Towards Finite Generation of the Canonical Ring without the Minimal Model Program

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This dissertation is submitted for the degree of
Doctor of Philosophy

December 2008

Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text. No part of this dissertation has been submitted for any other qualification.

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Summary

The purpose of this thesis is to make the first step in a project to prove finite generation of the canonical ring without the techniques of the Minimal Model Program. The proof of the finite generation by Birkar, Cascini, Hacon and M[c]Kernan exploits fully constructions of Mori theory, and is a part of a larger induction scheme in which several other conjectures of the theory are settled.

The route undertaken in this thesis is completely different, and the idea is to prove the finite generation directly, by induction on the dimension. A version of the hyperplane section principle is applied in order to restrict to carefully chosen log canonical centres. The biggest conceptual difficulty in attempts to obtain a proof by induction was the finite generation of the kernel of the restriction map. The idea to resolve the kernel issue in this thesis is to view the canonical ring as a subalgebra of a larger algebra, which would a priori contain generators of the kernel. In practice this means that the new algebra will have higher rank grading, and techniques to deal with these algebras are developed along the way.

The problem of finite generation is reduced to a property which should be easier to handle with analytic techniques. I also discuss the ultimate goal of the project – the finite generation in the case of pairs with log canonical singularities, as well as relations to Abundance Conjecture and the finite generation in positive characteristic.

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Chapter 1

Introduction

The topic of this thesis is to make the first of two steps in a project to prove finite generation of the canonical ring without using techniques of the Minimal Model Program.

Finite Generation Conjecture. *Let (X, Δ) be a projective log canonical pair. Then the canonical ring*

$$R(X, K_X + \Delta) = \bigoplus_{n \geq 0} H^0(X, \mathcal{O}_X(\lfloor n(K_X + \Delta) \rfloor))$$

is a finitely generated $\mathbb{C}$-algebra.

The ring above is often called the *log canonical ring*, to stress the log structure. I will drop the "log" part since there is no danger of ambiguity.

There has been a recent enormous progress in Mori Theory, starting with papers of Shokurov [Sho03] and Hacon and M$^{\text{c}}$Kernan [HM05], and culminating with the paper by Birkar, Cascini, Hacon and M$^{\text{c}}$Kernan [BCHM06], which settled several conjectures of the MMP for kawamata log terminal pairs: existence of flips and existence of minimal models for varieties of log general type. However, the picture is still incomplete – ultimately, we would like the programme to work for varieties with log canonical singularities and not necessarily of log general type. Certainly, finite generation of the canonical ring is a consequence of the existence of minimal models and of Basepoint Free theorem.

This thesis undertakes a different approach. The philosophy is that finite generation lies at the beginning, and that (almost) the whole Mori Theory can be reconstructed from the point of view of finite generation. The existence of flips is a straightforward consequence of the finite generation, and there is some work that suggests that other conjectures, including Abundance Conjecture, can be deduced from it. Actually, a form of abundance is the main technical obstacle to performing the procedure from [BCHM06] to complete the MMP in the case of klt singularities.

The paper [BCHM06] uses the bigness of boundary divisors almost everywhere, and it seems very difficult to avoid that fact. One of the motivations for this thesis, apart from the very appealing project of having a proof of finite generation which is conceptual, concise and by induction on the dimension, is to try use the bigness of the boundary as little as possible. At present, avoiding that assumption completely does not seem foreseeable, in particular because of the form of the extension results that we have at present which use bigness essentially. It seems reasonable to expect that analytic techniques could be involved in a similar manner to that of Siu's paper [Siu02] to get around the existence of an ample divisor in the boundary, but this seems a far-fetched task at the moment.

Let me outline the contents of this thesis; more is given at the beginning of individual chapters and sections. In Chapter 2 I survey the known properties of b-divisors, multiplier ideals, asymptotic invariants of linear systems and Diophantine approximation used in the following chapters. The common feature of all these concepts is that they measure the behaviour of certain objects in some limiting processes. I will use the techniques from Chapter 2 extensively throughout the thesis, and I have tried to make the presentation self-contained and to keep citing external sources to the minimum. The basic reference for b-divisors is [Cor07]. There are many references for standard multiplier ideal sheaves, but the presentation of multiplier ideals used in this thesis is closely following [HM08]. Asymptotic numerical invariants attached to linear series have been systematically investigated ever since the book [Nak04] appeared, and papers [ELM$^+$06, Bou04, Hac08] are the main references used in this work. As for Diophantine approximation, I draw upon results of [Cas57, BCHM06], apart from Lemma 2.28, where I have to make use of the precise quantity of the error term between the actual and approximated values.

Chapter 3 surveys the Minimal Model Program, and I sketch the central role of finite generation, in particular in the problem of the existence of flips. The standard literature on this is [KMM87, KM98]. The last part of the chapter concentrates on the existence of pl flips, and in particular we prove that a suitable statement in dimension one less implies finite generation of the restricted canonical ring. The presentation here follows closely, and is occasionally taken almost verbatim from, the paper [HM08], and I stress potential issues that will be equally observable in the general case of the finite generation in Chapter 6. Apart from Chapters 2 and 3, the thesis is my own original work.

