U,\Delta_U) \to (X',\Delta')$ and then $K_U + \Delta_U = h^*(K_{X'} + \Delta')$: indeed, locally in the analytic category both sides of this equation are trivial, which implies that all relevant discrepancies are zero. This is all implicit already in [KKMSD73]. The pair (U,Δ_U) is log smooth, and as in the proof of Theorem 2.18, one shows that K_U is linearly equivalent to an effective Cartier divisor. Therefore, if one prefers toroidal embeddings, one can avoid the use of dlt models; however, compare to [dFKX12, Section 5].

Finally we can prove our main results.

Proof of Theorem 2.1. Let (X,Δ) be a uniruled klt pair. By replacing (X,Δ) by its terminal model, cf. Theorem 2.4(a), we may assume that the pair (X,Δ) is terminal, and thus that K_X is not pseudoeffective by Theorem 1.5. Let $\tau = \tau(X,0;\Delta) = \min\{t \in \mathbb{R} \mid K_X + t\Delta \text{ is pseudoeffective}\}$. Since K_X is not pseudoeffective and $K_X + \Delta$ is pseudoeffective, we have $0 < \tau \leq 1$. If $\tau = 1$, then we conclude by Theorem 2.16.

Therefore, we may assume that $\tau < 1$, and hence by Theorem 2.16 there exists a $\mathbb{Q}$-divisor $D_\tau \geq 0$ such that $K_X + \tau\Delta \sim_{\mathbb{Q}} D_\tau$. This yields

$$K_X + \Delta \sim_{\mathbb{Q}} D \geq 0, \quad \text{where} \quad D = D_\tau + (1-\tau)\Delta.$$

In particular, $\operatorname{Supp}\Delta \subseteq \operatorname{Supp}D$. Let $f\colon Y \to X$ be a log resolution of the pair (X,D). Then we may write

$$K_Y + \Gamma \sim_{\mathbb{Q}} f^*(K_X + \Delta) + E,$$

where Γ and E are effective $\mathbb{Q}$-divisors with no common components, and $\Gamma = f_*^{-1}\Delta$ since (X,Δ) is a terminal pair. In particular, if we denote $D_Y = f^*D + E$, then $K_Y + \Gamma \sim_{\mathbb{Q}} D_Y$ and $\operatorname{Supp}\Gamma \subseteq \operatorname{Supp}D_Y$. We have

$$\kappa(X,K_X+\Delta) = \kappa(Y,K_Y+\Gamma) \quad \text{and} \quad \kappa_\sigma(X,K_X+\Delta) = \kappa_\sigma(Y,K_Y+\Gamma)$$

by Lemma 2.7, hence by replacing (X,Δ) by (Y,Γ) and D by D_Y, we may assume that (X,D) is a log smooth pair. Finally, by replacing Δ by $\Delta+\varepsilon D$ for $0 < \varepsilon \ll 1$, we may further assume that $\operatorname{Supp}\Delta = \operatorname{Supp}D$. We conclude by Theorem 2.18 and by Lemma 2.8. □

Proof of Theorem A. Let (X,Δ) be a uniruled klt pair. As in the proofs of Theorems 2.1 and 2.18, there exists a log smooth klt pair (T,Δ_T) such that $|K_T| \neq \emptyset$ and

$$\kappa(X,K_X+\Delta) = \kappa(T,K_T+\Delta_T) \geq 0 \quad \text{and} \quad \kappa_\sigma(X,K_X+\Delta) = \kappa_\sigma(T,K_T+\Delta_T).$$

In particular, T is not uniruled by Theorem 1.5. By Theorem 2.5, there exists a log terminal model $(T,\Delta_T) \dashrightarrow (T',\Delta_{T'})$ of (T,Δ_T), hence

$$\kappa(T',K_{T'}+\Delta_{T'}) = \kappa_\sigma(T',K_{T'}+\Delta_{T'})$$

since we assume the abundance conjecture for non-uniruled pairs. We conclude by Lemmas 2.7 and 2.8. □

Proof of Theorem B. Immediate from Theorem 2.1. □

Remark 2.21. Assume that for every smooth variety of dimension n with K_X pseudoeffective we have $\kappa(X,K_X) \geq 0$. Then the previous proofs show that if good models exist for log smooth klt pairs (X,Δ) of dimension n such that the linear system $|K_X|$ is not empty, then good models exist for klt pairs in dimension n.

Indeed, by Theorem B we only have to show that the assumptions imply the existence of good models for non-uniruled klt pairs in dimension n. Fix such a pair (X,Δ), and note that we may assume that the pair is terminal by Theorem 2.4. Then $\kappa(X,K_X) \geq 0$ by our assumption, hence there exists an effective divisor D' such that $K_X \sim_{\mathbb{Q}} D'$. In particular, by denoting $D = D' + \Delta$ we have $K_X + \Delta \sim_{\mathbb{Q}} D$ and $\operatorname{Supp}\Delta \subseteq \operatorname{Supp} D$. As in the proof of Theorem 2.1, by passing to a log resolution, we may assume that (X,D) is log smooth. By replacing Δ by $\Delta+\varepsilon D$ for $0<\varepsilon \ll 1$, we may further assume that $\operatorname{Supp}\Delta = \operatorname{Supp} D$, and we conclude by Theorem 2.18 and by Lemma 2.8.

This leads to the following result.

Lemma 2.22. *Let (X,Δ) be a $\mathbb{Q}$-factorial terminal pair and assume that $\kappa(X,K_X) \geq 0$. Then there exists a generically finite morphism $f\colon Y \to X$ from a smooth variety Y and an effective $\mathbb{Q}$-divisor Γ on Y with simple normal crossings support such that the pair (Y,Γ) is klt, $|K_Y| \neq \emptyset$ and*

$$\kappa(X,K_X+\Delta) = \kappa(Y,K_Y+\Gamma) \quad \textit{and} \quad \kappa_\sigma(X,K_X+\Delta) = \kappa_\sigma(Y,K_Y+\Gamma).$$

If $\Delta = 0$, we may additionally assume that $\Gamma = 0$.

Proof. The first claim follows from the proof of Theorem 2.1. When $\Delta = 0$, as in Remark 2.21 we may assume that X is smooth and that there exists a $\mathbb{Q}$-divisor $D \geq 0$ with simple normal crossings support such that $K_X \sim_{\mathbb{Q}} D'$. Setting $\Delta_X = \varepsilon D$ and $D = D' + \Delta_X$ for a rational number $0<\varepsilon \ll 1$, we have $K_X + \Delta_X \sim_{\mathbb{Q}} D$ and $0 < \operatorname{mult}_E \Delta_X < \operatorname{mult}_E D$ for every component E of D. Then with notation from the proof of Theorem 2.18, we obtain a generically finite map $(W,\Delta_W) \to (X,\Delta_X)$ such that the pair (W,Δ_W) is log smooth,

$$\kappa(W, K_W + \Delta_W) = \kappa(X, K_X + \Delta_X) \quad \text{and} \quad \kappa_\sigma(W, K_W + \Delta_W) = \kappa_\sigma(X, K_X + \Delta_X),$$

and $K_W \sim_{\mathbb{Q}} G_W$ for some Cartier divisor G_W such that – crucially – $\operatorname{Supp} G_W = \operatorname{Supp}(G_W + \Delta_W)$. In particular, by Lemma 2.9 this implies

$$\kappa(W, K_W) = \kappa(X, K_X) \quad \text{and} \quad \kappa_\sigma(W, K_W) = \kappa_\sigma(X, K_X).$$

Finally, one more étale cover allows to conclude as in the proof of Theorem 2.18. $\square$

Chapter 3

The Cone conjecture

3.1 Introduction

The results of this chapter are taken from [LP13, CL14, KKL12].

A *Calabi-Yau manifold* of dimension n is a projective manifold X with trivial canonical bundle $K_X \simeq \mathscr{O}_X$ such that $H^1(X, \mathscr{O}_X) = 0$. In particular, we do not require X to be simply connected. With notation and definitions from Section 1.5, it is well-known, see for instance [Ogu14, Proposition 2.4], that the group $\mathrm{Bir}(X)$ is finite if and only if $\mathscr{B}(X)$ is, and similarly for $\mathrm{Aut}(X)$ and $\mathscr{A}(X)$.

Based on and inspired by recent work of Oguiso [Ogu14] we prove the following results.

Theorem 3.1. *Let X be a Calabi-Yau manifold of Picard number 2. Then either $|\mathscr{A}(X)| \leq 2$, or $\mathscr{A}(X)$ is infinite; and either $|\mathscr{B}(X)| \leq 2$, or $\mathscr{B}(X)$ is infinite.*

In fact, we explicitly calculate the groups $\mathscr{A}(X)$ and $\mathscr{B}(X)$, and for more detailed information we refer to Section 3.4. The consequences for the Cone conjectures can be summarized as follows.

Theorem C. *Let X be a Calabi-Yau manifold with Picard number 2. If the group $\mathrm{Bir}(X)$ is infinite, then the Cone conjecture holds on X.*

Oguiso in [Ogu14] showed that there are indeed Calabi-Yau threefolds X with $\rho(X) = 2$ and with infinite $\mathrm{Bir}(X)$, see Example 1.7, as well as hyperkähler 4-folds X with $\rho(X) = 2$ and with infinite $\mathrm{Aut}(X)$.

In Section 3.2 we discuss several questions around the Cone conjecture in the general setting, which are of independent interest.

3.2 Motivation and discussion

Recall that certain amount of motivation for the Cone conjecture was given in the introduction, together with the evidence, mostly on surfaces. Here, we give some further motivation, first with analogy with Fano varieties, and then some known results on varieties of Calabi-Yau type.

Here, a projective variety X is said to be of *Calabi-Yau type* if there exists a $\mathbb{Q}$-divisor $\Delta \geq 0$ such that (X,Δ) is klt and $K_X + \Delta \equiv 0$. It is known that this condition is equivalent to $K_X + \Delta \sim_{\mathbb{Q}} 0$: the case when $\Delta = 0$ and X has canonical singularities was proved in [Kaw85a, Theorem 8.2], and the general case is treated in [CKP12, Theorem 0.1].

3.2.1 Fano varieties

We say that a klt pair (X,Δ) is a *log Fano pair* if $-(K_X + \Delta)$ is ample. Recall from Chapter 1, that in order to show that "on a Fano manifold the Cone conjecture holds", it suffices to prove the following result.

Theorem 3.2. *Let (X,Δ) be a log Fano klt pair. Then the cones* $\operatorname{Nef}(X)$ *and* $\overline{\operatorname{Mov}}(X)$ *are rational polyhedral and contained in* $\operatorname{Eff}(X)$.

We first need some serious preparation. We will see that Theorem 3.2 is essentially the following statement, once we equip ourselves with right tools.

Theorem 3.3. *Let (X,Δ) be a log Fano klt pair. Then* $\operatorname{Pic}(X)_{\mathbb{Q}} \simeq N^1(X)_{\mathbb{Q}}$*, and there is a basis $D_1,\dots,D_r$ of* $\operatorname{Pic}(X)_{\mathbb{Q}}$ *such that*

(i) $\overline{\operatorname{Eff}}(X) \subseteq \sum \mathbb{R}_+ D_i$,

(ii) the ring $R(X;D_1,\dots,D_r)$ is finitely generated.

Proof. First, we have $H^i(X,\mathscr{O}_X) = 0$ for all $i > 0$ by the Kawamata-Viehweg vanishing. The long exact sequence in cohomology associated to the exponential sequence

$$0 \to \mathbb{Z} \to \mathscr{O}_X \to \mathscr{O}_X^* \to 0$$

shows that $\operatorname{Pic}(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Q}}$. Let $D_1,\dots,D_r$ be a basis of $\operatorname{Pic}(X)_{\mathbb{Q}}$ such that $\overline{\operatorname{Eff}}(X) \subseteq \sum \mathbb{R}_+ D_i$, and pick a rational number $0 < \varepsilon \ll 1$ such that $A_i = \varepsilon D_i - (K_X + \Delta)$ is ample for every i. Then the ring

$$R(X;\varepsilon D_1,\dots,\varepsilon D_r) = R(X;K_X + \Delta + A_1,\dots,K_X + \Delta + A_r)$$

is finitely generated by Theorem 1.25, hence the ring $R(X;D_1,\dots,D_r)$ is finitely generated by Lemma 1.21. □

Part (i) of the following lemma is [CL13, Lemma 3.8]. Part (ii) is a result of Zariski and Wilson, cf. [Laz04, Theorem 2.3.15].

Lemma 3.4. *Let X be a normal projective variety and let D be a divisor in* $\mathrm{Div}_{\mathbb{Q}}(X)$*.*

(i) If $|D|_{\mathbb{Q}} \neq \emptyset$, then D is semiample if and only if $R(X,D)$ is finitely generated and $o_\Gamma(D) = 0$ for all geometric valuations Γ over X.

(ii) If D is nef and big, then D is semiample if and only if $R(X,D)$ is finitely generated.

Proof. If D is semiample, then some multiple of D is basepoint free, thus $R(X,D)$ is finitely generated by Lemma 1.21, and all $o_\Gamma(D) = 0$. Now, fix a point $x \in X$. If $R(X,D)$ is finitely generated and $o_x(D) = 0$, then $x \notin \mathbf{B}(D)$ by Theorem 1.23(4), which proves (i).

For (ii), let A be an ample divisor. Then $D + \varepsilon A$ is ample for any $\varepsilon > 0$, hence $o_\Gamma(D + \varepsilon A) = 0$ for any geometric valuation Γ over X. But then $o_\Gamma(D) = \lim\limits_{\varepsilon \to 0} o_\Gamma(D + \varepsilon A) = 0$ by Lemma 5.3, so we conclude by (i). □

Corollary 3.5. *Let X be a normal projective variety and let $D_1, \ldots, D_r$ be divisors in* $\mathrm{Div}_{\mathbb{Q}}(X)$*. Assume that the ring $\mathfrak{R} = R(X; D_1, \ldots, D_r)$ is finitely generated, and let $\operatorname{Supp}\mathfrak{R} = \bigcup_{i=1}^{N} \mathscr{C}_i$ be a finite rational polyhedral subdivision as in Theorem 1.23(3). Denote by $\pi\colon \mathrm{Div}_{\mathbb{R}}(X) \to N^1(X)_{\mathbb{R}}$ the natural projection.*

Then there is a set $I_1 \subseteq \{1, \ldots, N\}$ such that

$$\operatorname{Supp}\mathfrak{R} \cap \pi^{-1}\big(\overline{\mathrm{Mov}}(X)\big) = \bigcup_{i \in I_1} \mathscr{C}_i.$$

Assume further that $\operatorname{Supp}\mathfrak{R}$ *contains an ample divisor. Then there is a set $I_2 \subseteq \{1, \ldots, N\}$ such that the cone* $\operatorname{Supp}\mathfrak{R} \cap \pi^{-1}\big(\mathrm{Nef}(X)\big)$ *equals $\bigcup_{i \in I_2} \mathscr{C}_i$, and every element of this cone is semiample.*

Proof. For every prime divisor Γ on X denote $\mathscr{C}_\Gamma = \{D \in \operatorname{Supp}\mathfrak{R} \mid o_\Gamma(D) = 0\}$. If $\mathscr{C}_\Gamma$ intersects the interior of some $\mathscr{C}_\ell$, then $\mathscr{C}_\ell \subseteq \mathscr{C}_\Gamma$ since o_Γ is a linear non-negative function on $\mathscr{C}_\ell$. Therefore, there exists a set $I_\Gamma \subseteq \{1, \ldots, N\}$ such that $\mathscr{C}_\Gamma = \bigcup_{i \in I_\Gamma} \mathscr{C}_i$. Now the first claim follows since $\overline{\mathrm{Mov}}(X)$ is the intersection of all $\mathscr{C}_\Gamma$.

For the second claim, note that since $\operatorname{Supp}\mathfrak{R} \cap \pi^{-1}\big(\mathrm{Nef}(X)\big)$ is a cone of dimension $\dim \operatorname{Supp}\mathfrak{R}$, we can consider only maximal dimensional cones $\mathscr{C}_\ell$. Now, for every $\mathscr{C}_\ell$ whose interior contains an ample divisor, all asymptotic order functions o_Γ are identically zero on $\mathscr{C}_\ell$ similarly as above. Therefore, by Lemma 3.4, every element of $\mathscr{C}_\ell$ is semiample, and thus $\mathscr{C}_\ell \subseteq \operatorname{Supp}\mathfrak{R} \cap \pi^{-1}\big(\mathrm{Nef}(X)\big)$. The claim follows. □

Now Theorem 3.2 follows immediately from Corollary 3.5 once we take $D_1,\dots,D_r$ to be the basis of $\mathrm{Pic}(X)_{\mathbb{Q}}$ such that $\overline{\mathrm{Eff}}(X) \subseteq \sum \mathbb{R}_+ D_i$, and that the ring $R(X;D_1,\dots,D_r)$ is finitely generated, which we can according to Theorem 3.3.

3.2.2 Local shape inside of the big cone

Our goal is to show the following.

Theorem 3.6. *Let X be a variety of Calabi-Yau type.*

(1) The cone $\mathrm{Nef}(X) \cap \mathrm{Big}(X)$ *is locally rational polyhedral in* $\mathrm{Big}(X)$*, and every element of* $\mathrm{Nef}(X) \cap \mathrm{Big}(X)$ *is semiample.*

(2) The cone $\overline{\mathrm{Mov}}(X) \cap \mathrm{Big}(X)$ *is locally rational polyhedral in* $\mathrm{Big}(X)$.

Part (1) was first proved in [Kaw88, Theorem 5.7]. The problem of finding the shape of $\overline{\mathrm{Mov}}(X) \cap \mathrm{Big}(X)$ was posed in [Kaw88, Problem 5.10]. This was solved in [Kaw97, Corollary 2.7] for 3-folds, and in [KKL12, Theorem 3.8] in general.

The proof is very similar to that of Theorem 3.2. It is essentially the following statement.

Theorem 3.7. *Let X be a projective $\mathbb{Q}$-factorial variety of Calabi-Yau type, and let $B_1,\dots,B_q$ be big $\mathbb{Q}$-divisors on X. Then the ring*

$$R(X;B_1,\dots,B_q)$$

is finitely generated.

Proof. Let $\Delta \geq 0$ be a $\mathbb{Q}$-divisor such that (X,Δ) is klt and $K_X + \Delta \equiv 0$, and write $B_i = A_i + E_i$, where each A_i is ample and $E_i \geq 0$. Let $\varepsilon > 0$ be a rational number such that all pairs $(X, \Delta + \varepsilon E_i)$ are klt, and denote $A_i' = \varepsilon B_i - (K_X + \Delta + \varepsilon E_i)$. Then each A_i' is ample since $A_i' \equiv \varepsilon A_i$, hence the adjoint ring

$$R(X;K_X + \Delta + \varepsilon E_1 + A_1', \dots, K_X + \Delta + \varepsilon E_q + A_q') = R(X;\varepsilon B_1,\dots,\varepsilon B_q)$$

is finitely generated by Theorem 1.25. Therefore $R(X;B_1,\dots,B_q)$ is finitely generated by Lemma 1.21. □

Proof of Theorem 3.6. Let V be a relatively compact subset of the boundary of $\overline{\mathrm{Nef}}(X) \cap \mathrm{Big}(X)$, and denote by $\pi\colon \mathrm{Div}_{\mathbb{R}}(X) \to N^1(X)_{\mathbb{R}}$ the natural projection. Then we can choose finitely many big $\mathbb{Q}$-divisors $B_1,\dots,B_q$

such that $V \subseteq \pi(\sum_{i=1}^q \mathbb{R}_+ B_i)$. Theorem 3.7 implies that the ring $\mathfrak{R} = R(X;B_1,\dots,B_q)$ is finitely generated, and hence $\pi^{-1}\big(\overline{\mathrm{Nef}}(X)\big) \cap \operatorname{Supp}\mathfrak{R}$ is a rational polyhedral cone and its every element is semiample by Corollary 3.5. But then V is contained in finitely many rational hyperplanes. This shows (1), and the proof of (2) is similar. ☐

3.2.3 Number of good models

The main motivation for the Cone conjecture, in the realm of birational geometry, is that as a consequence it has finiteness of good models of any terminal variety. We prove that assertion in this section, together with some other predictions.

We first note the following most general version of finite generation of adjoint rings generalising Theorem 1.25, which is the expected consequence of the Minimal Model Program.

Theorem 3.8. *Assume the existence of good models for klt pairs in dimensions at most n. Let X be a $\mathbb{Q}$-factorial projective variety of dimension n, and let $\Delta_1,\dots,\Delta_r$ be $\mathbb{Q}$-divisors such that all pairs (X,Δ_i) are klt.*

Then the adjoint ring

$$R(X;K_X+\Delta_1,\dots,K_X+\Delta_r)$$

is finitely generated.

Proof. See [DHP13, Theorem 8.10]. A version of this result was proved in [SC11]. The difference is that the assumptions in [SC11] are stronger: the full force of the MMP was used, including termination of any sequence of flips. ☐

In particular, the finite generation of the adjoint rings is a theorem without any assumptions in dimensions up to 3.

Definition 3.9. Let $(X,\sum_{i=1}^p S_i)$ be a log smooth projective pair, where $S_1,\dots,S_p$ are distinct prime divisors, and let $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$. Given a birational contraction $f\colon X \dashrightarrow Y$, let $\mathscr{C}_f(V)$ denote the closure in $\mathscr{L}(V)$ in the standard topology of the set

$$\{\Delta \in \mathscr{E}(V) \mid f \text{ is a log terminal model of } (X,\Delta)\}.$$

The following is [DHP13, Theorem 8.10]; a similar result was proved in [SC11, Theorem 3.4], but as in the proof of Theorem 3.8, the assumptions were stronger.