Chapter 4 is devoted to developing techniques that will be used in Chapters 5 and 6 in order to prove that certain superlinear maps are in fact piecewise linear. The method developed requires deep techniques of Diophantine approximation and extensive use of Lipschitz continuity, and is one of the technically most demanding parts of the thesis which is not within the realm of algebraic geometry.

In Chapter 5 I develop necessary tools to deal with algebras of higher rank. There are two approaches: that algebras should be given by additive maps of adjoint divisors, or by superadditive maps of mobile b-divisors which satisfy a certain saturation condition in the sense of Shokurov. The former is undertaken successfully in Chapter 6 to prove finite generation of the canonical ring under certain assumptions, which is the core of this work. The latter is used in Chapter 5 to show that suitable higher rank analogues of Shokurov algebras that appeared in the context of 3-dimensional flips [Sho03, Cor07] stand, perhaps surprisingly, a good chance of being finitely generated, and the proof of this fact is given on curves. In particular, this method demonstrates that the saturation condition gives very strong numerical constraints on the divisors involved (not only rationality of divisors, but also a bound on the denominators).

Finally, Chapter 6 is the heart of this thesis. I prove finite generation of the canonical ring under a natural assumption on the convex geometry of the set of log canonical pairs with big boundaries in terms of divisorial components of the stable base loci, see Property $\mathcal{L}_A^G$ there. This property is a consequence of the MMP, however my hope is that there will very soon exist a proof of this statement obtained by techniques similar to those used in the proof of finite generation of the restricted algebra in Section 6.1 below.

Let me sketch the strategy for the proof of finite generation and present difficulties that arise on the way. The natural idea is to pick a smooth divisor S on X and to restrict the algebra to it. If we are very lucky, the restricted algebra will be finitely generated and we might hope that the generators lift to generators on X. There are several issues with this approach.

Firstly, in order to obtain something meaningful on S, S should be a log canonical centre of some pair (X, Δ') such that $R(X, K_X + \Delta)$ and $R(X, K_X + \Delta')$ share a common truncation. Secondly, even if the restricted algebra were finitely generated, the same might not be obvious for the kernel of the restriction map. Thirdly, the natural choice is to use the Hacon-M^{c}Kernan extension theorem, and hence we must be able to ensure that S does not belong to the stable base locus of $K_X + \Delta'$.

The idea to resolve the kernel issue is to view $R(X, K_X + \Delta)$ as a subalgebra of a larger algebra, which would a priori contain generators of the kernel. In practice this means that the new algebra will have higher rank grading. Namely, we will see that, roughly, the rank corresponds to the number of components of Δ. The proof then proceeds to employ the techniques from all previous chapters: generalities about higher rank finite generation allow me to deduce finite generation of initial algebras from that of bigger algebras, and finite generation of the image is dealt with by using difficult techniques revolving around Hacon-M^{c}Kernan methods of extending sections of adjoint line bundles.

Finally, it is my hope that the techniques of this thesis could be adapted to handle finite generation in the case of log canonical singularities, Abundance Conjecture and the case of positive characteristic. The Minimal Model Program and the finite generation in the case of log canonical singularities seem increasingly within reach, especially since the works of Ambro and Fujino [Amb03, Fuj07b]. One of the main obstacles is finding a suitable analogue of the canonical bundle formula of Fujino and Mori, which would allow us to restrict attention to the log general type case. Abundance Conjecture is closely related to a certain non-vanishing statement, which has been successfully proved without Mori Theory in [Pău08]; similar techniques appeared in [Hac08] and they are precisely those used here in order to prove finite generation of the restricted algebra. Finally, the case of positive characteristic is also an active field of research. The method presented here is mostly characteristic-free, apart from two important ingredients: resolution of singularities and the extension theorem of Hacon and M^{c}Kernan, which uses in its proof

Kawamata-Viehweg vanishing which is known only in characteristic zero. I expect that some of these projects will be completed by using the techniques developed in this thesis.

Acknowledgements

First and foremost, I would like to wholeheartedly thank my supervisor Alessio Corti for his encouragement, support and continuous inspiration. A paragraph here is not enough to express how much I have benefited from conversations with him and from his ideas. Among other things, I am indebted to him for the philosophical point that sometimes higher rank is better than rank 1 and that starting from simple examples is essential.

A big Thank-You to my official supervisor in Cambridge, Nick Shepherd-Barron, who has provided mathematical advice and support for me in the last three years.

I am very grateful to Christopher Hacon for suggesting that methods from [Hac08] might be useful in the context of finite generation of the restricted algebra. Anne-Sophie Kaloghiros has been a constant friend and colleague, from and with whom I have learned lots of good maths. Miles Reid and Burt Totaro have always been there to offer me good advice and a sanity check. I have benefited from conversations and exchanging ideas with many colleagues: Sébastien Boucksom, Paolo Cascini, Alex Küronya, James M^c^Kernan, Shigefumi Mori, Mihai Păun.

I owe a huge gratitude to Trinity College, Cambridge for supporting me through my thesis, and to the Department of Pure Mathematics and Mathematical Statistics of the University of Cambridge for funding trips that turned out to shape my mathematical knowledge more than I could have hoped for.