Theorem 3.10. *Assume the existence of good models for klt pairs in dimensions at most n. Let* $(X, \sum_{i=1}^{p} S_i)$ *be a log smooth projective pair, where* $S_1, \dots, S_p$ *are distinct prime divisors, and let* $V = \sum_{i=1}^{p} \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$.

Then there exist birational contractions $f_i \colon X \dashrightarrow Y_i$ *for* $i = 1, \dots, k$, *such that* $\mathscr{C}_{f_1}(V), \dots, \mathscr{C}_{f_k}(V)$ *are rational polytopes and*

$$\mathscr{E}(V) = \bigcup_{i=1}^{k} \mathscr{C}_{f_i}(V).$$

In particular, $\mathscr{E}(V)$ *is a rational polytope.*

Together with the relative version of the Cone conjecture [Kaw97], the relative version of the previous theorem implies finiteness of minimal models up to isomorphism. The following theorem is folklore, but we include the proof for the benefit of the reader. The proof below came out of discussions with C. Xu.

Theorem 3.11. *Assume the MMP in dimension n and the relative Cone conjecture in dimensions* $\leq n$. *Let X be a terminal projective variety of dimension n.*

Then the number of minimal models of X is finite up to isomorphism.

Proof. Replacing X by a minimal model, we may assume that K_X is semiample, and let $X \to S$ be the canonical model. If Y is another minimal model of X and $A \subseteq Y$ is a very ample divisor over S, then the map $\varphi \colon X \dashrightarrow Y$ is an isomorphism in codimension 1, the divisor $D = \varphi^* A \subseteq X$ is movable over S and $Y \simeq \mathrm{Proj}_S R(X/S, D)$. Let Π be a fundamental domain for the action of $\mathrm{Bir}(X/S)$ on the cone $\overline{\mathrm{Mov}}(X/S) \cap \mathrm{Eff}(X/S)$. Then there exists $g \in \mathrm{Bir}(X/S)$ such that $g^* D \in \Pi$, and we have $R(X/S, D) \simeq R(X/S, g^* D)$ since g is an isomorphism in codimension 1. Replacing D by $g^* D$, we may assume that $D \in \Pi$.

Let $D_1, \dots, D_r$ be effective divisors whose classes generate Π and let $S_1, \dots, S_k$ be all the prime divisors in the support of $\sum D_i$. Let $V = \sum_{i=1}^{p} \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$, and let $\Pi' \subseteq V$ be the inverse image of Π under the natural map $V \to N^1(X)_{\mathbb{R}}$. Note that D belongs to set $\Pi' \cap \mathbb{R}_+ \mathscr{L}(V)$ since the pair $(X, \varepsilon D)$ is klt for some $0 < \varepsilon \ll 1$. Since K_X is trivial over S, by [SC11, Theorem 3.4] and Theorem 5.9, there are finitely many cones $\mathscr{C}_i \subseteq V$ and contractions $f_i \colon X \dashrightarrow Z_i$ for $i = 1, \dots, k$, such that $\Pi' \cap \mathbb{R}_+ \mathscr{L}(V) = \bigcup \mathscr{C}_i$ and if $\Delta \in \mathscr{C}_i \cap \mathscr{L}(V)$, then f_i is the ample model of $K_X + \Delta$ over S. In particular, there exists a cone $\mathscr{C}_i$ which contains D, and hence $Y \simeq Z_i$. □

3.2.4 Effective versus rational

As mentioned in Chapter 1, it seems to be a believed conjecture that

$$\mathrm{Nef}(X)_+ = \mathrm{Nef}(X) \cap \mathrm{Eff}(X),$$

although it is not clear what the evidence for it is. In Theorem 3.13 we show that at least one part of it is true, that

$$\mathrm{Nef}(X) \cap \mathrm{Eff}(X) \subseteq \mathrm{Nef}(X)_+.$$

We need the following result of Shokurov and Birkar, [Bir11, Proposition 3.2].

Theorem 3.12. *Let X be a $\mathbb{Q}$-factorial projective variety, let $S_1,\dots,S_p$ be prime divisors on X and denote $V = \bigoplus_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$. Then the set*

$$\mathcal{N}(V) = \{\Delta \in V \mid (X,\Delta) \text{ is log canonical and } K_X + \Delta \text{ is nef}\}$$

is a rational polytope.

Theorem 3.13. *Let X be a variety of Calabi-Yau type. Then*

$$\mathrm{Nef}(X) \cap \mathrm{Eff}(X) \subseteq \mathrm{Nef}(X)_+.$$

Proof. Fix D be an effective nef $\mathbb{R}$-divisor, that is, a divisor whose class is in $\mathrm{Nef}(X) \cap \mathrm{Eff}(X)$, and let $V \subseteq \mathrm{Div}_R(X)$ be the vector space spanned by all the components $D_1,\dots,D_r$ of D. By replacing D by εD for $0 < \varepsilon \ll 1$, we may assume that (X,D) is a klt pair, and in particular, with notation from Theorem 3.12, $D \in \mathcal{N}(V)$. On the other hand, clearly $D \in \sum_{i=1}^r \mathbb{R}_+ D_i \subseteq V$. By Theorem 3.12, the set

$$\mathcal{N}(V) \cap \sum_{i=1}^r \mathbb{R}_+ D_i$$

is a rational polytope, hence D is spanned by nef $\mathbb{Q}$-divisors. □

3.3 Preliminaries

In this section we give some basic definitions and gather results which we need in the rest of this chapter.

Notation 3.14. Assume that a Calabi-Yau manifold X has Picard number $\rho(X) = 2$. We let ℓ_1, ℓ_2 be the two boundary rays of $\mathrm{Nef}(X)$, and let m_1, m_2 be the boundary rays of $\overline{\mathrm{Mov}}(X)$. We fix non-trivial elements $x_i \in \ell_i$ and $y_i \in m_i$.

Recall the following result [Ogu14, Proposition 3.1].

Proposition 3.15. *Let X be a Calabi-Yau manifold of dimension n such that $\rho(X) = 2$.*

(1) If n is odd, or if one of the ℓ_i is rational, then every non-trivial element of $\mathscr{A}(X)$ has order 2.

(2) If one of the m_i is rational, then every non-trivial element of $\mathscr{B}(X)$ has order 2.

As a consequence, by using Burnside's theorem, Oguiso obtains:

Theorem 3.16. *Let X be a Calabi-Yau manifold of dimension n such that $\rho(X) = 2$.*

(1) If n is odd, then $\mathrm{Aut}(X)$ is finite.

(2) If n is even and one of the rays ℓ_i is rational, then $\mathrm{Aut}(X)$ is finite.

(3) If one of the rays m_i is rational, then $\mathrm{Bir}(X)$ is finite.

Proposition 3.20 below makes this result more precise. In contrast to Theorem 3.16, Oguiso constructed an example of Calabi-Yau manifold with $\rho(X) = 2$ such that $\mathrm{Bir}(X)$ is infinite. In this example both rays m_i are irrational, and we recall it in Example 3.32.

If g is any element of $\mathscr{B}(X)$, then $\det g = \pm 1$ since g acts on the integral lattice $N^1(X)$. We introduce the notations

$$\mathscr{A}^+(X) = \{g \in \mathscr{A}(X) \mid \det g = 1\}$$

and

$$\mathscr{A}^-(X) = \{g \in \mathscr{A}(X) \mid \det g = -1\};$$

and similarly $\mathscr{B}^+(X)$ and $\mathscr{B}^-(X)$. Note that each $g \in \mathscr{A}(X)$ restricts to an action on the set $\ell_1 \cup \ell_2$, and each $g \in \mathscr{B}(X)$ restricts to an action on the set $m_1 \cup m_2$. Moreover, since the cone $\overline{\mathrm{Eff}}(X)$ does not contain lines, this "restricted" action completely determines g. Additionally, each $g \in \mathscr{A}(X)$ is completely determined by gx_1 since $\det g = \pm 1$. Similarly, each $g \in \mathscr{B}(X)$ is completely determined by gy_1.

We frequently and without explicit mention use the following well-known lemma, see for instance [Kaw97, Lemma 1.5].

Lemma 3.17. *Let X be a Calabi-Yau manifold. Then $g \in \mathrm{Bir}(X)$ is an automorphism if and only if there exists an ample divisor H on X such that g^*H is ample.*

3.4 Calculating Aut(X) and Bir(X)

In this section we calculate explicitly the groups $\mathscr{A}(X)$ and $\mathscr{B}(X)$ on a Calabi-Yau manifold with Picard number 2. We start with some elementary observations.

Lemma 3.18. *Let X be a Calabi-Yau manifold such that $\rho(X) = 2$. If $g \in \mathscr{B}^-(X)$, then $g^2 = \mathrm{id}$.*

Proof. By assumption there exist $\alpha > 0$ and $\beta > 0$ such that $gy_1 = \alpha y_2$ and $gy_2 = \beta y_1$. But then $g^2 y_1 = \alpha\beta y_1$ and $g^2 y_2 = \alpha\beta y_2$, and we have $g^2 \in \mathscr{A}^+(X)$. Therefore $\det(g^2) = (\alpha\beta)^2 = 1$, so $\alpha\beta = 1$. Thus, g^2 is the identity. □

Lemma 3.19. *Let X be a Calabi-Yau manifold such that $\rho(X) = 2$. Then $\mathscr{B}^-(X) = \mathscr{B}^+(X)g$ for any $g \in \mathscr{B}^-(X)$. Similarly, $\mathscr{A}^-(X) = \mathscr{A}^+(X)h$ for any $h \in \mathscr{A}^-(X)$.*

In particular, if $\mathscr{B}(X)$ is infinite, so is $\mathscr{B}^+(X)$; and if $\mathscr{A}(X)$ is infinite, so is $\mathscr{A}^+(X)$.

Proof. Let $g, g' \in \mathscr{B}^-(X)$. Then $g'g = f \in \mathscr{B}^+(X)$, and since $g^2 = \mathrm{id}$ by Proposition 3.15, we have $g' = fg \in \mathscr{B}^+(X)g$. The proof in the case of automorphisms is identical. □

Proposition 3.20. *Let X be a Calabi-Yau manifold such that $\rho(X) = 2$. If $\mathscr{A}(X)$ is finite, then $|\mathscr{A}^+(X)| = 1$ and $|\mathscr{A}(X)| \leq 2$. If $\mathscr{B}(X)$ is finite, then $|\mathscr{B}^+(X)| = 1$ and $|\mathscr{B}(X)| \leq 2$.*

In particular, if n is odd, or if one of the ℓ_i is rational, then $|\mathscr{A}(X)| \leq 2$.

Proof. Assume that $\mathscr{A}(X)$ is finite, and fix $g \in \mathscr{A}(X)$. If $g \in \mathscr{A}^+(X)$, then there exists $\alpha > 0$ such that $gx_1 = \alpha x_1$. Then $g^m = \mathrm{id}$ for some positive integer m, hence $\alpha^m = 1$, and therefore $\alpha = 1$ and $\mathscr{A}^+(X) = \{\mathrm{id}\}$. Now $|\mathscr{A}(X)| \leq 2$ by Lemma 3.19. The proof for $\mathscr{B}(X)$ is the same, and the last claim follows from Theorem 3.16. □

Proposition 3.20 can also be directly deduced from the following elementary lemma, simplifying calculations in [Ogu14].

Lemma 3.21. *Let X be an n-dimensional Calabi-Yau manifold with $\rho(X) = 2$. Assume that $|\mathscr{A}^+(X)| \neq 1$. Then*

$$x_1^m \cdot x_2^{n-m} = 0$$

for all m unless $n = 2m$.

If $n = 2m$, then $x_1^m \neq 0$ and $x_2^m \neq 0$.

Proof. Let f be a non-trivial element in $\mathscr{A}^+$. Then $fx_1 = \alpha x_1$ and $fx_2 = \alpha^{-1}x_2$ with $\alpha > 0$, $\alpha \neq 1$. Then

$$(fx_1)^m \cdot (fx_2)^{n-m} = \alpha^{2m-n} x_1^m \cdot x_2^{n-m}.$$

On the other hand,

$$(fx_1)^m \cdot (fx_2)^{n-m} = x_1^m \cdot x_2^{n-m},$$

hence $x_1^m \cdot x_2^{n-m} = 0$ unless $n = 2m$.

For the second statement, observe that $x_1 + x_2$ is an ample class, hence

$$0 < (x_1 + x_2)^n = \binom{n}{m} x_1^m \cdot x_2^m,$$

and therefore the classes x_i^m are non-zero. □

Corollary 3.22. *Let X be a Calabi-Yau manifold of dimension n such that $\rho(X) = 2$. If the group* $\mathrm{Aut}(X)$ *is infinite, then the following holds.*

(1) n is even and the rays ℓ_i are irrational.

(2) $\mathrm{Nef}(X) = \overline{\mathrm{Eff}}(X)$*, and* $\mathrm{Nef}(X) \cap \mathrm{Eff}(X) = \mathrm{Amp}(X)$.

(3) $c_{n-1}(X) = 0$ *in* $H^{2n-2}(X, \mathbb{Q})$.

Proof. Claim (1) is Oguiso's Theorem 2.3.

For the first part of (2), if $\mathrm{Nef}(X) \neq \overline{\mathrm{Eff}}(X)$, then at least one boundary ray of $\mathrm{Nef}(X)$ is rational by Theorem 3.6. This contradicts (1). For the second part of (2), without loss of generality it suffices to show that x_1 is not effective. Otherwise, we can write $x_1 = \sum \delta_j D_j \geq 0$ as a sum of at least two prime divisors, since x_1 is irrational. But then ℓ_1 is not an extremal ray of the cone $\mathrm{Nef}(X) = \overline{\mathrm{Eff}}(X)$, a contradiction.

For (3), note that $|\mathscr{A}^+(X)| \geq 2$ by Lemma 3.19. Pick a non-trivial element $f \in \mathscr{A}^+(X)$, and let $\alpha \neq 1$ be a positive number such that $fx_1 = \alpha x_1$. Then

$$\alpha x_1 \cdot c_{n-1}(X) = fx_1 \cdot c_{n-1}(X) = x_1 \cdot c_{n-1}(X)$$

since the Chern class $c_{n-1}(X)$ is invariant under f. Thus $x_1 \cdot c_{n-1}(X) = 0$; similarly we get $x_2 \cdot c_{n-1}(X) = 0$. Therefore $c_{n-1}(X) = 0$ as $\{x_1, x_2\}$ is a basis of $N^1(X)_{\mathbb{R}}$. □

Remark 3.23. (1) The same arguments as in Corollary 3.22 yield

$$c_{i_1}(X) \cdot \ldots \cdot c_{i_r}(X) = 0$$

if $i_1 + \ldots + i_r = n-1$.
(2) We do not know of any example of a simply connected Calabi-Yau manifold X in the strong sense (i.e. such that $H^q(X, \mathscr{O}_X) = 0$ for $1 \leq q \leq n-1$) of even dimension n such that $c_{n-1}(X) = 0$. One might wonder whether any simply connected irreducible projective manifold X of dimension n with $\omega_X \simeq \mathscr{O}_X$ and $c_{n-1}(X) = 0$ is a hyperkähler manifold.

In some further cases, the even dimensional case can be treated:

Theorem 3.24. *Let X be a Calabi-Yau manifold of even dimension n. If $\rho(X) = 2$ and if $c_2(X)$ can be represented by a positive closed $(2,2)$-form, then* Aut(X) *is finite.*

Proof. Arguing by contradiction, we suppose that there is an automorphism $f \in \mathscr{A}^+(X)$ of infinite order, cf. Lemma 3.19. Write $n = 2m$. Then $x_1^m \neq 0$ and $x_2^m \neq 0$ by Lemma 3.21.

Suppose that m is even, and write $m = 2k$. Then

$$x_1^{2k} \cdot c_2(X)^k > 0$$

by our positivity assumption on $c_2(X)$. On the other hand,

$$x_1^{2k} \cdot c_2(X)^k = (f x_1)^{2k} \cdot c_2(X)^k = \alpha^{2k} x_1^{2k} \cdot c_2(X)^k$$

since $c_2(X)$ is invariant under f. Since $\alpha \neq 1$, this is a contradiction.

If m is odd, we write $n = 4s+2$ and argue with $x_1^{2s} \cdot c_2(X)^{s+1}$. □

Notice that for every projective manifold X of dimension n with nef canonical bundle, the second Chern class $c_2(X)$ has the following positivity property (Miyaoka [Miy87]):

$$c_2(X) \cdot H_1 \ldots \cdot H_{n-2} \geq 0$$

for all ample line bundles H_j.

Concerning bounds for $\mathscr{B}(X)$, we have:

Proposition 3.25. *Let X be a Calabi-Yau manifold such that $\rho(X) = 2$. Assume that* Nef(X) $\not\subseteq$ Mov(X)*. Then $\mathscr{A}^+(X) = \mathscr{B}^+(X)$. In particular, if the dimension of X is odd, then $|\mathscr{B}(X)| \leq 2$.*

Proof. The condition Nef(X) $\not\subseteq$ Mov(X) implies that one of the rays ℓ_i is an extremal ray of $\overline{\text{Mov}}(X)$. Hence, without loss of generality, we may assume that $m_1 = \ell_1$. Let g be a non-trivial element of $\mathscr{B}^+(X)$. Then $g\ell_1 = gm_1 = m_1$, and m_1 is an extremal ray of the cone

$$\mathbb{R}_+ m_1 + \mathbb{R}_+ g\ell_2 = \mathbb{R}_+ g\ell_1 + \mathbb{R}_+ g\ell_2 = g\,\mathrm{Nef}(X).$$

This implies that $g\,\mathrm{Nef}(X)$ intersects the interior of $\mathrm{Nef}(X)$, and hence $g \in \mathscr{A}(X)$ by Lemma 2.4. This proves the first claim.

The second claim then follows from Proposition 3.20. □

Theorem 3.26. *Let X be a Calabi-Yau manifold such that $\rho(X) = 2$. Then either $|\mathscr{A}^+(X)| = 1$ or $\mathscr{A}^+(X) \simeq \mathbb{Z}$; and either $|\mathscr{B}^+(X)| = 1$ or $\mathscr{B}^+(X) \simeq \mathbb{Z}$.*

Proof. Assume that $|\mathscr{A}^+(X)| \geq 2$. For every $g \in \mathscr{A}^+(X)$, let α_g be the positive number such that $gy_1 = \alpha_g y_1$, and set

$$\mathscr{S} = \{\alpha_g \mid g \in \mathscr{A}^+(X)\}.$$

Note that $\mathscr{S}$ is a multiplicative subgroup of $\mathbb{R}^*$ and that the map

$$\mathscr{A}^+(X) \to \mathscr{S}, \quad g \mapsto \alpha_g$$

is an isomorphism of groups. We need to show that $\mathscr{S}$ is an infinite cyclic group.

We first show that $\mathscr{S}$ is, as a set, bounded away from 1. Otherwise, we can pick a sequence (g_i) in $\mathscr{A}^+(X)$ such that α_{g_i} converges to 1. Fix two integral linearly independent classes h_1 and h_2 in $N^1(X)_\mathbb{R}$. Then $g_i h_1$ converge to h_1 and $g_i h_2$ converge to h_2. Since $g_i h_1$ and $g_i h_2$ are also integral classes and $N^1(X)$ is a lattice in $N^1(X)_\mathbb{R}$, this implies that $g_i h_1 = h_1$ and $g_i h_2 = h_2$ for $i \gg 0$, and hence $g_i = \mathrm{id}$ for $i \gg 0$.

Hence, the set $\mathscr{S}' = \{\ln\alpha \mid \alpha \in \mathscr{S}\}$ is an additive subgroup of $\mathbb{R}$ which is discrete as a set. Then it is a standard fact that $\mathscr{S}'$, and hence $\mathscr{S}$, is isomorphic to $\mathbb{Z}$, cf. [For81, 21.1].

The proof for the birational automorphism group is the same. □

3.5 Structures of $\mathrm{Nef}(X)$ and $\overline{\mathrm{Mov}}(X)$

Proposition 3.27. *Let X be a Calabi-Yau manifold such that $\rho(X) = 2$. If $\mathscr{A}(X)$ is finite, then the weak Cone conjecture holds for $\mathrm{Nef}(X)$. If $\mathscr{B}(X)$ is finite, then the weak Cone conjecture holds for $\overline{\mathrm{Mov}}(X)$.*

Proof. We only prove the statement about the nef cone, since the other statement is analogous. By Proposition 3.20, we have $|\mathscr{A}(X)| \leq 2$, hence we may assume that $|\mathscr{A}(X)| = 2$. Fix an integral class $x \in \text{Nef}(X)$, let $g \in \mathscr{A}^-(X)$, and consider the class $y = x + gx \in \text{Nef}(X)$. Then y is fixed under the action of $\mathscr{A}(X)$. Since g acts on $N^1(X)$, both gx and y must be integral. It is then obvious that $\Pi = \ell_1 + \mathbb{R}_+ y$ is a fundamental domain for the action of $\mathscr{A}(X)$ on Nef(X). □

Remark 3.28. If X is a Calabi-Yau manifold of odd dimension such that $\rho(X) = 2$ and $\text{Nef}(X) \not\subseteq \text{Mov}(X)$, then the weak Cone conjecture holds for $\overline{\text{Mov}}(X)$. The proof is analogous to that of Proposition 3.27, using Proposition 3.25.