And essentially, I would like to thank my parents for all their love and support through highs and lows of the whole process called PhD. My friends have been there when I needed them most, and I could not have made this journey without them.

Notation and Conventions

Unless stated otherwise, varieties in this thesis are normal over $\mathbb{C}$ and projective over an affine variety Z. The group of Weil, respectively Cartier, divisors on a variety X is denoted by $\mathrm{WDiv}(X)$, respectively $\mathrm{Div}(X)$. I denote $\mathrm{WDiv}(X)^{\kappa\geq 0} = \{D \in \mathrm{WDiv}(X) : \kappa(X, D) \geq 0\}$, and similarly for $\mathrm{Div}(X)^{\kappa\geq 0}$, where κ is the Iitaka dimension. I write $\sim$ for linear equivalence of Weil divisors and $\equiv$ for numerical equivalence of Cartier divisors, and $\rho(X) = \operatorname{rk} N^1(X)$ is the Picard number of X. Similarly for relative versions. Subscripts denote either the ring in which the coefficients of divisors are taken or that the equivalence is relative to a specified morphism.

An ample $\mathbb{Q}$-divisor A on a variety X is *(very) general* if there is a sufficiently divisible positive integer k such that kA is very ample and kA is a (very) general section of the linear system $|kA|$. In particular we can assume that for some $k \gg 0$, kA is a smooth divisor on X. In practice, we fix k in advance, and generality is needed to ensure that A does not make singularities of pairs worse, as in Theorem 3.19.

If T is a prime divisor on X such that $T \not\subset \operatorname{Fix}|D|$, then $|D|_T$ denotes the image of the linear system $|D|$ under restriction to T.

For any two divisors $P = \sum p_i E_i$ and $Q = \sum q_i E_i$ on a variety X, set $P \wedge Q = \sum \min\{p_i, q_i\} E_i$.

I use the adjunction formula with *differents* as explained in [K$^+$92, Chapter 16].

The sets of non-negative (respectively non-positive) rational and real numbers are denoted by $\mathbb{Q}_+$ and $\mathbb{R}_+$ ($\mathbb{Q}_-$ and $\mathbb{R}_-$ respectively).

Geometry of pairs and valuations. In this thesis, a *log pair* (X, Δ) consists of a variety X and an effective divisor $\Delta \in \mathrm{WDiv}(X)_{\mathbb{R}}$ such that $K_X + \Delta$ is $\mathbb{R}$-Cartier. A pair (X, Δ) is *log smooth* if X is nonsingular and $\operatorname{Supp}\Delta$ has simple normal crossings. A *model* over X is a proper birational morphism $f\colon Y \to X$. A *log resolution* of (X, Δ) is a model $f\colon Y \to X$ such that the pair $(Y, f_*^{-1}\Delta + \operatorname{Exc} f)$ is log smooth. A *boundary* is a divisor $\Delta = \sum d_i D_i \in \mathrm{WDiv}(X)_{\mathbb{R}}$ such that $0 \leq d_i \leq 1$ for all i. A birational morphism $f\colon Y \to X$ is *small* if $\operatorname{codim}_Y \operatorname{Exc} f \geq 2$.

A valuation $\nu\colon k(X) \to \mathbb{Z}$ is *geometric* if $\nu = \operatorname{mult}_E$, where $E \subset Y$ is a prime divisor on a model $Y \to X$. In that case I denote the valuation also by E. The *centre* of a geometric valuation ν on X associated to a divisor E on a model $f\colon Y \to X$ is $c_X\nu := f(E)$.

Convex geometry. If $\mathcal{S} = \sum \mathbb{N}e_i$ is a submonoid of $\mathbb{N}^n$, I denote $\mathcal{S}_\mathbb{Q} = \sum \mathbb{Q}_+e_i$ and $\mathcal{S}_\mathbb{R} = \sum \mathbb{R}_+e_i$. A monoid $\mathcal{S} \subset \mathbb{N}^n$ is *saturated* if $\mathcal{S} = \mathcal{S}_\mathbb{R} \cap \mathbb{N}^n$.

If $\mathcal{S} = \sum_{i=1}^n \mathbb{N}e_i$ and $\kappa_1, \dots, \kappa_n$ are positive integers, the submonoid $\mathcal{S}' = \sum_{i=1}^n \mathbb{N}\kappa_i e_i$ is called a *truncation* of $\mathcal{S}$. If $\kappa_1 = \cdots = \kappa_n = \kappa$, I denote $\mathcal{S}^{(\kappa)} := \sum_{i=1}^n \mathbb{N}\kappa e_i$, and this truncation does not depend on a choice of generators of $\mathcal{S}$.

A submonoid $\mathcal{S} = \sum \mathbb{N}e_i$ of $\mathbb{N}^n$ (respectively a cone $\mathcal{C} = \sum \mathbb{R}_+e_i$ in $\mathbb{R}^n$) is *simplicial* if its generators e_i are $\mathbb{R}$-linearly independent. The e_i form a *basis* of $\mathcal{S}$ (respectively $\mathcal{C}$).

For a cone $\mathcal{C} \subset \mathbb{R}^n$, I denote $\mathcal{C}_\mathbb{Q} = \mathcal{C} \cap \mathbb{Q}^n$. The *dimension* $\dim \mathcal{C}$ of a cone $\mathcal{C} = \sum \mathbb{R}_+e_i$ is the dimension of the space $\sum \mathbb{R}e_i$. All cones considered are convex and strongly convex, that is they do not contain lines.

In this thesis the *relative interior* of a cone $\mathcal{C} = \sum \mathbb{R}_+e_i \subset \mathbb{R}^n$, denoted by $\operatorname{relint}\mathcal{C}$, is the topological interior of $\mathcal{C}$ in the space $\sum \mathbb{R}e_i$ union the origin. If $\dim \mathcal{C} = n$, we instead call it the *interior* of $\mathcal{C}$ and denote it by $\operatorname{int}\mathcal{C}$. The boundary of a closed set $\mathcal{C}$ is denoted by $\partial\mathcal{C}$.

Let $\mathcal{S} \subset \mathbb{N}^n$ be a finitely generated monoid, $\mathcal{C} \in \{\mathcal{S}, \mathcal{S}_\mathbb{Q}, \mathcal{S}_\mathbb{R}\}$ and V an $\mathbb{R}$-vector space. A function $f\colon \mathcal{C} \to V$ is: *positively homogeneous* if $f(\lambda x) = \lambda f(x)$ for $x \in \mathcal{C}, \lambda \geq 0$; *superadditive* if $f(x) + f(y) \leq f(x+y)$ for $x, y \in \mathcal{C}$; $\mathbb{Q}$*-superadditive* if $\lambda f(x) + \mu f(y) \leq f(\lambda x + \mu y)$ for $x, y \in \mathcal{C}, \lambda, \mu \in \mathbb{Q}_+$; $\mathbb{Q}$*-additive* if the previous inequality is an equality; *superlinear* if $\lambda f(x) + \mu f(y) \leq f(\lambda x + \mu y)$ for $x, y \in \mathcal{S}_\mathbb{R}, \lambda, \mu \in \mathbb{R}_+$, or equivalently, if and only if it is superadditive and positively homogeneous. Similarly for *additive, subadditive, sublinear*. It is *piecewise additive* if there is a finite polyhedral decomposition $\mathcal{C} = \bigcup \mathcal{C}_i$ such that $f_{|\mathcal{C}_i}$ is additive for every i; additionally, if each $\mathcal{C}_i$ is a rational cone, it is *rationally piecewise additive*. Similarly for (rationally) piecewise linear, abbreviated PL and $\mathbb{Q}$-PL. Assume furthermore that f is linear on $\mathcal{C}$ and $\dim \mathcal{C} = n$. The *linear extension of f to* $\mathbb{R}^n$ is the unique linear function $\ell\colon \mathbb{R}^n \to V$ such that $\ell_{|\mathcal{C}} = f$.

I often use without explicit mention that if $\lambda\colon \mathcal{M} \to \mathcal{S}$ is an additive surjective map between finitely generated saturated monoids, and if $\mathcal{C}$ is a rational polyhedral cone in $\mathcal{S}_{\mathbb{R}}$, then $\lambda^{-1}(\mathcal{S} \cap \mathcal{C}) = \mathcal{M} \cap \lambda^{-1}(\mathcal{C})$. In particular, the inverse image of a saturated finitely generated submonoid of $\mathcal{S}$ is a saturated finitely generated submonoid of $\mathcal{M}$.

For a polytope $\mathcal{P} \subset \mathbb{R}^n$, I denote $\mathcal{P}_{\mathbb{Q}} = \mathcal{P} \cap \mathbb{Q}^n$. A polytope is *rational* if it is the convex hull of finitely many rational points.

If $\mathcal{B} \subset \mathbb{R}^n$ is a convex set, then $\mathbb{R}_+\mathcal{B}$ will denote the set $\{rb : r \in \mathbb{R}_+, b \in \mathcal{B}\}$, the *cone over* $\mathcal{B}$. In particular, if $\mathcal{B}$ is a rational polytope, $\mathbb{R}_+\mathcal{B}$ is a rational polyhedral cone. The dimension of a rational polytope $\mathcal{P}$, denoted $\dim \mathcal{P}$, is the dimension of the smallest rational affine space containing $\mathcal{P}$.

Chapter 2

Asymptotic Techniques

In this chapter I present some recent techniques that will be useful in the rest of this thesis. Their common feature is that they describe asymptotic behaviour of certain objects attached to varieties.

2.1 b-Divisors

b-Divisors can be understood as limits of regular divisors on different birational models. In particular, mobile b-divisors are useful when dealing with finite generation issues as they help keep track of sections needed to generate an algebra. Further, the language of b-divisors makes, on occasion, mathematical texts more concise, which is already obvious in the proofs of basic properties of multiplier ideals.

Definition 2.1. An integral *b-divisor* $\mathbf{D}$ on X is an element of the group

$$\mathbf{Div}(X) = \varprojlim \mathrm{WDiv}(Y),$$

where the limit is taken over all models $f\colon Y \to X$ with the induced homomorphisms $f_*\colon \mathrm{WDiv}(Y) \to \mathrm{WDiv}(X)$. Thus $\mathbf{D}$ is a collection of divisors $\mathbf{D}_Y \in \mathrm{WDiv}(Y)$ compatible with push-forwards. Each $\mathbf{D}_Y$ is the *trace* of $\mathbf{D}$ on Y.