Proposition 3.29. *Let X be a Calabi-Yau manifold such that $\rho(X) = 2$. Assume that* $\text{Nef}(X) \subseteq \text{Mov}(X)$. *Then the Cone conjecture holds for* Nef(X).

Proof. By assumption, we have $\text{Nef}(X) \subseteq \text{Big}(X)$, and hence, the nef cone is rational polyhedral by Theorem 3.6. Then argue as in the proof of Proposition 3.27. □

Lemma 3.30. *Let X be a Calabi-Yau manifold with $\rho(X) = 2$. Assume that* Bir(X) *is infinite. Then* $\overline{\text{Mov}}(X) \cap \text{Eff}(X) = \text{Mov}(X)$.

Proof. The rays of $\overline{\text{Mov}}(X)$ are irrational by Proposition 3.15, and therefore $\overline{\text{Mov}}(X) = \overline{\text{Eff}}(X)$ by Theorem 3.6. We cannot have $y_1 \in \text{Eff}(X)$: otherwise, we can write $y_1 = \sum \delta_i D_i \geq 0$ as a sum of at least two different prime divisors, since m_1 is irrational. But then m_1 is not an extremal ray of the cone $\overline{\text{Mov}}(X) = \overline{\text{Eff}}(X)$, a contradiction. This concludes the proof. □

Theorem 3.31. *Let X be a Calabi-Yau manifold with $\rho(X) = 2$. If the group* Bir(X) *is infinite, then the Cone conjecture holds on X.*

Proof. (i) First we show that the Cone conjecture holds for Nef(X) in case Aut(X) is infinite.

Note that $\text{Nef}(X) = \overline{\text{Eff}}(X)$ and $\text{Nef}(X) \cap \text{Eff}(X) = \text{Amp}(X)$ by Corollary 3.22(2), and in particular we have $\mathscr{A}(X) = \mathscr{B}(X)$. By Lemma 3.19 and Theorem 3.26, we know that $\mathscr{A}(X) = \mathscr{A}^+(X) \cup \mathscr{A}^-(X)$, where $\mathscr{A}^+(X) \simeq \mathbb{Z}$ and $\mathscr{A}^-(X) = \mathscr{A}^+(X)g$ for any $g \in \mathscr{A}^-(X)$.

Assume first that $\mathscr{A}(X) = \mathscr{A}^+(X) \simeq \mathbb{Z}$. Let h be a generator of $\mathscr{A}(X)$, let x be any point in Amp(X), and denote

$$\Pi = \mathbb{R}_+ x + \mathbb{R}_+ hx.$$

It is then straightforward to check that Π is a fundamental domain for the action of $\mathscr{A}(X)$ on $\mathrm{Amp}(X)$. Indeed, it is clear that the cones $h^k\Pi$ have disjoint interiors, and to see that they cover $\mathrm{Amp}(X)$, it suffices to notice that the rays $\mathbb{R}_+h^kx$ converge to ℓ_1, respectively ℓ_2, when $k \to \pm\infty$.

Now assume that $\mathscr{A}^-(X) \neq \emptyset$. Let f be a generator of $\mathscr{A}^+(X)$, let τ be an element of $\mathscr{A}^-(X)$, and let x be an integral class in $\mathrm{Amp}(X)$. Set

$$z_1 = x + \tau x \quad \text{and} \quad z_2 = z_1 + f z_1,$$

and note that z_1 and z_2 are integral classes since τ and f act on $N^1(X)$. Denote $\theta = f\tau \in \mathscr{A}^-(X)$. Then $\tau^2 = \theta^2 = \mathrm{id}$ by Lemma 3.18, and hence

$$\theta\tau = (f\tau)\tau = f \qquad \text{and} \qquad \theta f = \theta(\theta\tau) = \tau.$$

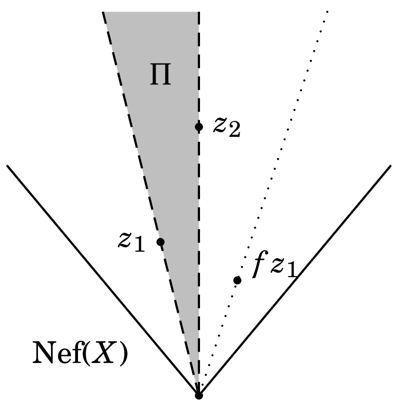
This implies

$$\tau z_1 = z_1, \qquad \theta z_1 = f z_1, \qquad \theta z_2 = z_2. \tag{3.1}$$

Now, let

$$\Pi = \mathbb{R}_+ z_1 + \mathbb{R}_+ z_2.$$

Then Π is a rational polyhedral cone, and we claim that Π is a fundamental domain for the action of $\mathscr{A}(X)$ on $\mathrm{Amp}(X)$.



First, by (3.1) we have

$$\theta\Pi = \mathbb{R}_+\theta z_1 + \mathbb{R}_+\theta z_2 = \mathbb{R}_+ f z_1 + \mathbb{R}_+ z_2,$$

and thus

$$\Pi \cup \theta\Pi = \mathbb{R}_+ z_1 + \mathbb{R}_+ f z_1.$$

This implies

$$\bigcup_{k\in\mathbb{Z}} f^k(\Pi \cup \theta\Pi) = \mathrm{Amp}(X)$$

as in the first part of the proof, and therefore,

$$\bigcup_{g\in\mathscr{A}(X)} g\Pi = \text{Amp}(X).$$

Second, assume that there exists $\lambda \in \mathscr{A}(X)$ such that $\text{int}\,\Pi \cap \text{int}\,\lambda\Pi \neq \emptyset$. Then, possibly after replacing λ by λ^{-1}, this implies that $\lambda z_1 \subseteq \text{int}\,\Pi$ or $\lambda z_2 \subseteq \text{int}\,\Pi$. If $\lambda z_1 \subseteq \text{int}\,\Pi$, then by Lemma 3.19 there exists $k \in \mathbb{Z}$ such that $\lambda = f^k\tau$, hence $\lambda z_1 = f^k z_1 \in \text{int}\,\Pi$ by (3.1), which is clearly impossible. Similarly, if $\lambda z_2 \subseteq \text{int}\,\Pi$, again by Lemma 3.19 there exists $\ell \in \mathbb{Z}$ such that $\lambda = f^\ell\theta$, hence $\lambda z_2 = f^\ell z_2 \in \text{int}\,\Pi$ by (3.1), a contradiction. This finishes the proof of (i).

(ii) Next we show that the Cone conjecture holds for Nef(X) if Aut(X) is finite but Bir(X) is infinite. Here Nef(X) $\subseteq$ Mov(X) by Lemma 3.19 and Proposition 3.25. Then the Cone conjecture for Nef(X) holds by Proposition 3.29.

(iii) Finally, note that $\overline{\text{Mov}}(X) \cap \text{Eff}(X) = \text{Mov}(X)$ by Lemma 3.30, hence the proof of the Cone conjecture for Mov(X) is the same as that of (i) by a simple adaption. ☐

Example 3.32. We recall [Ogu14, Proposition 6.1]. Oguiso constructs a Calabi-Yau 3-fold X with Picard number 2, obtained as the intersection of general hypersurfaces in $\mathbb{P}^3 \times \mathbb{P}^3$ of bi-degrees (1,1), (1,1), and (2,2), which has the following properties: x_1 and x_2 are rational, $y_1 = (3 + 2\sqrt{2})x_2 - x_1$, $y_2 = (3+2\sqrt{2})x_1 - x_2$, there are two birational involutions τ_1 and τ_2 such that $\tau_1\tau_2$ is of infinite order, and the group Bir(X) is generated by Aut(X) and by τ_1 and τ_2.

We now show that Example 3.32 is a typical example of a Calabi-Yau manifold with Picard number 2 and with infinite group of birational automorphisms.

Theorem 3.33. *Let X be a Calabi-Yau manifold of dimension n and with $\rho(X) = 2$. Assume that* Bir(X) *is infinite.*

(1) Let f be a generator of $\mathscr{B}^+(X)$, and let $\alpha > 0$ be the real number such that $f y_1 = \alpha y_1$. Then $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 2$.

(2) Let $\{v, w\}$ be any integral basis of $N^1(X)_\mathbb{R}$. Then $m_1 = \mathbb{R}_+(av + bw)$ and $m_2 = \mathbb{R}_+(cv + dw)$, where $a, b, c, d \in \mathbb{Q}(\alpha)$.

(3) There exist a birational automorphism τ (possibly the identity) such that $\tau^2 \in$ Aut(X)*, and a birational automorphism of infinite order σ such that the group* Bir(X) *is generated by* Aut(X) *and by τ and σ.*

Proof. By rescaling y_1 and y_2, we can assume that

$$h = y_1 + y_2$$

is a primitive integral class in $N^1(X)_{\mathbb{R}}$. Denote

$$h' = fh = \alpha y_1 + \frac{1}{\alpha} y_2 \quad \text{and} \quad h'' = f^2 h = \alpha^2 y_1 + \frac{1}{\alpha^2} y_2;$$

these are again primitive integral classes since $\mathscr{B}(X)$ preserves $N^1(X)$. Then an easy calculation shows that

$$h + h'' = \frac{\alpha^2 + 1}{\alpha} h',$$

and hence the number $\frac{\alpha^2+1}{\alpha} = \alpha + \frac{1}{\alpha}$ is an integer. Since

$$y_1 = \frac{1}{\alpha^2 - 1}(\alpha h' - h),$$

and y_1 is not rational by Theorem 3.16, the number α cannot be rational, and (1) follows.

For (2) fix an integral basis $\{v, w\}$ of $N^1(X)_{\mathbb{R}}$, and write

$$y_1 = av + bw \quad \text{and} \quad y_2 = cv + dw.$$

Then

$$h = (a+c)v + (b+d)w \quad \text{and} \quad h' = (\alpha a + c/\alpha)v + (\alpha b + d/\alpha)w.$$

Write $p = a + c$ and $q = \alpha a + c/\alpha$, and note that $p, q \in \mathbb{Z}$. Then an easy calculation shows that $a, c \in \mathbb{Q}(\alpha)$, and similarly for b and d.

Finally, for (3), note that by Theorem 3.26 and Lemma 3.19, we have $\mathscr{B}(X) = \mathscr{B}^+(X) \cup \mathscr{B}^-(X)$, where $\mathscr{B}^+(X)$ is infinite cyclic with generator σ', and $\mathscr{B}^-(X) = \mathscr{B}^+(X)\tau'$ for any $\tau' \in \mathscr{B}^-(X)$. Pick $\tau, \sigma \in \mathrm{Bir}(X)$ such that

$$r(\tau) = \tau' \quad \text{and} \quad r(\sigma) = \sigma',$$

see Notation 3.14. Since $r(\tau^2) = \tau'^2 = \mathrm{id}$ by Lemma 3.18, it follows that τ^2 is an isomorphism by [Ogu14, Proposition 2.4]. Now take an element θ is any element of $\mathrm{Bir}(X)$, then there exist integers k and ℓ such that $r(\theta) = \sigma'^k \tau'^\ell = r(\sigma^k \tau^\ell)$, and we conclude again by [Ogu14, Proposition 2.4]. □

Remark 3.34. One can obtain a similar description of the cone Nef(X) when the automorphism group of X is infinite.

Basically there are two types of simply connected irreducible Calabi-Yau manifolds: those which do not carry any holomorphic forms of intermediate degree – these manifolds are often simply called Calabi-Yau manifolds – and hyperkähler manifolds carrying a non-degenerate holomorphic 2-form. While in the hyperkähler case the nef cone can be irrational by [Ogu14, Proposition 1.3], it is believed that the nef cone of a "strict" Calabi-Yau manifold with, say, $\rho(X) = 2$, must be rational. The evidence is provided by the fact that in odd dimensions Aut(X) is finite, and then the Cone conjecture would imply the rationality. In even dimensions we saw that an infinite automorphism group on a strict Calabi-Yau manifold with Picard number two is possible only in very special circumstances.

Chapter 4

Topological considerations

4.1 Introduction

The results of this chapter are taken from [CL14]. They represent the first attempt to bound the number of minimal models of a given log smooth pair of dimension 3 with respect to the underlying topology as a complex manifold. Our main result is the following.

Theorem 4.1. *Let p and ρ be positive integers, and let ε be a positive rational number. Let $(X, \sum_{i=1}^{p} S_i)$ be a 3-dimensional log smooth pair such that:*

(i) X is not uniruled,

(ii) $S_1, \ldots, S_p$ are distinct prime divisor which are not contained in $\mathbf{B}(K_X + \sum_{i=1}^{p} a_i S_i)$ for all $0 \le a_i \le 1$,

(iii) the divisors S_i span $\mathrm{Div}_{\mathbb{R}}(X)$ up to numerical equivalence,

(iv) $\rho(X) \le \rho$ and $\rho(S_i) \le \rho$ for all $i = 1, \ldots, p$.

Let I be the total number of irreducible components of intersections of each two and each three of the divisors $S_1, \ldots, S_p$.

There exists a constant C that depends only on p, ρ, ε and I such that for any $\Delta = \sum_{i=1}^{p} \delta_i S_i$ with $\delta_i \in [\varepsilon, 1-\varepsilon]$ and (X, Δ) terminal, the number of log terminal models of (X, Δ) is at most C.

The proof is an easy consequence of our main technical result, Theorem 4.17 below. An immediate corollary is the following result announced in Chapter 1.

Theorem D. *Let ε be a positive number. Let $\mathfrak{X}$ be the collection of all log smooth 3-fold terminal pairs $(X, \Delta = \sum_{i=1}^{p} \delta_i S_i)$ such that:*

(1) X is not uniruled,

(2) $\varepsilon \leq \delta_i \leq 1-\varepsilon$ for all i,

(3) $S_1,\dots,S_p$ are distinct prime divisor not contained in

$$\mathbf{B}(K_X + \sum_{i=1}^{p} a_i S_i)$$

for all $0 \leq a_i \leq 1$, and

(4) S_i span $\mathrm{Div}_{\mathbb{R}}(X)$ up to numerical equivalence.

Then for every $(X_0, \Delta_0) \in \mathfrak{X}$ there exists a constant N such that for every $(X, \Delta) \in \mathfrak{X}$ of the topological type as (X_0, Δ_0), the number of log terminal models of (X, Δ) is bounded by N.

In the proofs we use the full force of the 3-dimensional MMP. Our main tools are Shokurov's log geography [Sho96] and the techniques involved in the proof of termination of 3-fold flips. The log geography has played an important role in studying the birational geometry of projective varieties: for instance, it was recently used to prove the Sarkisov Program for Mori fibre spaces [HM13]. We believe that a more accurate study of Fano threefolds combined with the results of this chapter will give a new insight on the classification of Fano threefolds [Cor09].

4.2 Preliminary results

The size of a set S is denoted by $\#S$. The notation $N = N(a_1,\dots,a_k)$ means that the constant N depends only on the parameters $a_1,\dots,a_k$.

4.2.1 Divisors, valuations and models

We will use the following lemma in Section 4.3.

Lemma 4.2. *Let $(X, \sum_{i=1}^{p} b_i S_i)$ be a log smooth terminal threefold pair, where $S_1,\dots,S_p$ are distinct prime divisors. Let*

$$f : X \dashrightarrow X'$$

be a birational contraction to a terminal threefold X'. Let S'_i be the proper transform of S_i in X' for every i. Let Y be a smooth variety, let $g\colon Y \to X$ be a birational morphism, and let $E \subseteq Y$ be an $(f\circ g)$-exceptional prime divisor such that the centre of E on X' is a curve. Then

$$a\Big(E, X', \sum_{i=1}^{p} b_i S'_i\Big) = a(E, X', 0) - \sum_{i=1}^{p} b_i \operatorname{mult}_E S'_i, \tag{4.1}$$

where $a(E,X',0)$ is an integer such that $0 < a(E,X',0) < \rho(Y/X')$.

Proof. It is easy to show the identity (4.1). Let $T \subseteq X'$ be a general ample surface, and let W be its proper transform on Y. Since X' is terminal and $c_{X'}(E)$ is a curve, after possibly replacing X with a smaller open subset of X, we may assume that $T \cap c_{X'}(E)$ is a smooth point of X' by [KM98, Corollary 5.39]. Then the induced map $W \to T$ is a birational morphism and W is obtained from T by blowing up $\rho(W/T)$ times.

Let $(p,q)\colon Z \to Y \times X'$ be a resolution of $f\circ g$. Then since T is general we have $T' := q^*T = q_*^{-1}T$, and hence

$$K_Z + T' = q^*(K_{X'} + T) + \Gamma$$

for some q-exceptional divisor $\Gamma \geq 0$. Restricting this equality to T' and pushing forward to X, we obtain $a(E,X',0) = a(W\cap E,T,0)$, which is clearly a positive integer. Since $T \cap c_{X'}(E)$ is smooth, it is easy to see from the discrepancy formulas that $a(W\cap E,T,0) \leq \rho(W/T)$. Finally, observe that since T is general, $\rho(W/T)$ is bounded by the number of $(f\circ g)$-exceptional divisors on Y, hence it is bounded by $\rho(Y/X')$. □

Lemma 4.3. *Let (X,Δ) be a canonical projective pair, and let $f\colon X \dashrightarrow Y$ be a $(K_X+\Delta)$-nonpositive birational contraction. Assume that f does not contract any component of Δ, and let $\Delta_Y = f_*\Delta$.*

Then (Y,Δ_Y) is canonical. Additionally, if f is $(K_X+\Delta)$-negative and (X,Δ) is terminal, then (Y,Δ_Y) is terminal.

Proof. This follows easily from the definitions. □

The following result is inspired by [KM98, Proposition 2.36] and by [AHK07, Lemma 1.5].

Lemma 4.4. *Let $(X, \Delta = \sum_{i=1}^{p} a_i S_i)$ be a 3-dimensional log smooth terminal pair with $0 < a_i < 1$, and let $Z \subseteq \sum_{i=1}^{p} S_i$ be a union of m curves. Let I be the total number of points of intersection of each three of the divisors $S_1, \dots, S_p$.*

Then there exists a constant $N = N(m,p,a_1,\dots,a_p,I)$ such that the number of geometric valuations E on X with $c_X(E) \subseteq Z$ and $a(E,X,\Delta) < 1$ is bounded by N. Furthermore, the number of blow-ups along smooth centres needed to realise the valuations is bounded by N.

Proof. After possibly replacing X by a smaller open subset, we may assume that $S_i \cap S_j \subseteq Z$ for any distinct $i,j \in \{1,\dots,p\}$. Since (X,Δ) is log smooth, by first blowing up intersections of triples of components S_i, and then intersections of each two of them, we obtain a composition of $M = M(m,p,I)$ blowups $f\colon Y \to X$ such that we may write

$$K_Y + \Gamma = f^*(K_X + \Delta) + E_Y,$$

where Γ and E_Y are effective $\mathbb{R}$-divisors with no common components, (Y,Γ) is log smooth, E_Y is f-exceptional and the components of Γ are pairwise disjoint. In particular, there are at most M prime divisors E on Y such that $a(E,X,\Delta) < 1$. Also, note that by discrepancy formulas, the discrepancies $a(E,X,\Delta)$ which lie in the interval $(0,1)$ are of the form $2-a_i-a_j-a_k$ or $1-a_i-a_j$ for some pairwise different i,j,k.

It remains to count valuations which are exceptional over Y. Let $g\colon W \to X$ be a log resolution which dominates Y, and let $W' \to W$ be a blowup along a smooth centre with exceptional divisor F. Then it is easy to see that if $a(F,X,\Delta) < 1$, then $c_W(F)$ is the intersection of the proper transform of some S_i and some prime divisor G on Y with $0 < a(G,X,\Delta) < 1$.

For each curve $C \subseteq Z$, if f^{-1} is an isomorphism at the generic point of C, let $C' \subseteq Y$ be the unique curve isomorphic to C at the generic point of C'; otherwise, let C' be the union of curves on Y which map onto C, and which are of the form $f_*^{-1}S_i \cap F$ for some prime divisor $F \subseteq Y$ with $0 < a(F,X,\Delta) < 1$. Hence, there are at most $m + mM$ such curves, let Z' be their union, and by shrinking X we may assume that all the curves in Z' are smooth. Then, similarly as in [AHK07, Example 1.4], there are at most $N' = N'(m,M,a_1,\dots,a_p)$ valuations over Y with discrepancy smaller than 1 and whose centres lie in Z'. Now set $N = N' + m$. □

Let (X,Δ) be a klt pair of dimension n, and let $f\colon X \dashrightarrow Y$ be a good model of (X,Δ). Then the prime divisors contracted by f are precisely those that are contained in $\mathbf{B}(K_X + \Delta)$. The following lemma, which will be extensively used in Section 4.3, establishes a similar link between ample models and the augmented base loci.

Lemma 4.5. *Let X be a smooth projective threefold and let D be a big $\mathbb{Q}$-divisor on X. Let $f\colon X \dashrightarrow Y$ be the ample model of D.*

Then $\mathbf{B}_+(D)$ *coincides with the exceptional locus of* f.

Proof. The result follows immediately from [BCL13, Theorem A]. □

In special circumstances, the restriction of an MMP for a pair (X,Δ) to a prime divisor S on X induces an MMP on S. The following lemma is just a minor reformulation of [BCHM10, Lemma 4.1], and follows from the proof of that result.