For every model $f\colon Y \to X$ the induced map $f_*\colon \mathbf{Div}(Y) \to \mathbf{Div}(X)$ is an isomorphism, so b-divisors on X can be identified with b-divisors on any model over X. For every open subset $U \subset X$ we naturally define the restriction $\mathbf{D}_{|U}$ of a b-divisor $\mathbf{D}$ on X.

Definition 2.2. The *b-divisor of a nonzero rational function* φ is

$$\mathbf{div}_X \varphi = \sum \operatorname{mult}_E \varphi \cdot E,$$

where E runs through geometric valuations with centre on X.

The *b-divisorial sheaf* $\mathcal{O}_X(\mathbf{D})$ associated to a b-divisor $\mathbf{D}$ is defined by

$$\Gamma(U, \mathcal{O}_X(\mathbf{D})) = \{\varphi \in k(X) : (\mathbf{div}_X \varphi + \mathbf{D})_{|U} \geq 0\}.$$

Definition 2.3. The *proper transform* b-divisor $\widehat{D}$ of an $\mathbb{R}$-divisor D has trace $\widehat{D}_Y = f_*^{-1} D$ on every model $f\colon Y \to X$.

The *Cartier closure* of an $\mathbb{R}$-Cartier divisor D on X is the b-divisor $\overline{D}$ with trace $\overline{D}_Y = f^* D$ on every model $f\colon Y \to X$.

A b-divisor $\mathbf{D}$ *descends* to a model $Y \to X$ if $\mathbf{D} = \overline{\mathbf{D}_Y}$; we then say that $\mathbf{D}$ is a *Cartier b-divisor*.

A b-divisor $\mathbf{M}$ on X is *mobile* if it descends to a model $Y \to X$, where $\mathbf{M}_Y$ is basepoint free.

Note that if a mobile b-divisor $\mathbf{M}$ descends to a model $W \to X$, then $\mathbf{M}_W$ is free and $H^0(X, \mathbf{M}) \simeq H^0(W, \mathbf{M}_W)$.

Cartier restriction. Let $\mathbf{D}$ be a Cartier b-divisor on X and let S be a normal prime divisor on X such that $S \not\subset \operatorname{Supp} \mathbf{D}_X$. Let $f\colon Y \to X$ be a log resolution of (X, S) such that $\mathbf{D}$ descends to Y. Define the *restriction* of $\mathbf{D}$ to S as

$$\mathbf{D}_{|S} := \overline{\mathbf{D}_{Y|\widehat{S}_Y}}.$$

This is a b-divisor on S via $(f_{|\widehat{S}_Y})_*$ and it does not depend on the choice of f. By definition, $\mathbf{D}_{|S}$ is a Cartier b-divisor that satisfies $(\mathbf{D}_1 + \mathbf{D}_2)_{|S} = \mathbf{D}_{1|S} + \mathbf{D}_{2|S}$, and $\mathbf{D}_{1|S} \geq \mathbf{D}_{2|S}$ if $\mathbf{D}_1 \geq \mathbf{D}_2$.

Definition 2.4. The *canonical* b-divisor $\mathbf{K}_X$ on X has a trace $(\mathbf{K}_X)_Y = K_Y$ on every model $Y \to X$. The *discrepancy* $\mathbf{A}(X, \Delta)$ of the pair (X, Δ) is

$$\mathbf{A}(X, \Delta) = \mathbf{K}_X - \overline{K_X + \Delta}.$$

To streamline several arguments in the thesis, we introduce the following.

Definition 2.5. Let (X, Δ) be a log pair. For a model $f \colon Y \to X$ we can write uniquely

$$K_Y + B_Y = f^*(K_X + \Delta) + E_Y,$$

where B_Y and E_Y are effective with no common components and E_Y is f-exceptional. The *boundary* b-divisor $\mathbf{B}(X, \Delta)$ is given by $\mathbf{B}(X, \Delta)_Y = B_Y$ for every model $Y \to X$. If the pair (X, Δ) is log smooth and Δ is a reduced divisor, define $\mathbf{A}^*(X, \Delta) = \mathbf{A}(X, \Delta) + \mathbf{B}(X, \Delta)$; this is an effective and exceptional integral b-divisor.

Lemma 2.6. *If (X, Δ) is a log pair, then the boundary b-divisor $\mathbf{B}(X, \Delta)$ is well-defined.*

Proof. Let $g \colon Y' \to X$ be a model such that there is a proper birational morphism $h \colon Y' \to Y$. Pushing forward $K_{Y'} + B_{Y'} = g^*(K_X + \Delta) + E_{Y'}$ via h_* yields

$$K_Y + h_* B_{Y'} = f^*(K_X + \Delta) + h_* E_{Y'},$$

and since $h_* B_{Y'}$ and $h_* E_{Y'}$ have no common components, $h_* B_{Y'} = B_Y$. □

The following result will be used several times in this thesis, and it will enable us to pass to more suitable models in order to apply extension results.

Lemma 2.7. *Let (X, Δ) be a log canonical pair. There exists a log resolution $Y \to X$ such that the components of $\{\mathbf{B}(X, \Delta)_Y\}$ are disjoint.*

Proof. See [KM98, Proposition 2.36] or [HM05, Lemma 6.7]. □

2.2 Multiplier Ideals

Multiplier ideal sheaves and their asymptotic versions have proved absolutely essential in recent major progress in Mori theory. The techniques as we know them today were introduced in the seminal paper [Siu98] in order to prove invariance of plurigenera. The form of multiplier ideas used in this thesis is not the most general, and with a bit of work some of the results of this chapter can be generalised to the context of [Fuj08], but I do not pursue this here.

Definition 2.8. Let (X, Δ) be a log smooth pair where Δ is a reduced divisor, and let V be a linear system whose base locus contains no log canonical centres of (X, Δ). Let $\mu\colon Y \to X$ be a log resolution of V and (X, Δ), and let $F = \operatorname{Fix} \mu^* V$. Then for any real number $c \geq 0$, define the *multiplier ideal sheaf*

$$\mathcal{J}_{\Delta, c\cdot V} := \mu_* \mathcal{O}_Y(\mathbf{A}^*(X, \Delta)_Y - \lfloor cF \rfloor).$$

If $\Delta = 0$ we will write $\mathcal{J}_{c\cdot V}$, and if $D = cG$, where $G > 0$ is a Cartier divisor, we define

$$\mathcal{J}_{\Delta, D} := \mathcal{J}_{\Delta, c\cdot V},$$

where $V = \{G\}$.

Lemma 2.9. *The multiplier ideal $\mathcal{J}_{\Delta, c\cdot V}$ in Definition 2.8 does not depend on the choice of a log resolution μ.*

Proof. Denote $\mathbf{A}^* = \mathbf{A}^*(X, \Delta)$ and $\mathbf{B} = \mathbf{B}(X, \Delta)$. Observe that the b-divisor $\mathbf{F}$ given by $\mathbf{F}_Z = \operatorname{Fix} \pi^* V$ for every model $\pi\colon Z \to X$, descends to Y. It is enough to show that $\mathbf{A}^* - \lfloor c\mathbf{F} \rfloor \geq \overline{\mathbf{A}^*_Y - \lfloor cF \rfloor}$. For this, let $f\colon Y' \to Y$ be a model. The inequality, on Y', is equivalent to

$$\lceil K_{Y'} - f^*(K_Y + \{cF\} + \mathbf{B}_Y) + \mathbf{B}_{Y'} \rceil \geq 0.$$

Observe that the log canonical centres of $(Y, \{cF\} + \mathbf{B}_Y)$ are exactly the intersections of components of $\mathbf{B}_Y$. Thus $\mathbf{B}_{Y'}$ is the locus of log canonical singularities on Y' for $(Y, \{cF\} + \mathbf{B}_Y)$, and the lemma follows. □

The basic properties of multiplier ideal sheaves are listed in the following result.

Lemma 2.10. *Let (X, Δ) be a log smooth pair where Δ is reduced, let V be a linear system whose base locus contains no log canonical centres of (X, Δ), and let G and*

D be effective $\mathbb{Q}$-Cartier divisors whose supports contain no log canonical centres of (X,Δ). Then:

(1) $\mathcal{J}_{\Delta,D} = \mathcal{O}_X$ if and only if $(X,\Delta+D)$ is divisorially log terminal and $\lfloor D \rfloor = 0$,

(2) if $0 \leq \Delta' \leq \Delta$ then $\mathcal{J}_{\Delta,c\cdot V} \subset \mathcal{J}_{\Delta',c\cdot V}$; in particular, $\mathcal{J}_{\Delta,c\cdot V} \subset \mathcal{J}_{c\cdot V} \subset \mathcal{O}_X$,

(3) if $\Sigma \geq 0$ is a Cartier divisor, $D - \Sigma \leq G$ and $\mathcal{J}_{\Delta,G} = \mathcal{O}_X$ then $\mathcal{I}_\Sigma \subset \mathcal{J}_{\Delta,D}$.

Proof. (1) and (2) follow easily from the definitions. To see (3), notice that as Σ is Cartier and $\mathcal{J}_{\Delta,G} = \mathcal{O}_X$, we have $\mathcal{J}_{\Delta,G}(-\Sigma) = \mathcal{O}_X(-\Sigma) = \mathcal{I}_\Sigma$. But since $D \leq G+\Sigma$, we also have $\mathcal{J}_{\Delta,G}(-\Sigma) = \mathcal{J}_{\Delta,G+\Sigma} \subset \mathcal{J}_{\Delta,D}$. □

The following is an extension of [Laz04, Theorem 9.4.8].

Theorem 2.11 (Nadel Vanishing)**.** *Let $\pi\colon X \to Z$ be a projective morphism to a normal affine variety Z. Let (X,Δ) be a log smooth pair where Δ is reduced, let D be an effective $\mathbb{Q}$-Cartier divisor whose support does not contain any log canonical centres of (X,Δ) and let N be a Cartier divisor. If $N - D$ is ample then*

$$H^i(X, \mathcal{J}_{\Delta,D}(K_X + \Delta + N)) = 0$$

for $i > 0$.