Lemma 4.6. *Let* $(X,S+B)$ *be a log smooth pair, where* S *is a prime divisor and* $\lfloor B\rfloor = 0$, *and let* $\varphi\colon X \dashrightarrow X'$ *be a weak log canonical model of* K_X+S+B. *Assume that* φ *does not contract* S, *let* $S' = \varphi_*S$ *and* $B' = \varphi_*B$, *and let* $\sigma\colon S \dashrightarrow S'$ *be the induced birational map. Define a divisor* Ψ *on* S' *by* $(K_{X'}+S'+B')|_{S'} = K_{S'}+\Psi$.

If $(S,B|_S)$ *is terminal, then there is a divisor* $\Xi \le B|_S$ *such that* $\sigma_*\Xi = \Psi$ *and* σ *is a weak log canonical model of* $K_S+\Xi$.

The next lemma, combined with Lemma 4.6, shows that under certain assumptions, the restriction of the ample model is again the ample model on the restriction.

Lemma 4.7. *Let* $(X,S+B)$ *be a plt pair, where* S *is a prime divisor and* $\lfloor B\rfloor = 0$. *Assume that* $D = K_X+S+B$ *is semiample, and let* $f\colon X \to Y$ *be the corresponding fibration. Assume that* $f(S) \neq Y$ *and let* $g = f|_S$.

Then g *is the semiample fibration associated to* $D_{|S}$.

Proof. Fix a sufficiently divisible positive integer m such that f is the map associated to the linear system $|mD|$, and let A be an ample $\mathbb{Q}$-divisor on Y such that $D = f^*A$. Then g is the map associated to the linear system $|mD|_S$, and it is enough to show that $|mD|_S = |mD_{|S}|$. From a long exact sequence in cohomology, this in turn is equivalent to showing that the map

$$H^1(X,mD-S) \to H^1(X,mD)$$

is injective. Since $mD-S = K_X+B+(m-1)f^*A$, this follows from [Kol95, (10.19.3)]. □

Remark 4.8. The assumption $f(S) \neq Y$ in Lemma 4.7 is necessary. Indeed, let Y be a curve of genus ≥ 2. Let $\mathscr{E}$ be a sufficiently ample vector bundle of rank 2 on Y, set $X = \mathbb{P}(\mathscr{E})$, and let $f\colon X \to Y$ be the projection map. Then, by assumption, the line bundle $\xi = c_1(\mathscr{O}(1))$ is very ample, and let $S \in |2\xi|$ be a general section. If $G = c_1(\mathscr{E})$, then $K_X+S = f^*(K_Y+G)$, and since K_Y+G is ample, f is the semiample fibration associated to K_X+S. However, the general fibre of f meets S in two points, thus $f|_S$ does not have connected fibres.

4.2.2 Convex geometry

Lemma 4.9. *Let $\mathscr{C} \subseteq \mathbb{R}^p$ be a rational polytope which is defined by half-spaces*

$$\left\{(x_1,\dots,x_p)\in\mathbb{R}^p \mid \textstyle\sum_{j=1}^p \alpha_{ij}x_j \ge \beta_i\right\}$$

for $i = 1,\dots,\ell$, where α_{ij} and β_i are integers. Let M be a positive integer such that

$$\alpha_{ij} \ge -M \qquad \text{and} \qquad |\beta_i| < M$$

for all i,j. Pick a positive real number $\varepsilon < 1$.

Then there exists a positive integer m which depends only on M, p and ε (but not on $\mathscr{C}$), such that for every extreme point v of $\mathscr{C}$ which is contained in $[\varepsilon,1]^p$, the point mv is integral.

Proof. Since $v = (v_1,\dots,v_p)$ is an extreme point of $\mathscr{C}$, after relabelling we may assume that $\sum_{j=1}^p \alpha_{ij}v_j = \beta_i$ for $i = 1,\dots,p$. Denoting by A the $(p\times p)$-matrix (α_{ij}), we may additionally assume that the rows of A are linearly independent over $\mathbb{R}$. In particular, $\det A \neq 0$ and Cramer's rule implies that $\det A\cdot v$ is integral. By assumption, we have

$$\sum_{\alpha_{ij}<0}\alpha_{ij} + \varepsilon\sum_{\alpha_{ij}>0}\alpha_{ij} \le \sum_{j=1}^p \alpha_{ij}v_j = \beta_i < M,$$

and since $\alpha_{ij} \ge -M$, we have

$$|\alpha_{ij}| < \frac{Mp}{\varepsilon} \qquad \text{for all} \quad i,j = 1,\dots,p.$$

Therefore, $\det A$ is bounded by a constant m_0 which depends on M, p and ε, and the claim follows by taking $m = m_0!$. □

Definition 4.10. Let $\mathscr{P}_1,\mathscr{P}_2 \subseteq \mathbb{R}^p$ be polytopes of dimension p. We say that $\mathscr{P}_i$ are *adjacent* if $\mathscr{P}_1\cap\mathscr{P}_2$ is a codimension one face of both $\mathscr{P}_1$ and $\mathscr{P}_2$.

Let $\mathscr{P} = \bigcup_{i=1}^k \mathscr{P}_i$ be a (not necessarily convex) finite union of polytopes. We say that $\mathscr{P}_i$ and $\mathscr{P}_j$ are *adjacent-connected* if there exist indices $i_1,\dots,i_q$ such that $i_1 = i$, $i_q = j$, and $\mathscr{P}_{i_s}$ and $\mathscr{P}_{i_{s+1}}$ are adjacent for every $s = 1,\dots,q-1$. The equivalence classes of this relation are called *adjacent-connected components*. If the whole $\mathscr{P}$ belongs to one such component, we say that $\mathscr{P}$ is also adjacent-connected. A *face* of $\mathscr{P}$ is a face of any $\mathscr{P}_i$ which is not contained in the interior of $\mathscr{P}$.

Lemma 4.11. *Let $\mathscr{Q} \subseteq [0,1]^p \subseteq \mathbb{R}^p$ be a polytope containing the origin, and let $\mathscr{C}_1,\dots,\mathscr{C}_\ell$ be p-dimensional polytopes with pairwise disjoint interiors such that $\mathscr{Q} = \bigcup_{i=1}^{\ell} \mathscr{C}_i$. Let $\mathscr{P}_1,\dots,\mathscr{P}_k \subseteq \mathscr{Q}$ be p-dimensional polytopes such that*

$$(\mathscr{P}_i + \mathbb{R}^p_+) \cap \mathscr{Q} \subseteq \mathscr{P}_i \tag{4.2}$$

for all i. For any subset $I \subseteq \{1,\dots,k\}$, denote by $\mathscr{R}_I$ the closure of the set $\bigcup_{i\in I} \mathscr{P}_i \setminus \bigcup_{j\notin I} \mathscr{P}_j$, and let $\mathscr{R}_0$ denote the closure of $\mathscr{Q} \setminus \bigcup_{i=1}^{k} \mathscr{P}_i$. Assume that each adjacent-connected component of every $\mathscr{R}_I$ and of $\mathscr{R}_0$ with respect to the covering $\mathscr{Q} = \bigcup_{i=1}^{\ell} \mathscr{C}_i$ is the union of at most m polytopes $\mathscr{C}_i$.

Then there exists a constant $M = M(k,m)$ such that $\ell \le M$.

Proof. If $x = (x_1,\dots,x_p) \in \mathscr{R}_0$ and $y = (y_1,\dots,y_p) \in \mathscr{Q}$ are such that $y_i \le x_i$ for all $i = 1,\dots,p$, then $y \in \mathscr{R}_0$ by (4.2). Therefore, the set $\mathscr{R}_0$ is adjacent-connected, and hence it contains at most m polytopes $\mathscr{C}_i$.

For any $d = 1,\dots,p$, let $\mathscr{J}_d$ be the set of codimension d faces of $\mathscr{R}_0$ which are not contained in the boundary of $\mathscr{Q}$. Since the polytopes $\mathscr{C}_i$ and $\mathscr{P}_j$ are convex, and $\mathscr{R}_0$ contains at most m polytopes $\mathscr{C}_i$, it follows that each $\mathscr{P}_j$ contains at most m elements of $\mathscr{J}_1$, and hence $\#\mathscr{J}_1 \le mk$. Now, if $d > 1$, each element of $\mathscr{J}_{d-1}$ contains at most $\#\mathscr{J}_{d-1}$ elements of $\mathscr{J}_d$, and therefore $\#\mathscr{J}_d \le (\#\mathscr{J}_{d-1})^2$. This shows that $\#\mathscr{J}_d \le (mk)^{2^{d-1}}$.

Since

$$\bigcup_{i\in I} \mathscr{P}_i \Big\backslash \bigcup_{j\notin I} \mathscr{P}_j = \bigcup_{i\in I} \Big(\mathscr{P}_i \Big\backslash \bigcup_{j\notin I} \mathscr{P}_j\Big),$$

it is enough to bound the number of adjacent-connected components of each set $\mathscr{P}_i \setminus \bigcup_{j\notin I} \mathscr{P}_j$. The statement is trivial for $k = 1$, hence by induction we may assume that $I = \{1,\dots,k\}$ and, without loss of generality, that $i = 1$. For any element $F \in \mathscr{J}_1$, set $F_1 = F \cap \mathscr{P}_1$ and by (4.2) we have that $\mathscr{F}_1 := (F_1 + \mathbb{R}^p_+) \cap \mathscr{Q} \subseteq \mathscr{P}_1$. Thus, it is easy to see that

$$\mathscr{P}_1 \Big\backslash \bigcup_{j=2}^{k} \mathscr{P}_j = \bigcup_{F\in\mathscr{J}_1} \Big(\mathscr{F}_1 \Big\backslash \bigcup_{j=2}^{k} \mathscr{P}_j\Big),$$

hence it is enough to bound the number of adjacent-connected components contained in $\mathscr{F}_1 \setminus \bigcup_{j=2}^{k} \mathscr{P}_j$. Again by (4.2), it is enough to bound the number of adjacent-connected components of $F_1 \setminus \bigcup_{j=2}^{k} \mathscr{P}_j$, with respect to the induced topology on F_1. Note that every codimension $d-1$ face of an adjacent-connected component of $F_1 \setminus \bigcup_{j=2}^{k} \mathscr{P}_j$ is an element of $\mathscr{J}_d$. Hence, the number of such adjacent-connected components is bounded by a constant which depends only on all $\#\mathscr{J}_d$, and the lemma follows. □

4.3 Minimal models of threefolds

Lemma 4.12. *Let $(X, S = \sum_{i=1}^{p} S_i)$ be a log smooth projective threefold, where $S_1, \dots, S_p$ are distinct prime divisors, and assume that $0 < \varepsilon \le 1/2$ is a rational number such that $(X, \varepsilon S)$ is terminal and $K_X + \varepsilon S$ is big. Assume that $S_i \not\subseteq \mathbf{B}_+(K_X + \varepsilon S)$ for every i. Let I be the total number of irreducible components of intersections of each two of the divisors $S_1, \dots, S_p$.*

Then for any i, the number of curves contained in

$$\mathbf{B}_+(K_X + \varepsilon S) \cap S_i$$

is bounded by a constant which depends on $\rho(X)$, $\rho(S_i)$, ε and I.

Proof. Fix an index i. Then there exists a sequence of $(K_X + \varepsilon S)$-flips and divisorial contractions

$$f\colon X = X^0 \dashrightarrow \cdots \dashrightarrow X^k \to X^{k+1} \tag{4.3}$$

such that X^k is a log terminal model of $(X, \varepsilon S)$ and X^{k+1} is the ample model $(X, \varepsilon S)$. Since $S_i \not\subseteq \mathbf{B}_+(K_X + \varepsilon S)$, the divisor S_i is not contracted by this MMP by Lemma 4.5. Let S_ℓ^j and $\overline{S}_\ell^j$ denote the proper transform of S_ℓ in X^j and its normalisation for every $\ell = 1, \dots, p$, and set $S^j = \sum_{\ell=1}^{p} S_\ell^j$. Thus, there are induced sequences

$$g\colon S_i = S_i^0 \dashrightarrow S_i^1 \dashrightarrow \cdots \dashrightarrow S_i^k \dashrightarrow S_i^{k+1}$$

and

$$\overline{g}\colon S_i = \overline{S}_i^0 \dashrightarrow \overline{S}_i^1 \dashrightarrow \cdots \dashrightarrow \overline{S}_i^k \dashrightarrow \overline{S}_i^{k+1}$$

By Lemma 4.5, if C is a curve contained in $\mathbf{B}_+(K_X + \varepsilon S) \cap S_i$, then $C \subseteq \operatorname{Exc}(f)$.

We first assume that g is an isomorphism at the generic point of C. Then there exists an f-exceptional prime divisor $E \subseteq X$ containing C such that $f(C) = f(E) \subseteq X^{k+1}$; otherwise, the exceptional set of f would be 1-dimensional in a neighbourhood of C, hence g would not be an isomorphism at the generic point of C. In particular, since $(X^{k+1}, \varepsilon S^{k+1})$ is canonical and $f(C)$ is contained in S^{k+1}, it follows that X^{k+1} is terminal at the general point of $f(C)$. By Lemmas 4.3 and 4.2, we have

$$0 \le a(E, X^{k+1}, \varepsilon S^{k+1}) \le \rho(X) - \varepsilon \operatorname{mult}_E S^{k+1} \le \rho(X) - \varepsilon \operatorname{mult}_{f(E)} S^{k+1},$$

and in particular

$$\operatorname{mult}_{f(E)} S_i^{k+1} < \rho(X)/\varepsilon.$$

Therefore, for each f-exceptional divisor E, there are at most $\rho(X)/\varepsilon$ curves in $E \cap S_i$ which map to $f(E)$. Since there are at most $\rho(X/X^{k+1})$ such divisors E, the number of curves $C \subseteq \mathbf{B}_+(K_X + \varepsilon S) \cap S_i$ which are not contracted by g is at most $\rho(X)^2/\varepsilon$.

It remains to count the curves $C \subseteq \mathbf{B}_+(K_X + \varepsilon S) \cap S_i$ such that g is not an isomorphism at the generic point of C, and it suffices to count the curves contracted by each of the maps $g_j \colon S_i^j \dashrightarrow S_i^{j+1}$. Let $\overline{g}_j \colon \overline{S}_i^j \dashrightarrow \overline{S}_i^{j+1}$ be the induced maps of normalisations, and let N_j is the number of curves *extracted* by $\overline{g}_j$. First note that for each curve contracted by g_j there exists at least one curve contracted by $\overline{g}_j$. Thus, there are at most $\rho(\overline{S}_i^j) - \rho(\overline{S}_i^{j+1}) + N_j$ curves contracted by g_j, and we need to bound the number $\rho(S_i) + \sum_{j=0}^k N_j$.

If $N_j \neq 0$, then $X^j \dashrightarrow X^{j+1}$ must be a flip (hence necessarily $j < k$), and furthermore, N_j is the number of flipped curves contained in S_i^{j+1}. For each such a curve Γ, let E_Γ be the exceptional divisor obtained by blowing up Γ which dominates Γ. Then, by Lemma 4.3, X^{j+1} is terminal and therefore it is smooth at the generic point of Γ by [KM98, Corollary 5.39]. Thus,

$$0 \leq a(E_\Gamma, X, \varepsilon S) < a(E_\Gamma, X^{j+1}, \varepsilon S^{j+1}) = 1 - \varepsilon \operatorname{mult}_\Gamma S^{j+1} \leq 1 - \varepsilon, \qquad (4.4)$$

where the last inequality follows from $\operatorname{mult}_\Gamma S_i^{j+1} \geq 1$.

Let $\mathcal{V}$ be the set of all valuations which are either f-exceptional prime divisors on X, or obtained as the exceptional divisor on the blow-up of a curve in $S_\ell \cap S_i$ for each $\ell \neq i$; then it is clear that $\#\mathcal{V} \leq \rho(X) + I$. Viewing each E_Γ as a valuation, we first claim that $E_\Gamma \in \mathcal{V}$ for all Γ. Indeed, assume that the centre of E_Γ on X is a point $x \in X$. If E_Γ is obtained by blowing up x, then as (X,S) is log smooth, we have

$$a(E_\Gamma, X, \varepsilon S) = 2 - \varepsilon \operatorname{mult}_x S \geq 2 - 3\varepsilon \geq 1 - \varepsilon,$$

which is a contradiction with (4.4). The case when E_Γ is obtained by blowing up a point on a birational model of X also follows since the discrepancies increase by blowing up, as $(X, \varepsilon S)$ is terminal. Therefore, the centre of E_Γ on X is either a divisor or a curve, and then the rest of the claim follows by analogous computations. In particular, we have $N_j \leq \#\mathcal{V} \leq \rho(X) + I$ for each j.

Next we want to estimate how many times it happens that $N_j \neq 0$. In other words, we want to find an upper bound on the number of varieties X^{j+1} on which a valuation in $\mathcal{V}$ is realised as the exceptional divisor of a blow-up of a flipped curve on X^{j+1}. Fix $E \in \mathcal{V}$, and consider the number

$$M_E^{j+1} = \operatorname{mult}_E S^{j+1} \in \mathbb{N}.$$

If E is realised as the exceptional divisor on the blow-up of a flipped curve on X^{j+1}, then

$$0 \leq a(E, X^{j+1}, \varepsilon S^{j+1}) = 1 - \varepsilon M_E^{j+1},$$

and hence $M_E^{j+1} \leq 1/\varepsilon$ for all j. Since at each step of (4.3) the discrepancies are increasing, the sequence M_E^{j+1} is decreasing. Therefore, each $E \in \mathcal{V}$ is realised as an exceptional divisor on the blow-up of a flipped curve at most $1/\varepsilon$ times, hence

$$\sum_{j=0}^{k} N_j \leq \frac{\rho(X) + I}{\varepsilon}.$$

Putting all this together, we get that the number of curves contained in $\mathbf{B}_+(K_X + \varepsilon S) \cap S_i$ is at most

$$\rho(S_i) + \frac{\rho(X)^2 + \rho(X) + I}{\varepsilon},$$

which proves the lemma. □

Definition 4.13. Let X be a projective $\mathbb{Q}$-factorial variety, let $S_1, \dots, S_p$ be distinct prime divisors on X, and denote $V = \sum_{i=1}^{p} \mathbb{R} S_i \subseteq \operatorname{Div}_{\mathbb{R}}(X)$. For each $\varepsilon \geq 0$, define

$$\mathcal{L}_\varepsilon(V) = \Big\{ \sum_{i=1}^{p} a_i S_i \in V \mid a_i \in [\varepsilon, 1 - \varepsilon] \Big\}.$$

Similarly as in Definition 1.24, it is easy to check that for each ε, the set $\mathcal{L}_\varepsilon(V)$ is a rational polytope.

Lemma 4.14. *Let $(X, S = \sum_{i=1}^{p} S_i)$ be a log smooth projective threefold, where $S_1, \dots S_p$ are distinct prime divisors, and denote $V = \sum_{i=1}^{p} \mathbb{R}_+ S_i \subseteq \operatorname{Div}_{\mathbb{R}}(X)$. Assume that $S_j \not\subseteq \mathbf{B}_+(K_X + B)$ for all $B \in \mathcal{L}(V)$ such that $K_X + B$ is big and for all j. Let I be the total number of irreducible components of intersections of each two of the divisors $S_1, \dots, S_p$.*

Then for any j, and for every rational number $\varepsilon > 0$ such that $(X, \varepsilon S)$ is terminal and $K_X + \varepsilon S$ is big, the number of curves contained in

$$\bigcup_{B \in \mathcal{L}_\varepsilon(V)} \mathbf{B}_+(K_X + B) \cap S_j$$

is bounded by a constant which depends on $\rho(X)$, $\rho(S_j)$, p, ε and I.

Proof. By Lemma 4.12 there exists a constant $M = M(\varepsilon, I, \rho(X), \rho(S_j))$ such that the number of curves in $\mathbf{B}_+(K_X + \varepsilon S) \cap S_j$ is bounded by M.

Without loss of generality, we may assume that $\varepsilon < 1/2$. Let

$$\mathscr{L}'(V) = \{B = \sum a_i S_i \mid a_i \in [\varepsilon, 1]\},$$

and let $B_1, \dots, B_{2^p}$ be the extreme points of $\mathscr{L}'(V)$. Since $\mathscr{L}_\varepsilon(V) \subseteq \mathscr{L}'(V)$, it follows that

$$\bigcup_{B \in \mathscr{L}_\varepsilon(V)} \mathbf{B}_+(K_X + B) \subseteq \bigcup_{i=1}^{2^p} \mathbf{B}_+(K_X + B_i).$$

Hence, it is enough to bound the number of curves in $\mathbf{B}_+(K_X + B_i) \cap S_j$ for every $i = 1, \dots, 2^p$. Fix i, and note that $\operatorname{mult}_{S_j} B_i \in \{\varepsilon, 1\}$. We distinguish two cases.

If $\operatorname{mult}_{S_j} B_i = 1$, set $T = \varepsilon \sum_{k \neq j} S_k + S_j$. Then $(S_j, (\varepsilon \sum_{k \neq j} S_k)|_{S_j})$ is terminal, and let $f\colon X \dashrightarrow X'$ be the ample model of $K_X + T$. By assumption and by Lemma 4.5, f does not contract S_j and by Lemmas 4.6 and 4.7, the MMP for (X, T) induces an MMP for some terminal pair (S_j, Θ). In particular, since S_j is a surface, this induced MMP contracts at most $\rho(S_j)$ curves. Further, if a curve $C \subseteq \mathbf{B}_+(K_X + T) \cap S_j$ is not contracted by the MMP for (S_j, Θ), then similarly as in Lemma 4.12, there exists a f-exceptional divisor E on X such that $f(E) = f(C)$. Since the pair (X, T) is plt, the strict transform $S'_j = f_* S_j$ is normal, hence $\operatorname{mult}_{f(C)} S'_j = 1$. Therefore, for each f-exceptional divisor E, there is at most one curve in $E \cap S_i$ which maps to $f(E)$. Since there are at most $\rho(X/X')$ such divisors E, the number of such curves C is at most $\rho(X)$.