Proof. By [Sza94], there is a log resolution $\mu\colon Y \to X$ of $(X,\Delta+D)$ which is an isomorphism over the generic point of each log canonical centre of (X,Δ). Denote $\Gamma = \mathbf{B}(X,\Delta)_Y$ and $E = \mathbf{A}^*(X,\Delta)_Y$. Since $(Y,\Gamma+\mu^*D)$ is log smooth and Γ and μ^*D have no common components, $(Y,\Gamma+\{\mu^*D\})$ is divisorially log terminal. Therefore we may pick an effective μ-exceptional divisor H such that $K_Y+\Gamma+\{\mu^*D\}+H$ is divisorially log terminal, and $\mu^*(N-D)-H$ and $-H$ are μ-ample. As

$$E - \lfloor \mu^*D \rfloor - (K_Y + \Gamma + \{\mu^*D\} + H) = -\mu^*(K_X + \Delta + D) - H$$

is μ-ample, Kawamata-Viehweg vanishing implies that

$$R^i\mu_*\mathcal{O}_Y(E - \lfloor \mu^*D \rfloor + \mu^*(K_X+\Delta+N)) = R^i\mu_*\mathcal{O}_Y(E - \lfloor \mu^*D \rfloor) \otimes \mathcal{O}_X(K_X+\Delta+N) = 0$$

for $i > 0$. As

$$E - \lfloor \mu^* D \rfloor + \mu^*(K_X + \Delta + N) - (K_Y + \Gamma + \{\mu^* D\} + H) = \mu^*(N - D) - H$$

is ample, Kawamata-Viehweg vanishing again implies that

$$H^i(Y, E - \lfloor \mu^* D \rfloor + \mu^*(K_X + \Delta + N)) = 0$$

for $i > 0$. Since the Leray spectral sequence degenerates, this proves the result. $\square$

Lemma 2.12. *Let $\pi\colon X \to Z$ be a projective morphism to a normal affine variety Z. Let (X, Δ) be a log smooth pair where Δ is reduced, S a component of Δ, D an effective $\mathbb{Q}$-Cartier divisor whose support does not contain any log canonical centres of (X, Δ) and denote $\Theta = (\Delta - S)_{|S}$. Then there is a short exact sequence*

$$0 \to \mathcal{J}_{\Delta - S, D+S} \to \mathcal{J}_{\Delta, D} \to \mathcal{J}_{\Theta, D_{|S}} \to 0. \tag{2.1}$$

If N is a Cartier divisor such that $N - D$ is ample, then the restriction map

$$H^0(X, \mathcal{J}_{\Delta, D}(K_X + \Delta + N)) \to H^0(S, \mathcal{J}_{\Theta, D_{|S}}(K_X + \Delta + N)) \tag{2.2}$$

is surjective.

Proof. Let $\mu\colon Y \to X$ be a log resolution as in the proof of Theorem 2.11, and denote $\Gamma = \mathbf{B}(X, \Delta)_Y$, $E = \mathbf{A}^*(X, \Delta)_Y$ and $T = \mu_*^{-1} S$. There is a short exact sequence

$$0 \to \mathcal{O}_Y(E - \lfloor \mu^* D \rfloor - T) \to \mathcal{O}_Y(E - \lfloor \mu^* D \rfloor) \to \mathcal{O}_T(E - \lfloor \mu^* D \rfloor) \to 0.$$

Now $\mu_* \mathcal{O}_Y(E - \lfloor \mu^* D \rfloor) = \mathcal{J}_{\Delta, D}$, and since

$$E - \mu^* D - T = (K_Y + \Gamma - T) - \mu^*(K_X + \Delta - S + (D + S)),$$

we have $\mu_* \mathcal{O}_Y(E - \lfloor \mu^* D \rfloor - T) = \mathcal{J}_{\Delta - S, D+S}$, and similarly $\mu_* \mathcal{O}_T(E - \lfloor \mu^* D \rfloor) = \mathcal{J}_{\Theta, D_{|S}}$. As in the proof of Theorem 2.11, we may pick an effective μ-exceptional divisor H such that $K_Y + \Gamma - T + \{\mu^* D\} + H$ is divisorially log terminal and $-H$

is μ-ample. As

$$E - \lfloor \mu^* D \rfloor - T - (K_Y + \Gamma - T + \{\mu^* D\} + H) = -\mu^*(K_X + \Delta + D) - H$$

is μ-ample, Kawamata-Viehweg vanishing implies that

$$R^1 \mu_* \mathcal{O}_Y(E - \lfloor \mu^* D \rfloor - T) = 0,$$

and this gives (2.1). Now (2.2) follows from (2.1) and Theorem 2.11. □

Now we turn to asymptotic multiplier ideal sheaves. Firstly, if D is a divisor on a normal variety X, an *additive sequence of linear systems associated to D* is a sequence $V_\bullet$ such that $V_m \subset \mathbb{P}(H^0(X, mD))$ and $V_i + V_j \subset V_{i+j}$.