It follows that the number of curves inside $\mathbf{B}_+(K_X + T) \cap S_j$ is bounded by $\rho(S_j) + \rho(X)$. We have

$$\begin{aligned}\mathbf{B}_+(K_X + B_i) \cap S_j &\subseteq (\mathbf{B}_+(K_X + T) \cup \operatorname{Supp}(B_i - T)) \cap S_j, \\ &\subseteq \Big(\mathbf{B}_+(K_X + T) \cup \bigcup\nolimits_{k \neq j} S_k\Big) \cap S_j,\end{aligned}$$

and hence the number of curves inside $\mathbf{B}_+(K_X + B_i) \cap S_j$ is at most $\rho(S_j) + \rho(X) + I$.

Finally, if $\operatorname{mult}_{S_j} B_i = \varepsilon$, then, since $B_i \geq \varepsilon S$, we have

$$\begin{aligned}\mathbf{B}_+(K_X + B_i) \cap S_j &\subseteq \big(\mathbf{B}_+(K_X + \varepsilon S) \cup \mathbf{B}_+(B_i - \varepsilon S)\big) \cap S_j \\ &\subseteq \Big(\mathbf{B}_+(K_X + \varepsilon S) \cup \bigcup\nolimits_{k \neq j} S_k\Big) \cap S_j.\end{aligned}$$

Thus, the number of curves in $\mathbf{B}_+(K_X + B_i) \cap S_j$ is bounded by $M + I$ and the result follows. ☐

Definition 4.15. Let $(X, \sum_{i=1}^p S_i)$ be a log smooth projective pair of dimension n, where $S_1, \dots, S_p$ are distinct prime divisors and denote $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$. For each $\varepsilon \geq 0$, define

$$\mathcal{L}_\varepsilon^{\mathrm{can}}(V) = \{\Delta \in \mathcal{L}_\varepsilon(V) \mid (X, \Delta) \text{ is canonical}\}.$$

This is easily seen to be a rational polytope, and note that if the dimension of $\mathcal{L}_\varepsilon^{\mathrm{can}}$ is p and Δ is contained in its interior, then (X, Δ) is terminal.

If $f\colon X \dashrightarrow Z$ is a birational contraction, and if $\mathscr{C} = \mathscr{C}_f(V) \cap \mathcal{L}_\varepsilon^{\mathrm{can}}(V)$ a polytope of dimension p which intersects the interior of $\mathcal{L}_\varepsilon^{\mathrm{can}}(V)$, then $\mathscr{C}$ is called a *terminal chamber* in V. Now, assume the existence of good models in dimension n. Then, with notation from Theorem 3.10, there are finitely many terminal chambers

$$\mathscr{C}_i = \mathscr{C}_{f_i}(V) \cap \mathcal{L}_\varepsilon^{\mathrm{can}}(V).$$

Lemma 4.16. *Let $(X, \sum_{i=1}^p S_i)$ be a 3-dimensional log smooth pair such that K_X is pseudoeffective, $S_1, \dots, S_p$ are distinct prime divisor, and let $V = \sum_{i=1}^p \mathbb{R}_+ S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$. Assume that $S_i \nsubseteq \mathbf{B}(K_X + B)$ for all $B \in \mathcal{L}_\varepsilon(V)$ and every $i = 1, \dots, p$. Let $F_1, \dots, F_\ell$ be all the prime divisors contained in $\mathbf{B}(K_X)$, and for every $\nu \subseteq \{1, \dots, \ell\}$, define*

$$\mathscr{B}_\nu = \{B \in \mathcal{L}_\varepsilon^{\mathrm{can}}(V) \mid F_i \subseteq \mathbf{B}(K_X + B) \textit{ if and only if } i \in \nu\}.$$

Let $\mathscr{C}_i$ be the terminal chambers in V (cf. Definition 4.15), for $1 \leq i \leq k$. Assume that each adjacent-connected component of every $\mathscr{B}_\nu$ with respect to the covering by $\mathscr{C}_i$ is the union of at most m polytopes $\mathscr{C}_i$.

Then there exists a constant $M = M(\ell, m)$ such that $k \leq M$.

Proof. For any $B \in \mathcal{L}_\varepsilon^{\mathrm{can}}(V)$ we have $\mathbf{B}(K_X + B) \subseteq \mathbf{B}(K_X) \cup \mathbf{B}(B)$, hence by assumptions, any prime divisor in $\mathbf{B}(K_X + B)$ must be one of F_j. For each $1 \leq i \leq \ell$ denote

$$\mathscr{P}_i = \{B \in \mathcal{L}_\varepsilon(V) \mid F_i \nsubseteq \mathbf{B}(K_X + B)\}.$$

Then for any $\nu \subsetneq \{1, \dots, \ell\}$, the set $\mathscr{B}_\nu$ is the closure of $\bigcup_{i \notin \nu} \mathscr{P}_i \setminus \bigcup_{j \in \nu} \mathscr{P}_j$, and $\mathscr{B}_{\{1,\dots,\ell\}}$ is the closure of $\mathcal{L}_\varepsilon^{\mathrm{can}}(V) \setminus \bigcup_{i=1}^\ell \mathscr{P}_i$. It is clear that every $\mathscr{P}_i$ satisfies the relation (4.2) on page 73, and we conclude by Lemma 4.11. $\square$

Theorem 4.17. *Let p and ρ be positive integers, and let ε be a positive rational number. Let $(X, \sum_{i=1}^p S_i)$ be a 3-dimensional log smooth pair such that:*

(i) K_X *is pseudoeffective;*

(ii) $S_1,\dots,S_p$ *are distinct prime divisor which are not contained in* $\mathbf{B}(K_X+B)$ *for all* $B\in\mathscr{L}(V)$*,*

(iii) the vector space $V=\sum_{i=1}^p \mathbb{R}S_i\subseteq \mathrm{Div}_{\mathbb{R}}(X)$ *spans* $\mathrm{Div}_{\mathbb{R}}(X)$ *up to numerical equivalence,*

(iv) $\rho(X)\le\rho$ *and* $\rho(S_i)\le\rho$ *for all* $i=1,\dots,p$*.*

Let I *be the total number of irreducible components of intersections of each two and each three of the divisors* $S_1,\dots,S_p$*.*

Then there exists a constant $N=N(p,\rho,\varepsilon,I)$ *such that the number of terminal chambers in V which intersect the interior of* $\mathscr{L}_\varepsilon(V)$ *is at most* N*.*

Proof. Let $(X,\sum_{i=1}^p S_i)$ be a 3-dimensional log smooth pair satisfying the conditions (i)–(iv). Note that K_X+B is big for every $B\in\mathscr{L}_\varepsilon(V)$. Let $C_1,\dots,C_q$ be all the curves contained in

$$\bigcup_{B\in\mathscr{L}_\varepsilon(V)} \mathbf{B}_+(K_X+B)\cap S.$$

Then q is bounded by a constant depending on p, ρ, ε and I by Lemma 4.14. By Lemma 4.4, there are finitely many geometric valuations $E_1,\dots,E_m$ such that $c_X(E_j)\subseteq\bigcup_{i=1}^q C_i$ for all j and $a(E_j,X,B)<1$ for some $B\in\mathscr{L}_\varepsilon(V)$, and $m\le M=M(q,\rho,\varepsilon,I)$.

Let $F_1,\dots,F_\ell$ be all the prime divisors in $\mathbf{B}(K_X)$. Then by (ii), for every $B\in\mathscr{L}_\varepsilon(V)$ the divisorial part of $\mathbf{B}(K_X+B)$ is contained in $\sum F_i$. Let $f=f_B\colon X\dashrightarrow X_B$ be a log terminal model of (X,B). For every $\nu\subseteq\{1,\dots,\ell\}$, let

$$\mathscr{B}_\nu=\{B\in\mathscr{L}_\varepsilon(V)\mid F_i \text{ is contracted by } f_B \text{ if and only if } i\in\nu\}.$$

Then by Lemma 4.16, it is enough to bound the number of terminal chambers which intersect each adjacent-connected component of each $\mathscr{B}_\nu$.

Hence, from now on we fix such ν and we assume, as we may, that each $\mathscr{B}_\nu$ is adjacent-connected. We will show that the number of terminal chambers which intersect B_ν is bounded by a constant depending only on p, ρ and ε, which is enough to conclude.

Set $\mu=\rho+M$; then μ depends only on p,ρ, ε and I by above. Let $\mathscr{S}$ be the set of all p-tuples $(m_1,\dots,m_p)\in\mathbb{N}^p$ such that $m_i<\mu/\varepsilon$ for every i. Then $\#\mathscr{S}<(\mu/\varepsilon)^p$. Let $\mathscr{H}$ be the set of all hyperplanes $\langle\Sigma_1-$

$\Sigma_2,\mathbf{x}\rangle = r$, where $\Sigma_1 \neq \Sigma_2$ are elements of $\mathscr{S}$ and $-\mu < r < \mu$ is an integer. Then $\#\mathscr{H} \leq 2\mu\binom{(\mu/\varepsilon)^p}{2}$. The elements of $\mathscr{H}$ subdivide $\mathscr{B}_v$ into at most $2^{\#\mathscr{H}}$ polytopes, and by replacing $\mathscr{B}_v$ by any of these polytopes, we may assume that none of the elements of $\mathscr{H}$ intersects the interior of $\mathscr{B}_v$. It is now enough to show that there is exactly one terminal chamber whose interior intersects $\mathscr{B}_v$.

Assume that there are two adjacent terminal chambers $\mathscr{C}'$ and $\mathscr{C}''$ whose interiors intersect $\mathscr{B}_v$. Let X' and X'' be the corresponding log terminal models, let $B = \sum_{i=1}^p b_i S_i$ be a divisor in $\mathscr{C}''$, and let B' and S_i', respectively B'' and S_i'' be the proper transforms of B and S_i on X' and X''. Note that X' and X'' are terminal by Lemma 4.3. Denote $\mathbf{b} = (b_1,\dots,b_p)$ and let $\langle,\rangle$ denote the standard scalar product on V. For each geometric valuation E on X, define

$$\Sigma_{E,\mathscr{C}'} = (\operatorname{mult}_E S_1',\dots,\operatorname{mult}_E S_p'),\quad \Sigma_{E,\mathscr{C}''} = (\operatorname{mult}_E S_1'',\dots,\operatorname{mult}_E S_p'').$$

By the definition of $\mathscr{B}_v$, and possibly by relabelling the chambers, we may assume that the induced map $X' \dashrightarrow X''$ is the flip of (X',B'). Note that X' is the ample model of (X,Δ) for any Δ in the interior of $\mathscr{C}'$, and similarly for $\mathscr{C}''$. Let $C \subseteq X''$ be a flipped curve, and let E be the valuation on X'' obtained by blowing up C which dominates C. Since X'' is smooth at the generic point of C by [KM98, Corollary 5.39], we have

$$0 < a(E,X,B) < a(E,X'',B'') = 1 - \langle \Sigma_{E,\mathscr{C}''},\mathbf{b}\rangle \leq 1. \tag{4.5}$$

It is easy to see from the discrepancy formulas that then $c_X(E)$ belongs to some of the divisors $S_1,\dots,S_p$ since (X,B) is terminal and $a(E,X,B) < 1$. Moreover, if B belongs to the interior of $\mathscr{C}''$, then $X'' = \operatorname{Proj} R(X,K_X+B)$. Hence, $c_X(E)$ is contained in $\mathbf{B}_+(K_X+B)$ by Lemma 4.5, and this shows that E is one of the valuations $E_1,\dots,E_m$.

Furthermore, by Lemma 4.2 we have

$$0 < a(E,X',B') = \mu_{E,B} - \langle \Sigma_{E,\mathscr{C}'},\mathbf{b}\rangle \tag{4.6}$$

for some integer $0 < \mu_{E,B} < \mu$. Since $b_i \geq \varepsilon$ for all i, we have

$$0 \leq \operatorname{mult}_E S_i' < \mu/\varepsilon \quad \text{for all } i,$$

and in particular, $\Sigma_{E,\mathscr{C}'} \in \mathscr{S}$.

Now, if $B \in \mathscr{C}' \cap \mathscr{C}''$, then by Lemma 1.27 we have

$$a(E,X',B') = a(E,X'',B'').$$

Together with (4.5), (4.6) and the fact that none of the elements $\mathscr{H}$ intersects the interior of $\mathscr{B}_V$, this implies that $\Sigma_{E,\mathscr{C}'} = \Sigma_{E,\mathscr{C}''}$.

On the other hand, if B belongs to the interior of $\mathscr{C}''$, then Lemma 1.27 again gives

$$a(E,X',B') < a(E,X'',B''),$$

and this together with (4.5) and (4.6) implies $\mu_{E,B} < 1$, which is a contradiction. □

We are now ready to give proofs of our main results.

Proof of Theorem 4.1. The number of terminal chambers inside of the set

$$\{\sum a_i S_i \mid a_i \in [\varepsilon/2, 1-\varepsilon/2]\}$$

is bounded by a constant $N = N(p,\rho,\varepsilon/2)$ by Theorem 4.17. We set $C = N$. □

Proof of Theorem D. It is clear that the total number of irreducible components of intersections of each two and each three of the components of Δ_0 and Δ is the same under a homeomorphism which preserves the topological type of (X,Δ_0). Therefore, the result follows immediately from Theorem 4.17. □

Chapter 5

Geography of models

5.1 Introduction

The results of this chapter are taken from [KKL12].

As mentioned in Chapter 1, there are two classes of projective varieties whose birational geometry is particularly interesting and rich. The first family consists of varieties where the classical Minimal Model Program (MMP) can be performed successfully with the current techniques. The other class is that of Mori Dream Spaces. We now know that, in both cases, the geometry of birational contractions from the varieties in question is entirely determined by suitable finitely generated divisorial rings.

More precisely, let X be a $\mathbb{Q}$-factorial projective variety that belongs to one of these two classes. Then, there are effective $\mathbb{Q}$-divisors $D_1,\dots,D_r$ strongly related to the geometry of X such that the multigraded divisorial ring

$$\mathfrak{R} = R(X;D_1,\dots,D_r)$$

is finitely generated. In the first case, $\mathfrak{R}$ is an adjoint ring; in the second, it is a Cox ring. Then, for any divisor D in the span $\mathscr{S} = \sum \mathbb{R}_+ D_i$, finite generation implies the existence of a birational map $\varphi_D\colon X \dashrightarrow X_D$, where φ_D is a composition of elementary surgery operations that can be fully understood. Both X_D and $(\varphi_D)_*D$ have good properties: X_D is projective and $\mathbb{Q}$-factorial, and $(\varphi_D)_*D$ is semiample.

In addition, there is a decomposition of $\mathscr{S} = \bigcup \mathscr{S}_j$ into finitely many rational polyhedral cones, together with birational maps $\varphi_j\colon X \dashrightarrow X_j$, such that the pushforward under φ_j of every divisor in $\mathscr{S}_j$ is a nef divisor on X_j. We say that these models $\varphi_j\colon X \dashrightarrow X_j$ are *optimal*, see Definition 1.16. By analogy with the classical case, the map $\varphi_j\colon X \dashrightarrow X_j$ is called a *D-MMP*. After Shokurov, the decomposition of $\mathscr{S}$ above is called a *geography* of optimal models associated to $\mathfrak{R}$.

The goal of this chapter is twofold. On the one hand, we want to put these two families of varieties under the same roof. That is to say, we want to identify the maximal class of varieties and divisors on them where a suitable MMP can be performed. On the other hand, we want to understand why this class is the right one, i.e. what the key ingredients that make the MMP work are.

Let D be a $\mathbb{Q}$-divisor on a variety X in one of the two families above. The D-MMP has two significant features, which we would like to extend to a more general setting:

(i) all varieties in the MMP are $\mathbb{Q}$-factorial,

(ii) the section ring $R(X,D)$ is preserved under the operations of the MMP.

Condition (ii) is by now well understood: contracting maps that preserve sections of D are D-nonpositive. Somewhat surprisingly, preserving $\mathbb{Q}$-factoriality is the main obstacle to extending the MMP to arbitrary varieties X and divisors D, even when the rings $R(X,D)$ are finitely generated; this is explained in Section 5.3.

As mentioned in Chapter 1, we introduce the notion of *gen* divisors, see Definition 1.20. Ample divisors are examples of gen divisors. As we explain in Section 5.3, in the situations of interest to us, these form essentially the only source of examples: indeed, all gen divisors there come from ample divisors on the end products of some MMP. However, one should bear in mind that semiample divisors are not necessarily gen.

As announced in Chapter 1, the main result of this chapter is the following.

Theorem E. *Let X be a projective $\mathbb{Q}$-factorial variety, let $D_1,\dots, D_r$ be effective $\mathbb{Q}$-divisors on X, and assume that the numerical classes of D_i span $N^1(X)_\mathbb{R}$. Assume that the ring $R(X;D_1,\dots,D_r)$ is finitely generated, that the cone $\sum \mathbb{R}_+D_i$ contains an ample divisor, and that every divisor in the interior of this cone is gen.*

Then there is a finite decomposition

$$\sum \mathbb{R}_+D_i = \coprod \mathscr{N}_i$$

into cones having the following properties:

(1) each $\overline{\mathscr{N}_i}$ is a rational polyhedral cone,

(2) *for each i, there exists a $\mathbb{Q}$-factorial projective variety X_i and a birational contraction $\varphi_i\colon X \dashrightarrow X_i$ such that φ_i is a good model for every divisor in $\mathcal{N}_i$.*

Our work has been influenced by several lines of research. The original idea that geographies of various models are the right thing to look at is due to Shokurov [Sho96], and the first unconditional results were proved in [BCHM10]. Similar decompositions were considered in the context of Mori Dream Spaces by Hu and Keel [HK00], and as we demonstrate here, these are closely related to the study of asymptotic valuations in [ELM$^+$06]. Theorem E reproves and generalises some of the main results from these papers. We obtain in Corollary 5.21 the finiteness of models due to [BCHM10] by using the main theorem from [CL12]. Further, in Corollary 5.22 we prove a characterisation of Mori Dream Spaces in terms of the finite generation of their Cox rings due to [HK00] without using GIT techniques.

We spend a few words on the organisation of the chapter. Section 5.2 sets the notation and gathers some preliminary results. In Section 5.3, we show the existence of a decomposition $\sum \mathbb{R}_+ D_i = \coprod \mathcal{A}_i$ similar to that from Theorem E, where all divisors in a given chamber $\mathcal{A}_i$ have a common *ample model*, see Theorem 5.9. We study the geography of ample models. The main drawback of this decomposition is that the corresponding models are not $\mathbb{Q}$-factorial in general. Moreover, we show in Example 5.14 that the conditions of Theorem 5.9 are not sufficient to ensure the existence of optimal models as in Theorem E. We explain why the presence of gen divisors is essential to the proof of Theorem E. However, we give a short proof that some of these models are indeed $\mathbb{Q}$-factorial in the case of adjoint divisors in Theorem 5.12.

In Section 5.4, we define what is meant by the MMP in our setting; it is easy to see that this generalises the classical MMP constructions. We then prove Theorem 5.19, which is a strengthening of Theorem E. The main technical result is Theorem 5.17, and the presence of gen divisors is essential to its proof. We mention here that this reveals the philosophical role of the gen condition: it enables one to prove a version of the classical Basepoint free theorem, which is why we can then run the Minimal Model Program and preserve $\mathbb{Q}$-factoriality in the process. We end the chapter with several corollaries that recover quickly some of the main results from [BCHM10] and [HK00].

5.2 Preliminary results

Asymptotic valuations. The following definition is due to Nakayama.

Definition 5.1. Let X be a normal projective variety, let D be an $\mathbb{R}$-Cartier divisor such that $|D|_{\mathbb{R}} \neq \emptyset$, and let Γ be a geometric valuation over X. If D is a big divisor, we define

$$N_\sigma(D) = \textstyle\sum_\Gamma o_\Gamma(D)\cdot\Gamma \quad \text{and} \quad P_\sigma(D) = D - N_\sigma(D),$$

where the sum runs over all prime divisors Γ on X.

Remark 5.2. On a surface X, the construction above gives the classical Zariski decomposition: this is a unique decomposition $D = P_\sigma(D) + N_\sigma(D)$, where $P_\sigma(D)$ is nef, and $N_\sigma(D) = \sum \gamma_i\Gamma_i$ is an effective divisor such that $P_\sigma(D)\cdot\Gamma_i = 0$ for all i, and the matrix $(\Gamma_i\cdot\Gamma_j)$ is negative definite. We use this characterisation in Example 5.14.

Lemma 5.3. *Let X be a $\mathbb{Q}$-factorial projective variety, let D be a big $\mathbb{R}$-divisor, and let Γ be a prime divisor. Then $o_\Gamma(D)$ depends only on the numerical class of D. The function o_Γ is homogeneous of degree one, convex and continuous on* Big(X)*. The formal sum $N_\sigma(D)$ is an $\mathbb{R}$-divisor, the divisor $P_\sigma(D)$ is movable, and for any $\mathbb{R}$-divisor $0 \le F \le N_\sigma(D)$ we have $N_\sigma(D-F) = N_\sigma(D) - F$. If $E \ge 0$ is an $\mathbb{R}$-divisor on X such that $D - E \in \overline{\mathrm{Mov}}(X)$, then $E \ge N_\sigma(D)$.*

Proof. See [Nak04, §III.1]. □

In certain situations we have more information on the divisor $P_\sigma(D)$.

Lemma 5.4. *Let X be a $\mathbb{Q}$-factorial projective variety, and let D be a big $\mathbb{Q}$-divisor on X. Assume that the cone $\overline{\mathrm{Mov}}(X)$ is rational polyhedral.*

Then $P_\sigma(D)$ is a $\mathbb{Q}$-divisor, and $R(X,D)$ is finitely generated if and only if $R(X,P_\sigma(D))$ is finitely generated.

Proof. Let Γ_i be the components of $N_\sigma(D)$, and denote

$$\mathscr{H} = D - \sum \mathbb{R}_+\Gamma_i \quad \text{and} \quad \mathscr{G} = P_\sigma(D) - \sum \mathbb{R}_+\Gamma_i.$$

Then we have $\overline{\mathrm{Mov}}(X) \cap \mathscr{H} \subseteq \mathscr{G}$ by Lemma 5.3. Since $\overline{\mathrm{Mov}}(X) \cap \mathscr{H}$ is an intersection of finitely many rational half-spaces, and as $P_\sigma(D) \in \overline{\mathrm{Mov}}(X)$ is an extremal point of $\mathscr{G}$, we conclude that $P_\sigma(D)$ is a $\mathbb{Q}$-divisor.

For the second statement, we may assume that D is an integral divisor and that $|D| \neq \emptyset$, so the claim follows from $P_\sigma(mD) \ge \mathrm{Mob}(mD)$ for every positive integer m. □

The proof of the following lemma is analogous to that of [CL13, Lemma 5.2], and it will be used in Section 5.4 to ensure that a certain MMP terminates.

Lemma 5.5. *Let $f\colon X \dashrightarrow Y$ be a birational contraction between projective $\mathbb{Q}$-factorial varieties, and let $\mathscr{C} \subseteq \operatorname{Div}_{\mathbb{R}}(X)$ be a cone such that f is D-nonpositive for all $D \in \mathscr{C}$. Let Γ be a geometric valuation on $k(X)$.*

Then o_Γ is linear on $\mathscr{C}$ if and only if it is linear on the cone $f_*\mathscr{C} \subseteq \operatorname{Div}_{\mathbb{R}}(Y)$.

Proof. Let $(p,q)\colon W \to X \times Y$ be a resolution of f. Then for every $D \in \mathscr{C}$ we have $p^*D = q^*f_*D + E_D$, where $E_D \geq 0$ is a q-exceptional divisor. This implies that f_* restricts to an isomorphism between $|D|_{\mathbb{R}}$ and $|f_*D|_{\mathbb{R}}$. Denote

$$V_D = \{D_X - D \mid D_X \in |D|_{\mathbb{R}}\} \quad \text{and} \quad W_D = \{D_Y - f_*D \mid D_Y \in |f_*D|_{\mathbb{R}}\}.$$

By the above, we have the isomorphism $f_*|_{V_D}\colon V_D \simeq W_D$, and we also have $\operatorname{mult}_\Gamma P_X = \operatorname{mult}_\Gamma f_*P_X$ for every $P_X \in V_D$ by [CL13, Lemma 5.1(2)]. Therefore

$$\begin{aligned} o_\Gamma(D) - \operatorname{mult}_\Gamma D &= \inf_{P_X \in V_D} \operatorname{mult}_\Gamma P_X \\ &= \inf_{P_X \in V_D} \operatorname{mult}_\Gamma f_*P_X = o_\Gamma(f_*D) - \operatorname{mult}_\Gamma f_*D, \end{aligned}$$

hence the function $o_\Gamma(\cdot) - o_\Gamma\big(f_*(\cdot)\big)\colon \mathscr{C} \to \mathbb{R}$ is equal to the linear map $\operatorname{mult}_\Gamma(\cdot) - \operatorname{mult}_\Gamma f_*(\cdot)$. The lemma follows. ☐

Finite generation and the stable base locus. As Example 5.6 below shows, the stable base locus and finite generation of section rings are not, in general, numerical invariants. However, we prove in Lemma 5.7 that under some finite generation hypotheses, the stable base loci of numerically equivalent big divisors coincide.

Example 5.6. We recall [Laz04, Example 10.3.3]. Let B be a smooth elliptic curve, and let A be an ample divisor of degree 1 on B. Let $X = \mathbb{P}(\mathscr{O}_B \oplus \mathscr{O}_B(A))$ be a projective bundle with the natural map $p\colon X \to B$. Let P_1 be a torsion divisor on B, let P_2 be a non-torsion degree 0 divisor on B, and consider $L_i = \mathscr{O}_X(1) \otimes p^*\mathscr{O}_B(P_i)$. Then L_1 and L_2 are numerically equivalent nef and big line bundles with $\emptyset = \mathbf{B}(L_1) \neq \mathbf{B}(L_2)$, and $R(X, L_1)$ is finitely generated while $R(X, L_2)$ is not by Lemma 3.4(2).

Lemma 5.7. *Let X be a $\mathbb{Q}$-factorial projective variety, and let D_1 and D_2 be big $\mathbb{Q}$-divisors such that $D_1 \equiv D_2$. Assume that the rings $R(X,D_i)$ are finitely generated, and consider the maps $\varphi_i\colon X \dashrightarrow \operatorname{Proj} R(X,D_i)$.*

Then we have $\mathbf{B}(D_1) = \mathbf{B}(D_2)$, and there is an isomorphism

$$\eta\colon \operatorname{Proj} R(X,D_1) \to \operatorname{Proj} R(X,D_2)$$

such that $\varphi_2 = \eta \circ \varphi_1$.

Proof. Since finite generation holds, we have $\mathbf{B}(D_i) = \{x \in X \mid o_x(D_i) > 0\}$, so the first claim follows immediately from Lemma 5.3.

For the second claim, by passing to a resolution and by Theorem 1.23, we may assume that there is a positive integer k such that $\operatorname{Mob}(kD_i)$ are basepoint free, and $\operatorname{Mob}(pkD_i) = p\operatorname{Mob}(kD_i)$ for all positive integers p. Note that then $P_\sigma(D_i) = \frac{1}{k}\operatorname{Mob}(kD_i)$, and that

$$P_\sigma(D_1) \equiv P_\sigma(D_2) \tag{5.1}$$

since $N_\sigma(D_1) = N_\sigma(D_2)$ by Lemma 5.3. Thus φ_i is given by the linear system $|kpP_\sigma(D_i)|$ for some $p \gg 0$. But then (5.1) shows that φ_1 and φ_2 contract the same curves, which implies the claim. □

5.3 Geography of ample models

In this section we study the geography of ample models associated to a finitely generated divisorial ring $\mathfrak{R} = R(X;D_1,\dots,D_r)$. More precisely, there is a decomposition $\operatorname{Supp}\mathfrak{R} = \coprod \mathscr{A}_i$ into finitely many chambers together with contracting maps $\varphi_i\colon X \dashrightarrow X_i$, such that φ_i is the ample model for every divisor in $\mathscr{A}_i$. We study these ample models in the special case of adjoint divisors; then, the varieties X_i are $\mathbb{Q}$-factorial when the numerical classes of the elements of $\mathscr{A}_i$ span $N^1(X)_\mathbb{R}$. This is a highly desirable feature which we would like to preserve in the general case. We then formally introduce the gen condition, and show – both by analysis and by example – that it is necessary in order to perform a Minimal Model Program in a more general setting.

We first recall the following important result [Rei80, Proposition 1.2]. We follow closely the proof of [Deb01, Lemma 7.10].

Lemma 5.8. *Let X be a smooth variety and let D be a big divisor on X. Assume that, for every positive integer m, the divisor $M_m = \operatorname{Mob}(mD)$ is basepoint free, that $M_m = mM_1$, and that $\operatorname{Fix}|D|$ has simple normal crossings. Let $\varphi\colon X \to Y$ be the semiample fibration associated to M_1.*

Then every component of $\operatorname{Fix}|D|$ *is contracted by* φ. *In particular, we have* $R(X,D) \simeq R(Y,\varphi_*D)$.

Proof. Denote $n = \dim X$. We may assume that φ is the morphism associated to M_1, and then $\mathcal{O}_X(M_1) = \varphi^*\mathcal{O}_Y(1)$ for a very ample line bundle $\mathcal{O}_Y(1)$ on Y. Let Γ be a component of $\operatorname{Fix}|D|$. We need to show that $h^0(\varphi(\Gamma),\mathcal{O}_{\varphi(\Gamma)}(m)) \leq O(m^{n-2})$.

Since $\mathcal{O}_X(M_m) = \varphi^*\mathcal{O}_Y(m)$ and the natural map $\mathcal{O}_{\varphi(\Gamma)} \to \varphi_*\mathcal{O}_\Gamma$ is injective, we have

$$h^0(\varphi(\Gamma),\mathcal{O}_{\varphi(\Gamma)}(m)) \leq h^0(\varphi(\Gamma),\mathcal{O}_Y(m)\otimes\varphi_*\mathcal{O}_\Gamma) = h^0(\Gamma,\mathcal{O}_\Gamma(M_m)). \tag{5.2}$$

Write $\Gamma_{|\Gamma} \sim G^+ - G^-$, where $G^+, G^- \geq 0$ are Cartier divisors on Γ. Consider the exact sequences

$$0 \to H^0(\Gamma, M_{m|\Gamma} - G^-) \to H^0(\Gamma, M_{m|\Gamma}) \to H^0(G^-, M_{m|G^-}) \tag{5.3}$$

and

$$H^0(X,M_m) \to H^0(X,M_m+\Gamma) \to H^0(\Gamma,(M_m+\Gamma)_{|\Gamma}) \to H^1(X,M_m). \tag{5.4}$$

Since $\operatorname{Fix}|mD| = m\operatorname{Fix}|D|$, the divisor Γ is a component of $\operatorname{Fix}|mD|$, hence the first map in (5.4) is an isomorphism and the last map in (5.4) is an injection. Therefore, from (5.2), (5.3) and (5.4) we have

$$\begin{aligned} h^0(\varphi(\Gamma),\mathcal{O}_{\varphi(\Gamma)}(m)) &\leq h^0(\Gamma, M_{m|\Gamma}) \\ &\leq h^0(\Gamma, M_{m|\Gamma} - G^-) + h^0(G^-, M_{m|G^-}) \\ &\leq h^0(\Gamma,(M_m+\Gamma)_{|\Gamma}) + h^0(G^-, M_{m|G^-}) \\ &\leq h^1(X,M_m) + h^0(G^-, M_{m|G^-}). \end{aligned}$$

As $h^0(G^-, M_{m|G^-}) \leq O(m^{n-2})$ for dimension reasons, it is enough to show that $h^1(X,M_m) \leq O(m^{n-2})$. To this end, consider the Leray spectral sequence

$$H^p(Y, R^{1-p}\varphi_*\mathcal{O}_X(M_m)) \Rightarrow H^1(X,\mathcal{O}_X(M_m)).$$

The terms $H^1(Y,\varphi_*\mathcal{O}_X(M_m)) = H^1(Y,\mathcal{O}_Y(m))$ vanish for $m \gg 0$ by Serre vanishing, so we need to prove

$$h^0(Y, R^1\varphi_*\mathcal{O}_X(M_m)) \leq O(m^{n-2}). \tag{5.5}$$

Let $U \subseteq Y$ be the maximal open subset over which φ is an isomorphism. By [Har77, III.11.2], for each m the sheaf $R^1\varphi_*\mathcal{O}_X(M_m)$ is supported on the set $Y \setminus U$ of dimension at most $n-2$, hence

$$\chi(Y, R^1\varphi_*\mathcal{O}_X(M_m)) \leq O(m^{n-2}).$$

But by Serre vanishing again, all the higher cohomology groups of the sheaf $R^1\varphi_*\mathcal{O}_X(M_m)$ vanish for $m \gg 0$, and this implies (5.5). ☐

The following is the main result of this section – the geography of ample models.

Theorem 5.9. *Let X be a projective $\mathbb{Q}$-factorial variety, and let $\mathscr{C} \subseteq \operatorname{Div}_{\mathbb{R}}(X)$ be a rational polyhedral cone such that the ring $\mathfrak{R} = R(X, \mathscr{C})$ is finitely generated. Assume that* $\operatorname{Supp}\mathfrak{R}$ *contains a big divisor. Then there is a finite decomposition*

$$\operatorname{Supp}\mathfrak{R} = \coprod \mathscr{A}_i$$

into cones such that the following holds:

(1) each $\overline{\mathscr{A}_i}$ is a rational polyhedral cone,

(2) for each i, there exists a normal projective variety X_i and a rational map $\varphi_i \colon X \dashrightarrow X_i$ such that φ_i is the ample model for every $D \in \mathscr{A}_i$,

(3) if $\mathscr{A}_j \subseteq \overline{\mathscr{A}_i}$, then there is a morphism $\varphi_{ij} \colon X_i \to X_j$ such that the diagram

$$\begin{array}{ccc} X & \overset{\varphi_i}{\dashrightarrow} & X_i \\ & {\scriptstyle\varphi_j}\searrow \quad \swarrow{\scriptstyle\varphi_{ij}} & \\ & X_j & \end{array}$$

commutes.

(4) if $\mathscr{A}_i$ contains a big divisor, then φ_i is a semiample model for every $D \in \overline{\mathscr{A}_i}$.

Proof. Let $\operatorname{Supp}\mathfrak{R} = \bigcup \mathscr{C}_i$ be a finite rational polyhedral decomposition as in Theorem 1.23, and let $\mathscr{A}_i$ be the relative interior of $\mathscr{C}_i$ for each i. We show that this is the required decomposition.

Let $f \colon \widetilde{X} \to X$ be a resolution and let d be a positive integer as in Theorem 1.23. For each i, fix $D_i \in \mathscr{A}_i \cap \operatorname{Div}(X)$, and denote

$$M_i = \operatorname{Mob} f^*(dD_i) \quad \text{and} \quad F_i = \operatorname{Fix}|f^*(dD_i)|.$$

Then M_i is basepoint free, and let $\psi_i \colon \widetilde{X} \to X_i$ be the semiample fibration associated to M_i. Let $\varphi_i \colon X \dashrightarrow X_i$ be the induced map.

$$\begin{array}{ccc} \widetilde{X} & & \\ {\scriptstyle f}\downarrow & \searrow{\scriptstyle\psi_i} & \\ X & \overset{\varphi_i}{\dashrightarrow} & X_i \end{array}$$

Claim 5.10. *Assume that $\mathscr{A}_j \subseteq \overline{\mathscr{A}_i}$, and let $C \subseteq \widetilde{X}$ be a curve such that $M_i \cdot C = 0$. Then $M_j \cdot C = 0$. In other words, all curves contracted by ψ_i are contracted by ψ_j.*

Indeed, since $\mathscr{A}_i$ is relatively open, there exist a divisor $D^\circ \in \mathscr{A}_i \cap \operatorname{Div}(X)$ and positive integers k_i, k_j, k° such that $k_i D_i = k^\circ D^\circ + k_j D_j$. By the definition of f and d, the divisor $M^\circ = \operatorname{Mob} f^*(dD^\circ)$ is basepoint free, and we have $k_i M_i = k^\circ M^\circ + k_j M_j$. In particular, if $C \subseteq \widetilde{X}$ is a curve such that $M_i \cdot C = 0$, then $M^\circ \cdot C = M_j \cdot C = 0$, which shows the claim.

The claim immediately implies that $\varphi_j = \varphi_{ij} \circ \varphi_i$ for some morphism $\varphi_{ij} \colon X_i \to X_j$, which shows (3). In particular, when $i = j$ and since the divisors D_i are arbitrary, this shows that the definition of φ_i is independent of the choice of D_i up to isomorphism.

Finally, we prove (2) and (4). For any j, pick an index i such that $\mathscr{A}_j \subseteq \overline{\mathscr{A}_i}$ and $\mathscr{A}_i$ contains a big divisor, and let E be the sum of all f-exceptional prime divisors. Since $\operatorname{Mob}(f^*(dD_i) + E) = M_i$ and $\operatorname{Fix}|f^*(dD_i) + E| = F_i + E$, the divisors F_i and E are ψ_i-exceptional by Lemma 5.8, and in particular, φ_i is a contraction.

Let D be any divisor in $\mathscr{A}_j$; without loss of generality, we may assume that $D = D_j$. Since all functions o_Γ are linear on $\overline{\mathscr{A}_i}$, we have $\operatorname{Supp} F_j \subseteq \operatorname{Supp} F_i$, hence F_j is ψ_i-exceptional by the argument above. As $M_j = \psi_j^* \mathscr{O}_{X_j}(1)$, by (3) we have

$$f^*(dD_j) = \psi_i^*(\varphi_{ij}^* \mathscr{O}_{X_j}(1)) + F_j,$$

and the divisor $(\varphi_i)_*(dD_j) = (\psi_i)_* M_j = \varphi_{ij}^* \mathscr{O}_{X_j}(1)$ is basepoint free. We conclude that φ_i is a semiample model for D_j, and φ_j is the ample model for D_j. □

An immediate corollary is the following result from [HK00]; we prove the converse statement in the next section.

Corollary 5.11. *Let X be a $\mathbb{Q}$-factorial projective variety. If X is a Mori Dream Space, then its Cox ring is finitely generated.*

Proof. We first show that the divisorial ring $R(X, \overline{\operatorname{Mov}}(X))$ is finitely generated. Indeed, with notation from Definition 1.15, we have that

$$\overline{\operatorname{Mov}}(X) = \bigcup \mathscr{C}_j, \quad \text{where} \quad \mathscr{C}_j = f_j^* \operatorname{Nef}(X_j),$$

and hence it is enough to show that each ring $R(X, \mathscr{C}_j) \simeq R(X_j, \operatorname{Nef}(X_j))$ is finitely generated. But this is clear because each $\operatorname{Nef}(X_j)$ is spanned by finitely many semiample divisors.

Let $\mathscr{F}_i$ be all the faces of all $\mathscr{C}_j$ with the property that $\mathscr{F}_i \subseteq \partial\overline{\mathrm{Mov}}(X)$ and $\mathscr{F}_i \cap \mathrm{Big}(X) \neq \emptyset$. Let $\varphi_i : X \dashrightarrow X_i$ be the ample models associated to interiors of $\mathscr{F}_i$, cf. Theorem 5.9, and let E_{ik} be the exceptional divisors of φ_i. Denote $\mathscr{D}_i = \mathscr{F}_i + \sum_k \mathbb{R}_+ E_{ik}$, and note that each $\mathscr{D}_i$ is a rational polyhedral cone.

We claim that

$$\overline{\mathrm{Eff}}(X) = \overline{\mathrm{Mov}}(X) \cup \bigcup_i \mathscr{D}_i.$$

To see this, let $D \in \mathrm{Big}(X) \setminus \overline{\mathrm{Mov}}(X)$ be a $\mathbb{Q}$-divisor. Then $P_\sigma(D)$ is a big $\mathbb{Q}$-divisor which belongs to $\partial\overline{\mathrm{Mov}}(X)$ by Lemma 5.4, and hence the ring $R(X,D)$ is finitely generated by the above. There is a face $\mathscr{F}_{i_0}$ which contains $P_\sigma(D)$ in its relative interior, and φ_{i_0} is the ample model of $P_\sigma(D)$ by Theorem 5.9. The divisor $N_\sigma(D)$ is contracted by φ_{i_0} by Lemma 5.8, and thus $D \in \mathscr{D}_{i_0}$. Therefore, we have

$$\mathrm{Big}(X) \subseteq \overline{\mathrm{Mov}}(X) \cup \bigcup_i \mathscr{D}_i,$$

and by taking closures we obtain $\overline{\mathrm{Eff}}(X) \subseteq \overline{\mathrm{Mov}}(X) \cup \bigcup_i \mathscr{D}_i$. The converse inclusion is obvious.

In particular, the cone $\overline{\mathrm{Eff}}(X)$ is rational polyhedral, and the ring $R(X, \overline{\mathrm{Eff}}(X))$ is a Cox ring of X. Fix an index i and pick generators $G_1, \dots, G_p$ of $\mathscr{D}_i$. It is enough to show that the ring $R(X; G_1, \dots, G_p)$ is finitely generated. The map φ_i is a semiample model for each G_ℓ by Theorem 5.9(4), and thus $G_\ell = \varphi_i^* M_\ell + F_\ell$, where M_ℓ is a semiample $\mathbb{Q}$-divisor on X_i and F_ℓ is φ_i-exceptional. But then

$$R(X; G_1, \dots, G_p) \simeq R(X_i; M_1, \dots, M_p),$$

and the finite generation follows. □

The next theorem shows that in the classical setting of adjoint divisors, some of the ample models X_i from Theorem 5.9 are $\mathbb{Q}$-factorial. This is a known consequence of the classical Minimal Model Program [HM13, Theorem 3.3], however here we obtain the result directly.

Theorem 5.12. *Let X be a projective $\mathbb{Q}$-factorial variety, and let $\Delta_1, \dots, \Delta_r$ be big $\mathbb{Q}$-divisors such that all pairs (X, Δ_i) are klt. Let*

$$\mathfrak{R} = R(X; K_X + \Delta_1, \dots, K_X + \Delta_r),$$

and note that $\mathfrak{R}$ is finitely generated by Theorem 1.25. Assume that $\mathrm{Supp}\,\mathfrak{R}$ *contains a big divisor. Then there exist a finite decomposition* $\mathrm{Supp}\,\mathfrak{R} = \coprod \mathscr{A}_i$ *and maps $\varphi_i : X \dashrightarrow X_i$ as in Theorem 5.9, such that:*

(i) if φ_i is birational, then X_i has rational singularities,

(ii) if the numerical classes of the elements of $\overline{\mathscr{A}_i}$ span $N^1(X)_{\mathbb{R}}$, then X_i is $\mathbb{Q}$-factorial.

Proof. We assume the notation from the proof of Theorem 5.9. For (i), pick a big $\mathbb{Q}$-divisor Δ such that (X,Δ) is klt and $K_X+\Delta\in\mathscr{A}_i$. Then $(X_i,(\varphi_i)_*\Delta)$ is also klt because φ_i is $(K+\Delta)$-nonpositive, hence X_i has rational singularities.

We now show (ii). Let B be a Weil divisor on X_i, and let $\widetilde{B}$ be its proper transform on $\widetilde{X}$. As $\widetilde{X}$ is smooth, $\widetilde{B}$ is $\mathbb{Q}$-Cartier. Let $E_1,\dots,E_k$ be all the f-exceptional prime divisors on $\widetilde{X}$. Since f is a resolution, we have

$$N^1(\widetilde{X})_{\mathbb{R}} = f^*N^1(X)_{\mathbb{R}} \oplus \bigoplus_{j=1}^{k} \mathbb{R}[E_j]. \tag{5.6}$$

Let $B_1,\dots,B_r$ be integral divisors in $\mathscr{A}_i$ whose numerical classes generate $N^1(X)_{\mathbb{R}}$. Then, by (5.6) there are rational numbers p_j,r_j such that

$$\widetilde{B}\equiv\sum p_jf^*(dB_j)+\sum r_jE_j.$$

Denote

$$M=\sum p_j\operatorname{Mob}f^*(dB_j) \quad\text{and}\quad F=\sum p_j\operatorname{Fix}|f^*(dB_j)|+\sum r_jE_j.$$

By Theorem 5.9(4), there exist ample $\mathbb{Q}$-divisors A_j on X_i such that $\operatorname{Mob}f^*(dB_j)=\psi_i^*A_j$, hence $M\equiv_{X_i}0$. Therefore

$$\widetilde{B}-F\equiv_{X_i}0.$$

Observe that $\operatorname{Supp}F\subseteq\operatorname{Supp}(F_i+\sum E_j)$, and that the divisor $F_i+\sum E_j$ is ψ_i-exceptional by Lemma 5.8. By (i) and by [KM92, Proposition 12.1.4], there is a divisor $T\in\operatorname{Div}_{\mathbb{Q}}(X_i)$ such that $\widetilde{B}-F\sim_{\mathbb{Q}}\psi_i^*T$, and thus the divisor $B=(\psi_i)_*\widetilde{B}\sim_{\mathbb{Q}}T$ is $\mathbb{Q}$-Cartier. □

It is natural to ask whether the conclusion on $\mathbb{Q}$-factoriality from Theorem 5.12 can be extended to the general situation of Theorem 5.9. We argue below that such a statement is, in general, not true, and we pin down precisely the obstacle to $\mathbb{Q}$-factoriality. The astonishing conclusion is that, in some sense, $\mathbb{Q}$-factoriality of ample models is essentially a condition on the numerical equivalence classes of the divisors in $\operatorname{Supp}\mathfrak{R}$.

With the notation from Theorem 5.9, what we are aiming for is the following statement. We would like to have a (possibly finer) decomposition $\operatorname{Supp}\mathfrak{R}=\coprod\mathscr{N}_i$ together with birational maps $\varphi_i\colon X\dashrightarrow X_i$ such that φ_i is an optimal model for every $D\in\mathscr{N}_i$, and in particular, every

X_i is $\mathbb{Q}$-factorial. It is immediate that, if the numerical classes of the elements of $\mathcal{N}_i$ span $N^1(X)_\mathbb{R}$, then φ_i is also the ample model for every $D \in \mathcal{N}_i$.

The following easy result gives us a necessary condition for the ample model of a big divisor to be $\mathbb{Q}$-factorial.

Lemma 5.13. *Let X be a $\mathbb{Q}$-factorial projective variety, and let D be a big $\mathbb{Q}$-divisor such that the ring $R(X,D)$ is finitely generated, and the map $\varphi\colon X \dashrightarrow \operatorname{Proj} R(X,D)$ is D-nonpositive. Let D' be a $\mathbb{Q}$-divisor such that $D \equiv D'$.*

Then the ring $R(X,D')$ is finitely generated if and only if the $\mathbb{Q}$-divisor $\varphi_* D'$ is $\mathbb{Q}$-Cartier.

Proof. If $R(X,D')$ is finitely generated, then by Lemma 5.7, φ is equal to the map $X \dashrightarrow \operatorname{Proj} R(X,D')$ up to isomorphism. Therefore $\varphi_* D'$ is ample, and in particular $\mathbb{Q}$-Cartier.

We now prove the converse implication. Denote $Y = \operatorname{Proj} R(X,D)$ and let $(p,q)\colon W \to X \times Y$ be a resolution of φ. By Lemma 1.28, we have

$$p^*(D - D') = q^* \varphi_*(D - D'),$$

hence $\varphi_* D \equiv \varphi_* D'$. Since $\varphi_* D$ is ample, so is $\varphi_* D'$, hence the ring $R(Y, \varphi_* D')$ is finitely generated. By Lemma 5.8, the divisor $E = p^* D - q^* \varphi_* D$ is effective and q-exceptional, and since $E = p^* D' - q^* \varphi_* D'$, we have $R(X,D') \simeq R(Y, \varphi_* D')$. □

Therefore, in the notation of Lemma 5.13, if the ample model of D is $\mathbb{Q}$-factorial, then the ring $R(X,D')$ is finitely generated for every $\mathbb{Q}$-divisor D' in the numerical class of D. This motivates the key definition of *gen* divisors as in Definition 1.20. There are three main examples of gen divisors of interest to us:

(i) ample $\mathbb{Q}$-divisors are gen,

(ii) every adjoint divisor $K_X + \Delta + A$ is gen, where A is an ample $\mathbb{Q}$-divisor on X, and the pair (X, Δ) is klt; indeed, this follows from Theorem 1.25,

(iii) if $\operatorname{Pic}(X)_\mathbb{Q} = N^1(X)_\mathbb{Q}$, then every divisor with a finitely generated section ring is gen.

As we show in Section 5.4, having lots of gen divisors is essentially equivalent to being able to run a Minimal Model Program. We have seen above that this is a necessary condition for the models to be optimal, and in particular $\mathbb{Q}$-factorial. We show in Theorem 5.19 that, remarkably, this is also a sufficient condition. This, together with (ii) and (iii), explains *precisely* why we are able to run the MMP for adjoint divisors and on Mori Dream Spaces, and the details are worked out in Corollaries 5.21 and 5.22.

We conclude this section with an example where all the conditions of Theorem 5.9 are satisfied, but the absence of gen divisors implies that there is no decomposition of $\operatorname{Supp}\mathfrak{R}$ into regions of divisors that share an optimal model. In particular, we cannot run the MMP as explained in Section 5.4, and therefore the conditions from Theorems 5.17 are 5.19 are not only sufficient, but they are optimal. The example shows that the finite generation of a divisorial ring in itself is not sufficient to perform the Minimal Model Program.

Example 5.14. Let X, L_1 and L_2 be as in Example 5.6, and note that X is a smooth surface with $\dim N^1(X)_\mathbb{R} = 2$. We show that there exist a big divisor D and an ample divisor A on X such that the ring $\mathfrak{R} = R(X;D,A)$ is finitely generated, the divisor L_1 belongs to the interior of the cone $\operatorname{Supp}\mathfrak{R} = \mathbb{R}_+D + \mathbb{R}_+A$, and *none* of the divisors in the cone $\mathbb{R}_+D + \mathbb{R}_+L_1 \subseteq \operatorname{Supp}\mathfrak{R}$ is gen. In particular, we cannot perform the MMP for D.

We first claim that there exists an irreducible curve C on X such that

$$L_1 \cdot C = 0 \quad \text{and} \quad C^2 < 0. \tag{5.7}$$

Indeed, since L_1 is semiample but not ample, there exists an irreducible curve $C \subseteq X$ such that $L_1 \cdot C = 0$. Since L_1 is big and nef, we have $L_1^2 > 0$, so the Hodge index theorem then implies $C^2 < 0$.

Now, set $D = L_1 + C$. Since $\dim N^1(X)_\mathbb{R} = 2$ and D is not nef, it is immediate that there exists an ample divisor A on X such that $L_1 \in \mathbb{R}_+D + \mathbb{R}_+A$. In order to show that $\mathfrak{R}$ is finitely generated, it is enough to show that the rings $R(X;D,L_1)$ and $R(X;L_1,A)$ are finitely generated, and this latter ring is finitely generated since both L_1 and A are semi-ample.

For $k_1, k_2 \in \mathbb{N}$, consider the divisor

$$D_{k_1,k_2} = k_1D + k_2L_1 = (k_1 + k_2)L_1 + k_1C.$$

Then (5.7) implies that $P_\sigma(D_{k_1,k_2}) = (k_1 + k_2)L_1$, hence $H^0(X, D_{k_1,k_2}) \simeq H^0(X,(k_1 + k_2)L_1)$. Therefore the ring

$$R(X;D,L_1) \simeq R(X;L_1,L_1)$$

is finitely generated.

Finally, note that $D_{k_1,k_2} \equiv (k_1+k_2)L_2+k_1C$, and that $P_\sigma((k_1+k_2)L_2+k_1C) = (k_1+k_2)L_2$. Therefore the ring

$$R(X,(k_1+k_2)L_2+k_1C) \simeq R(X,(k_1+k_2)L_2)$$

is not finitely generated, thus the divisor D_{k_1,k_2} is not gen.

Remark 5.15. The notion of genness is a very subtle one. For instance, every $\mathbb{Q}$-divisor D with $\kappa_\sigma(D) = 0$ is gen (for the definition and properties of κ_σ see [Nak04]). Indeed, for every $\mathbb{Q}$-divisor $D' \equiv D$ we have $\kappa(D') \leq \kappa_\sigma(D') = 0$, hence the ring $R(X,D)$ is isomorphic to either $\mathbb{C}$ or to the polynomial ring $\mathbb{C}[T]$.

5.4 Running the D-MMP

Let X be a projective $\mathbb{Q}$-factorial variety, and let $\mathscr{C} \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ be a rational polyhedral cone such that the divisorial ring $\mathfrak{R} = R(X,\mathscr{C})$ is finitely generated. Then by Theorem 5.9 we know that $\mathrm{Supp}\,\mathfrak{R}$ has a decomposition into finitely many rational polyhedral cones giving the geography of ample models associated to $\mathfrak{R}$.

In this section we explain how, when all divisors in the interior of $\mathrm{Supp}\,\mathfrak{R}$ are gen, the aforementioned decomposition can be refined to give a geography of optimal models. As indicated in the previous sections, the main technical obstacle is to prove $\mathbb{Q}$-factoriality of models, and this is the point where the gen condition on divisors plays a crucial role.

We assume that $\mathrm{Supp}\,\mathfrak{R}$ contains an ample divisor, and fix a divisor $D \in \mathrm{Supp}\,\mathfrak{R}$. Then we can run the Minimal Model Program for D as follows.

We define a certain finite rational polyhedral decomposition $\mathscr{C} = \bigcup \mathscr{N}_i$ in Theorem 5.19. If D is not nef, we show in Theorem 5.17 that there is a D-negative birational map $\varphi\colon X \dashrightarrow X^+$ such that X^+ is $\mathbb{Q}$-factorial, and φ is elementary – this corresponds to contractions of extremal rays in the classical MMP. We also show that there is a rational polyhedral subcone $D \in \mathscr{C}' \subseteq \mathscr{C}$ which is a union of *some, but not all* of the cones $\mathscr{N}_i$, such that $R(X,\mathscr{C}') \simeq R(X^+,\varphi_*\mathscr{C}')$ and the cone $\varphi_*\mathscr{C}' \subseteq \mathrm{Div}_{\mathbb{R}}(X^+)$ contains an ample divisor. Now we replace X by X^+, D by φ_*D, and $\mathscr{C}$ by $\varphi_*\mathscr{C}'$, and we repeat the procedure. Since there are only finitely many cones $\mathscr{N}_i$, this process must terminate with a variety X_D on which the proper transform of D is nef, and this is the optimal model for D. It is then automatic that X_D is also an optimal model for all divisors in the cone $\mathscr{N}_{i_0} \ni D$. The details are given in Theorem 5.19.

In the context of adjoint divisors and the classical MMP, we can additionally *direct* the MMP by an ample $\mathbb{Q}$-divisor A on X, as in [CL13]. The proofs of Theorems 5.17 and 5.19 can be easily modified to obtain the *D-MMP with scaling of A*, however we do not pursue this here.

First we define elementary contractions.

Definition 5.16. A birational contraction $\varphi\colon X \dashrightarrow Y$ between normal projective varieties is *elementary* if it not an isomorphism, and it is either an isomorphism in codimension 1, or a morphism whose exceptional locus is a prime divisor on X.

The following theorem is the key result: it shows that in our situation elementary contractions exist.

Theorem 5.17. *Let X be a projective $\mathbb{Q}$-factorial variety and let $\mathscr{C} \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ be a rational polyhedral cone. Denote by $\pi\colon \mathrm{Div}_{\mathbb{R}}(X) \to N^1(X)_{\mathbb{R}}$ the natural projection. Assume that the ring $\mathfrak{R} = R(X,\mathscr{C})$ is finitely generated, that $\mathrm{Supp}\,\mathfrak{R}$ contains an ample divisor, that $\pi(\mathrm{Supp}\,\mathfrak{R})$ spans $N^1(X)_{\mathbb{R}}$, and that every divisor in the interior of $\mathrm{Supp}\,\mathfrak{R}$ is gen. Let $\mathrm{Supp}\,\mathfrak{R} = \bigcup \mathscr{C}_i$ be a decomposition as in Theorem 1.23. Let $D \in \mathrm{Supp}\,\mathfrak{R}$ be a $\mathbb{Q}$-divisor which is not nef. Then:*

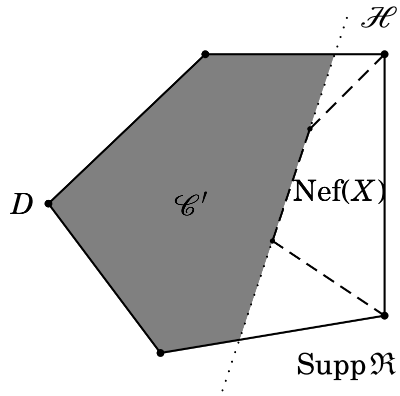
(1) the cone $\mathrm{Supp}\,\mathfrak{R} \cap \pi^{-1}\big(\mathrm{Nef}(X)\big)$ is rational polyhedral, and every element of this cone is semiample,

(2) there exists a rational hyperplane $\mathscr{H} \subseteq N^1(X)_{\mathbb{R}}$ which intersects the interior of $\pi(\mathrm{Supp}\,\mathfrak{R})$ and contains a codimension 1 face of the cone $\pi(\mathrm{Supp}\,\mathfrak{R}) \cap \mathrm{Nef}(X)$, such that $\pi(D)$ and $\mathrm{Nef}(X)$ are on the opposite sides of $\mathscr{H}$,

(3) let $\mathscr{W} \subseteq N^1(X)_{\mathbb{R}}$ be the half-space bounded by $\mathscr{H}$ which does not contain $\mathrm{Nef}(X)$, and let $\mathscr{C}' = \mathrm{Supp}\,\mathfrak{R} \cap \pi^{-1}(\mathscr{W})$. Then there exists a $\mathbb{Q}$-factorial projective variety X^+ together with an elementary contraction $\varphi\colon X \dashrightarrow X^+$, such that φ is W-nonpositive for every $W \in \mathscr{C}'$, and it is W-negative for every $W \in \mathscr{C}' \backslash \pi^{-1}(\mathscr{H})$,

(4) we have $R(X,\mathscr{C}') \simeq R(X^+,\mathscr{C}^+)$, where $\mathscr{C}^+ = \varphi_*\mathscr{C}' \subseteq \mathrm{Div}_{\mathbb{R}}(X^+)$, and $\mathscr{C}^+$ contains an ample divisor,

(5) for every cone $\mathscr{C}_i$ and for every geometric valuation Γ over X, the function o_Γ is linear on the cone $\varphi_*(\mathscr{C}' \cap \mathscr{C}_i) \subseteq \mathscr{C}^+$.



Proof. Step 1. The statement (1) follows immediately from Corollary 3.5, statement (4) follows from (3) and from the construction below, while (5) follows from (3) by Lemma 5.5. To show (2), let α be any ample class in the interior of $\pi(\operatorname{Supp}\mathfrak{R}) \subseteq N^1(X)$, and let β be the intersection of the segment $[\pi(D), \alpha]$ with $\partial\operatorname{Nef}(X)$. Then β lies in the interior of $\pi(\operatorname{Supp}\mathfrak{R})$, and by (1) there is a rational codimension 1 face of $\pi(\operatorname{Supp}\mathfrak{R}) \cap \operatorname{Nef}(X)$ containing β. We define $\mathscr{H}$ to be the rational hyperplane containing that face.

Step 2. It remains to show (3). By Corollary 3.5, there are cones $\mathscr{C}_j \nsubseteq \pi^{-1}(\operatorname{Nef}(X))$ and $\mathscr{C}_k \subseteq \pi^{-1}(\operatorname{Nef}(X))$ such that $\dim \pi(\mathscr{C}_j) = \dim \pi(\mathscr{C}_k) = \dim N^1(X)_{\mathbb{R}}$ and $\pi(\mathscr{C}_j) \cap \pi(\mathscr{C}_k) \subseteq \mathscr{H}$; denote $\mathscr{C}_{jk} = \mathscr{C}_j \cap \mathscr{C}_k$. Let $\varphi\colon X \dashrightarrow X^+$ and $\theta\colon X \dashrightarrow Y$ be the ample models associated to relative interiors of $\mathscr{C}_j$ and $\mathscr{C}_{jk}$ as in the proof of Theorem 5.9, and note that θ is a morphism by (1) since

$$\mathscr{C}_{jk} \subseteq \operatorname{Supp}\mathfrak{R} \cap \pi^{-1}\big(\operatorname{Nef}(X)\big).$$

Then, by Theorem 5.9(3), there is a morphism $\theta^+\colon X^+ \to Y$ such that the diagram

$$\begin{array}{ccc} X & \overset{\varphi}{\dashrightarrow} & X^+ \\ & {\scriptstyle\theta}\searrow \quad \swarrow{\scriptstyle\theta^+} & \\ & Y & \end{array}$$

is commutative. The following is the key claim:

Claim 5.18. *Let F be an $\mathbb{R}$-divisor on X such that $\pi(F) \in \mathscr{H}$. Then $F \sim_{\mathbb{R}} \theta^* F_Y$ for some $F_Y \in \operatorname{Div}_{\mathbb{R}}(Y)$. If additionally $\pi(F) \in \pi(\mathscr{C}_{jk})$, then F_Y is ample. In particular, a curve C is contracted by θ if and only if $C \cdot \delta = 0$ for every $\delta \in \mathscr{H}$.*

Pick $\mathbb{Q}$-divisors $B_1,\dots,B_r$ in $\mathscr{C}_{jk}$ and nonzero real numbers λ_i such that $\pi(B_i)$ span $\mathscr{H}$ and $\pi(F)=\sum\lambda_i\pi(B_i)$. We may assume that $\lambda_i\geq 0$ for all i when $\pi(F)\in\pi(\mathscr{C}_{jk})$. Hence, there is a $\mathbb{Q}$-divisor $B_1'\equiv B_1$ such that

$$F=\lambda_1B_1'+\sum_{i\geq 2}\lambda_iB_i.$$

Note that, by the definition of θ, there are ample divisors A_i on Y such that $B_i\sim_{\mathbb{Q}}\theta^*A_i$ for all $i\geq 2$.

Since B_1 is gen, the ring $R(X,B_1')$ is finitely generated, and therefore B_1' is semiample by Lemma 3.4(2) as it is nef and big. Denote by $\theta'\colon X\to Y'$ the semiample fibration associated to B_1'. By Lemma 5.7, there is an isomorphism $\eta\colon Y\to Y'$ such that $\theta'=\eta\circ\theta$. Since $B_1'\sim_{\mathbb{Q}}(\theta')^*A_1'$ for an ample divisor A_1' on Y', we have $B_1'\sim_{\mathbb{Q}}\theta^*A_1$, where $A_1=\eta^*A_1'$. Therefore $F\sim_{\mathbb{R}}\theta^*(\sum\lambda_iA_i)$, which proves the claim.

Step 3. We next show that X^+ is $\mathbb{Q}$-factorial.

Consider a Weil divisor P^+ on X^+, and let P be its proper transform on X. Since X is $\mathbb{Q}$-factorial, the divisor P is $\mathbb{Q}$-Cartier. Since $\dim\pi(\mathscr{C}_j)=\dim N^1(X)_{\mathbb{R}}$, there exist a $\mathbb{Q}$-divisor $G\in\mathscr{C}_j$ and $\alpha\in\mathbb{Q}$ such that $\pi(P+\alpha G)\in\mathscr{H}$. By Claim 5.18, there exists $M\in\operatorname{Div}_{\mathbb{Q}}(Y)$ such that $P+\alpha G\sim_{\mathbb{Q}}\theta^*M$. Let $(p,q)\colon\widetilde{X}\to X\times X^+$ be a resolution of φ. By the definition of φ and by Theorem 5.9, there is an ample $\mathbb{Q}$-divisor A on X^+ and a q-exceptional $\mathbb{Q}$-divisor E on $\widetilde{X}$ such that $p^*G=q^*A+E$. It follows that

$$p^*P\sim_{\mathbb{Q}}(\theta\circ p)^*M-\alpha(q^*A+E)=(\theta^+\circ q)^*M-\alpha q^*A-\alpha E.$$

Since φ is a contraction, we have $P^+=q_*p^*P$, and therefore the divisor

$$P^+\sim_{\mathbb{Q}}(\theta^+)^*M-\alpha A$$

is $\mathbb{Q}$-Cartier.

Step 4. In this step we show that φ is an elementary map.

If θ is an isomorphism in codimension 1, then so are φ and θ^+ as φ is a contraction.

Hence, we may assume that there exists a θ-exceptional prime divisor E. Let C be a curve contracted by θ, and let R be a ray in $N_1(X)_{\mathbb{R}}$ orthogonal to the hyperplane $\mathscr{H}$. Then the class of C belongs to R by Claim 5.18, and so $E\cdot R<0$ by Lemma 1.27. In particular, we have $E\cdot C<0$, thus $C\subseteq E$, and the exceptional locus of θ equals E. Therefore, θ is an elementary contraction.

Observe that $\pi(E)$ and $\operatorname{Nef}(X)$ lie on opposite sides of $\mathscr{H}$. This implies that there is a $\mathbb{Q}$-divisor G_E in the relative interior of $\mathscr{C}_j$ such that $\pi(G_E-$

E) belongs to the relative interior of $\mathscr{C}_{jk}$. Then, by Claim 5.18, there exists an ample divisor $M_E \in \mathrm{Div}_{\mathbb{Q}}(Y)$ such that $G_E - E \sim_{\mathbb{Q}} \theta^* M_E$, and thus

$$H^0(X, mG_E) \simeq H^0(X, m\theta^* M_E) \tag{5.8}$$

for every positive integer m. Since φ is the map $X \dashrightarrow \mathrm{Proj}\, R(X, G_E)$ by definition, we may assume that $X^+ = Y$ and $\varphi = \theta$ by (5.8), which shows that φ is an elementary contraction.

Step 5. The only thing left to prove is the last statement in (3). For $W \in \mathscr{C}'$, there exists an $\mathbb{R}$-divisor $G_W \in \mathscr{C}_j$ such that $\pi(W - G_W) \in \mathscr{H}$. Thus $W \equiv_Y G_W$ by Claim 5.18. Since φ is G_W-nonpositive by Theorem 5.9(4), this implies that φ is W-nonpositive by Corollary 1.28. If φ is an isomorphism in codimension 1, it is automatic that it is then also W-negative.

If $W \in \mathscr{C}' \setminus \pi^{-1}(\mathscr{H})$ and φ contracts a divisor E, there exists a positive rational number λ such that $\pi(W - \lambda E) \in \mathscr{H}$. Again by Claim 5.18, and since $X^+ = Y$ and $\varphi = \theta$, there is a divisor $M_W \in \mathrm{Div}_{\mathbb{R}}(X^+)$ such that $W - \lambda E \sim_{\mathbb{R}} \varphi^* M_W$. But then it is clear that φ is W-negative. □

The following is the main result of this chapter – the geography of optimal models.

Theorem 5.19. *Let X be a projective $\mathbb{Q}$-factorial variety, and let $\mathscr{C} \subseteq$* $\mathrm{Div}_{\mathbb{R}}(X)$ *be a rational polyhedral cone. Denote by $\pi\colon \mathrm{Div}_{\mathbb{R}}(X) \to N^1(X)_{\mathbb{R}}$ the natural projection. Assume that the ring $\mathfrak{R} = R(X, \mathscr{C})$ is finitely generated, that* $\mathrm{Supp}\,\mathfrak{R}$ *contains an ample divisor, that $\pi(\mathrm{Supp}\,\mathfrak{R})$ spans $N^1(X)_{\mathbb{R}}$, and that every divisor in the interior of* $\mathrm{Supp}\,\mathfrak{R}$ *is gen.*

Then for any $\mathbb{Q}$-divisor $D \in \mathscr{C}$, we can run a D-MMP which terminates.

Furthermore, there is a finite decomposition

$$\mathrm{Supp}\,\mathfrak{R} = \coprod \mathscr{N}_i$$

into cones having the following properties:

(1) each $\overline{\mathscr{N}_i}$ is a rational polyhedral cone,

(2) for each i, there exists a $\mathbb{Q}$-factorial projective variety X_i and a birational contraction $\varphi_i\colon X \dashrightarrow X_i$ such that φ_i is a good model for every divisor in $\mathscr{N}_i$.

Proof. Denote by $V \subseteq \mathrm{Div}_{\mathbb{R}}(X)$ the minimal vector space containing $\mathscr{C}$, and define $\mathscr{C}^1 = \mathrm{Supp}\,\mathfrak{R}$. Let $\mathscr{C}^1 = \bigcup_{i \in I_1} \mathscr{C}^1_i$ be the rational polyhedral decomposition as in Theorem 1.23. By subdividing $\mathscr{C}^1$ further, we may assume that the following property is satisfied:

(♮) let $\mathscr{G} \subseteq V$ be any hyperplane which contains a codimension 1 face of some $\mathscr{C}^1_{i_0}$. Then every $\mathscr{C}^1_i$ is contained in one of the two half-spaces of V bounded by $\mathscr{G}$.

For each $i \in I_1$, let $\mathscr{N}_i$ be the relative interior of $\mathscr{C}_i$. We claim that $\mathscr{C}^1 = \coprod_{i \in I_1} \mathscr{N}_i$ is the desired decomposition.

Let D be a point in some $\mathscr{N}_{i_0}$. If D is nef, then every divisor in $\mathscr{N}_{i_0}$ is semiample by Corollary 3.5, so the theorem follows.

Therefore, we may assume that D is not nef. Denote $Y_1 = X$ and $D_1 = D$. We show that there exists a D_1-MMP which terminates.

By Theorem 5.17, the cone $\mathscr{C}^1 \cap \pi^{-1}\big(\mathrm{Nef}(Y_1)\big)$ is rational polyhedral. Let $\mathscr{H} \subseteq N^1(Y_1)_{\mathbb{R}}$ be a rational hyperplane as in Theorem 5.17, and let $\mathscr{C}^1_\ell$, for $\ell \in I_2 \subsetneq I_1$, be those cones for which $\pi(\mathscr{C}^1_\ell)$ and $\pi(D)$ are on the same side of $\mathscr{H}$, cf. (♮). Let $f_1 \colon Y_1 \dashrightarrow Y_2$ be an elementary map as in Theorem 5.17(3), and denote $D_2 = (f_1)_* D_1$. Define rational polyhedral cones $\mathscr{C}^2_\ell = (f_1)_* \mathscr{C}^1_\ell \subseteq \mathrm{Div}_{\mathbb{R}}(Y_2)$, and set

$$\mathscr{C}^2 = \textstyle\bigcup_{\ell \in I_2} \mathscr{C}^2_\ell. \tag{5.9}$$

Then the ring $\mathfrak{R}^2 = R(Y_2, \mathscr{C}^2)$ is finitely generated by Theorem 5.17(4). By Theorem 5.17(5), the relation (5.9) gives a decomposition of $\mathscr{C}^2$ as in Theorem 1.23. Also note that $(f_1)_*(\mathscr{N}_{i_0}) \subseteq \mathscr{C}^2$.

In this way we construct a sequence of divisors D_p on $\mathbb{Q}$-factorial varieties Y_p. Since the size of the index sets I_p drops with each step, this process must terminate with a model X_{p_0} on which the divisor D_{p_0} is nef. Similarly as above, X_{p_0} is an optimal model for all divisors in $\mathscr{N}_{i_0}$, and the proper transform on Y_{p_0} of every element of $\mathscr{N}_{i_0}$ is semiample. □

Corollary 5.20. *Let X be a projective $\mathbb{Q}$-factorial variety, let $S_1, \dots, S_p$ be distinct prime divisors on X, denote $V = \sum_{i=1}^p \mathbb{R} S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$, and let A be an ample $\mathbb{Q}$-divisor on X. Let $\mathscr{C} \subseteq \mathscr{L}(V)$ be a rational polytope such that for every $\Delta \in \mathscr{C}$, the pair (X, Δ) is klt.*

Then there exists a positive integer M such that for every $\Delta \in \mathscr{C} \cap \mathscr{E}_A(V)$, there is a $(K_X + \Delta)$-MMP consisting of at most M steps.

Proof. By enlarging V and $\mathscr{C}$, we may assume that the numerical classes of the elements of $\mathscr{C} \cap \mathscr{E}_A(V)$ span $N^1(X)_{\mathbb{R}}$. The set $\mathscr{C} \cap \mathscr{E}_A(V)$ is a rational polytope by Corollary 1.26, and let $B_1, \dots, B_r$ be its vertices. Choose a positive integer $\lambda \gg 0$ such that all $K_X + A + B_i + \lambda A$ are ample. Denote

$$\mathscr{D} = \sum \mathbb{R}_+(K_X + A + B_i) + \sum \mathbb{R}_+(K_X + A + B_i + \lambda A).$$

Then the ring $\mathfrak{R} = R(X, \mathscr{D})$ is finitely generated by Theorem 1.25, and we have $\mathbb{R}_+(K_X + A + \mathscr{C} \cap \mathscr{E}_A(V)) \subseteq \mathrm{Supp}\,\mathfrak{R}$. Let $\mathrm{Supp}\,\mathfrak{R} = \coprod_{i=1}^N \mathscr{N}_i$ be the decomposition as in Theorem 5.19. Then it is immediate from the proof of Theorem 5.19 that we can set $M = N$. □

The following corollary is finiteness of models, cf. [BCHM10, Lemma 7.1].

Corollary 5.21. *Let X be a projective $\mathbb{Q}$-factorial variety, let $S_1,\dots,S_p$ be distinct prime divisors on X, denote $V = \sum_{i=1}^p \mathbb{R}S_i \subseteq \mathrm{Div}_{\mathbb{R}}(X)$, and let A be an ample $\mathbb{Q}$-divisor on X. Let $\mathscr{C} \subseteq \mathscr{L}(V)$ be a rational polytope such that for every $\Delta \in \mathscr{C}$, the pair (X,Δ) is klt.*

Then there are finitely many rational maps $\varphi_i\colon X \dashrightarrow Y_i$, with the property that if $\Delta \in \mathscr{C} \cap \mathscr{E}_A(V)$, then there is an index i such that φ_i is a log terminal model of $K_X + \Delta$.

Proof. By enlarging V and $\mathscr{C}$, we may assume that the numerical classes of the elements of $\mathscr{C} \cap \mathscr{E}_A(V)$ span $N^1(X)_{\mathbb{R}}$, and that there exists a divisor $B \in \mathscr{C} \cap \mathscr{E}_A(V)$ such that $K_X + A + B$ is ample. The ring $R(X, \mathbb{R}_+(K_X + A + \mathscr{C} \cap \mathscr{E}_A(V)))$ is finitely generated by Corollary 1.26, so the result follows immediately from Theorem 5.19. □

Finally, we recover one of the main results of [HK00].

Corollary 5.22. *Let X be a $\mathbb{Q}$-factorial projective variety and assume that $\mathrm{Pic}(X)_{\mathbb{Q}} = N^1(X)_{\mathbb{Q}}$. Then X is a Mori Dream Space if and only if its Cox ring is finitely generated.*

In particular, if (X,Δ) is a klt log Fano pair, then X is a Mori Dream Space.

Proof. Let $D_1,\dots,D_r$ be a basis of $\mathrm{Pic}(X)_{\mathbb{Q}}$ such that $\overline{\mathrm{Eff}}(X) \subseteq \sum \mathbb{R}_+ D_i$. The associated divisorial ring $\mathfrak{R} = R(X; D_1,\dots,D_r)$ is a Cox ring of X. Corollary 5.11 shows that if X is a Mori Dream Space, then $\mathfrak{R}$ is finitely generated. We now prove the converse statement.

Assume that $\mathfrak{R}$ is finitely generated, and let $\mathrm{Supp}\,\mathfrak{R} = \coprod_{i=1}^N \mathscr{N}_i$ be the decomposition from Theorem 5.19. Then $\mathrm{Nef}(X)$ is the span of finitely many semiample divisors by Corollary 3.5, and by the definition of the sets $\mathscr{N}_i$ and by Corollary 3.5, there is a set $I \subseteq \{1,\dots,N\}$ such that

$$\overline{\mathrm{Mov}}(X) = \bigcup_{i\in I} \overline{\mathscr{N}_i}.$$

By taking a smaller index set I, we may assume that the dimension of $\overline{\mathscr{N}_i}$ equals $\dim N^1(X)_{\mathbb{R}}$ for all $i \in I$. For $i \in I$, let $\varphi_i : X \dashrightarrow X_i$ be the maps as in Theorem 5.19. Then $\overline{\mathscr{N}_i} \subseteq \varphi_i^*\big(\mathrm{Nef}(X_i)\big)$, and hence

$$\overline{\mathrm{Mov}}(X) \subseteq \bigcup_{i\in I} \varphi_i^*\big(\mathrm{Nef}(X_i)\big).$$

Each φ_i is an optimal model for every divisor in $\mathscr{N}_i$, thus each φ_i is an isomorphism in codimension 1. Therefore, $R(X_i;(\varphi_i)_*D_1,\dots,(\varphi_i)_*D_r)$ is a Cox ring of X_i, and it is finitely generated since it is isomorphic to $\mathfrak{R}$. In particular, every $\operatorname{Nef}(X_i)$ is spanned by finitely many semiample divisors by above, and hence

$$\overline{\operatorname{Mov}}(X) \supseteq \bigcup\nolimits_{i\in I} \varphi_i^*\big(\operatorname{Nef}(X_i)\big).$$

This shows that X is a Mori Dream Space.

The last claim now follows from Theorem 3.3. □

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Abstract

The ‘Minimal Model Program’ is a classification procedure in higher dimensional algebraic geometry, which aims to decompose algebraic varieties into their basic building blocks. It is a central project in algebraic geometry ever since Mori’s Field Medal in 1990, awarded for his work which settled the three-dimensional case. The Minimal Model Program has seen tremendous progress in the last decade, during which many results were settled and our knowledge of the theory was hugely advanced.

We currently know that the Minimal Model Program for a smooth projective variety X leads to its goal (‘terminates’) if either K_X is a big divisor or if it is not pseudoeffective. In particular, all varieties with big canonical bundle have a birational model on which a multiple of the canonical divisor is basepoint point free. Furthermore, the number of such good models is finite.

The main outstanding problem in birational geometry is to generalise these results to as many varieties as possible, that is to prove that good models exist for certain varieties not necessarily of general type. Progress towards a solution of this problem is the topic of this thesis. There are four main results of this work.

(a) The existence of good models for klt pairs (X,Δ) with $K_X+\Delta$ pseudoeffective is the main outstanding conjecture in the Minimal Model Program for projective klt pairs in characteristic zero. It is well known that the existence of good models implies the Abundance conjecture, which predicts that the canonical bundle on a minimal model is actually semiample.

Ouf first result reduces the problem of existence of good models for non-uniruled pairs to the case of smooth varieties with effective canonical class. More precisely, assuming the existence of good models for klt pairs in dimensions at most $n-1$, we show that the existence of good models for non-uniruled klt pairs in dimension n implies the existence of good models for uniruled klt pairs in dimension n. This is a proper

generalisation of the strategy employed for threefolds, and is the first reduction step towards the proof of the existence of good models.

(b) If X is a variety, it is a basic question what the shape of interesting cones in its Néron-Severi space $N^1(X)_{\mathbb{R}}$ is. From the point of view of birational geometry, the interesting cones are the cone of nef divisors and the cone of movable divisors. The Cone conjecture of Morrison and Kawamata predicts that on a Calabi-Yau manifold these cones are rational polyhedral up to the action of natural groups acting on them.

In this work we prove the Cone conjecture for Calabi-Yau n-folds with Picard number 2 and infinite group $\mathrm{Bir}(X)$. This is one of the first results to treat the Cone conjecture in such a generality, and the first result to confirm it for a wide class of threefolds.

(c) It is an important and long-standing conjecture that the number of minimal models of a smooth projective variety is finite up to isomorphism. It is implied by a positive answer to the Cone conjecture together with the existence of good models. This gives the main motivation for the Cone conjecture in the realm of birational geometry. One might speculate that the number of minimal models of a smooth projective variety is bounded with respect to its underlying topology as a complex manifold.

Our third result shows that under certain conditions depending on the geometry of a log smooth threefold pair, the number of its minimal models depends only on its topological type. Here, two log smooth pairs (X_1,Δ_1) and (X_2,Δ_2) are said to be of the same topological type if there is a homeomorphism $\varphi\colon X_1 \to X_2$ which is a homeomophism between the supports of Δ_1 and Δ_2.

(d) There are two classes of projective varieties whose birational geometry is particularly interesting and rich. The first family consists of varieties where the classical Minimal Model Program can be performed successfully with the current techniques. The other class is that of so called Mori Dream Spaces. We now know that, in both cases, their birational geometry is entirely determined by suitable finitely generated divisorial rings, and there is a priori no clear connection between these rings.

In this thesis we put these two families of varieties under the same roof. We thus identify the maximal class of varieties and divisors on them where a suitable MMP can be performed.

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INTERESTS

Algebraic Geometry, birational classification, Calabi-Yau and hyperkähler manifolds

EDUCATION

07.2009	PhD, University of Cambridge; dissertation *Towards Finite Generation of the Canonical Ring without the Minimal Model Program*, supervisor Prof. Alessio Corti (viva date: 28.01.2009)
06.2005	Distinction, Certificate of Advanced Study in Mathematics, University of Cambridge
10.2003	B.A., Faculty of Mathematics, University of Belgrade

FELLOWSHIPS AND POSITIONS

09.2013 – present	Emmy-Noether-Nachwuchsgruppenleiter, Universität Bonn
11.2011 – 08.2013	Wissenschaftlicher Mitarbeiter, Universität Bayreuth
10.2011	Research visitor, Universität Bayreuth
10.2010 – 09.2011	Chapman Fellow, Imperial College London

Fellowships and positions (continued)

09.2010	Research visitor, Max-Planck-Institut für Mathematik, Bonn
09.2009 – 08.2010	European Post-Doctoral Institute (EPDI) fellow, Max-Planck-Institut für Mathematik, Bonn
04.2009 – 08.2009	Postdoc, Université Joseph Fourier, Grenoble
02.2009 – 03.2009	Research Visitor, Max-Planck-Institut für Mathematik, Bonn
12.2003 – 09.2004	Research Assistant Trainee, Mathematical Institute SANU, Belgrade

Funding

- DFG Emmy-Noether-Programm *Gute Strukturen in der höherdimensionalen birationalen Geometrie*, 09/2013-08/2018

Postdoctoral associates

11.2013 – 08.2014	Tobias Dorsch, Postdoc within the DFG Emmy-Noether-Programm *Gute Strukturen in der höherdimensionalen birationalen Geometrie*

Peer service

- Referee for: Duke Mathematical Journal, Mathematical Research Letters, Annales de l'Institut Fourier, Mathematische Zeitschrift, Bulletin of the London Mathematical Society, Manuscripta Mathematica, Compositio Mathematica, Journal für die reine und angewandte Mathematik, International Mathematics Research Notices, Publications of the Research Institute for Mathematical Sciences
- Reviewer for: MathSciNet, Zentralblatt Math

Organisation: long term events

01.2014 – 04.2014	*Birational and hyperkähler geometry* group at the *Junior Hausdorff Trimester in Algebraic Geometry*, Hausdorff Research Institute for Mathematics, Bonn, organisers: Daniel Greb (Bochum), Anne-Sophie Kaloghiros (Imperial College London), Vladimir Lazić (Bonn), Chenyang Xu (Beijing)

ORGANISATION: CONFERENCES, WORKSHOPS, LECTURE SERIES, SEMINARS

- *Calf seminar* (Junior Cambridge-Oxford-Warwick seminar), 2008, local organiser in Cambridge
- *Rob Lazarsfeld's Felix Klein Lectures 2014*, Hausdorff Research Institute for Mathematics, Bonn, 6-22 January 2014, organisers: Daniel Huybrechts (Bonn), Vladimir Lazić (Bonn)
- Workshop *Birational geometry and foliations*, Hausdorff Research Institute for Mathematics, Bonn, 24-28 February 2014, organisers: Daniel Greb (Bochum), Anne-Sophie Kaloghiros (Imperial College London), Vladimir Lazić (Bonn), Chenyang Xu (Beijing)
- Conference *Complex Analysis and Geometry (in honour of Thomas Peternell's 60th birthday)*, Freiburg Institute for Advanced Studies, Freiburg, 21-23 August 2014, organisers: Andreas Höring (Nice), Stefan Kebekus (Freiburg), Vladimir Lazić (Bonn), Gianluca Pacienza (Strasbourg)
- *Conference for Young Researchers in Arithmetic and Algebraic Geometry*, Universität Bonn, 6-8 October 2014, organisers: Vladimir Lazić (Bonn), Nikita Semenov (Mainz)
- *Winter school "Birational methods in hyperkähler geometry"*, Universität Bonn, 1-5 December 2014, organiser: Vladimir Lazić (Bonn)