Lemma 2.13. *Let (X, Δ) be a log smooth pair where Δ is reduced, and let $V_\bullet$ be an additive sequence of linear systems associated to a divisor D on X. Assume that there is a positive integer k such that no log canonical centre of (X, Δ) is contained in the base locus of V_k. If c is a positive real number, and p and q are positive integers such that k divides q and q divides p, then*

$$\mathcal{J}_{\Delta, \frac{c}{p} \cdot V_p} \subset \mathcal{J}_{\Delta, \frac{c}{q} \cdot V_q}.$$

Proof. If p divides q then pick a common log resolution $\mu \colon Y \to X$ of V_p, V_q and (X, Δ), and note that $\frac{1}{q} F_q \leq \frac{1}{p} F_p$, where $F_p = \operatorname{Fix} \mu^* V_p$ and $F_q = \operatorname{Fix} \mu^* V_q$. Therefore $\mathcal{J}_{\Delta, \frac{c}{p} \cdot V_p} \subset \mathcal{J}_{\Delta, \frac{c}{q} \cdot V_q}$. □

Definition 2.14. Let (X, Δ) be a log smooth pair where Δ is reduced, and let $V_\bullet$ be an additive sequence of linear systems associated to a divisor D on X. Assume that there is a positive integer k such that no log canonical centre of (X, Δ) is contained in the base locus of V_k. If c is a positive real number, the *asymptotic multiplier ideal sheaf* of $V_\bullet$, given by

$$\mathcal{J}_{\Delta, c \cdot V_\bullet} = \bigcup_{p > 0} \mathcal{J}_{\Delta, \frac{c}{p} \cdot V_p},$$

is equal to $\mathcal{J}_{\Delta, \frac{c}{p} \cdot V_p}$ for p sufficiently divisible by Lemma 2.13 and Noetherian condition. If we take $V_m = |mD|$, then define $\mathcal{J}_{\Delta, c\|D\|} = \mathcal{J}_{\Delta, c \cdot V_\bullet}$, and if S is a component of Δ, $\Theta = (\Delta - S)_{|S}$ and $W_m = |mD|_S$, define $\mathcal{J}_{\Theta, c\|D\|_S} = \mathcal{J}_{\Theta, c \cdot W_\bullet}$.

The next result is analogous to [Laz04, Proposition 11.2.10].

Lemma 2.15. *Let (X,Δ) be a log smooth pair with Δ reduced, let $D \in \mathrm{Div}(X)^{\kappa\geq 0}$ and assume that there is a positive integer p such that no log canonical centre of (X,Δ) is contained in $\mathrm{Bs}\,|pD|$. Then*

$$H^0(X, \mathcal{J}_{\Delta,\|D\|}(D)) = H^0(X,D).$$

Proof. Let $\mathcal{J}_{\Delta,\|D\|} = \mathcal{J}_{\Delta,\frac{1}{k}|kD|}$, let $f\colon Y \to X$ be a log resolution of (X,Δ), $|f^*kD|$ and $|f^*D|$, and denote $E = \mathbf{A}^*(X,\Delta)_Y$ and $F_p = \mathrm{Fix}\,|f^*pD|$ for every p. Since $\lceil F_k/k\rceil \leq F_1$, $\lfloor -F_k/k\rfloor \leq \lceil E - F_k/k\rceil$ and $\lceil E\rceil$ is effective and f-exceptional,

$$\begin{aligned} H^0(X,D) &= H^0(X, f_*\mathcal{O}_Y(f^*D - F_1)) \\ &\subset H^0(X, f_*\mathcal{O}_Y(f^*D + \lceil E - F_k/k\rceil)) = H^0(X, \mathcal{J}_{\Delta,\|D\|}(D)) \\ &\subset H^0(X, f_*\mathcal{O}_Y(f^*D + \lceil E\rceil)) = H^0(X,D). \end{aligned}$$

This concludes the proof. □

The following lemma is a weak version of Mumford's theorem [Laz04, Theorem 1.8.5].

Lemma 2.16. *Let $\pi\colon X \to Z$ be a projective morphism, where X is smooth of dimension n, Z is affine and let H be a very ample divisor on X. If $\mathcal{F}$ is a coherent sheaf on X such that $H^i(X,\mathcal{F}(mH)) = 0$ for $i > 0$ and for all $m \geq -n$, then $\mathcal{F}$ is globally generated.*

Proof. Pick $x \in X$. Let $\mathcal{T} \subset \mathcal{F}$ be the torsion subsheaf supported at x, and let $\mathcal{G} = \mathcal{F}/\mathcal{T}$. Then $H^i(X,\mathcal{G}(mH)) = 0$ for $i > 0$ and for all $m \geq -n$, and $\mathcal{F}$ is globally generated if and only if $\mathcal{G}$ is globally generated. Replacing $\mathcal{F}$ by $\mathcal{G}$ we may therefore assume that $\mathcal{T} = 0$.

Pick a general element $Y \in |H|$ containing x. As $\mathcal{T} = 0$ there is an exact sequence

$$0 \to \mathcal{F}(-Y) \to \mathcal{F} \to \mathcal{Q} \to 0,$$

where $\mathcal{Q} = \mathcal{F} \otimes \mathcal{O}_Y$. As $H^i(Y,\mathcal{Q}(mH)) = 0$ for $i > 0$ and for all $m \geq -(n-1)$, $\mathcal{Q}$ is globally generated by induction on the dimension. As $H^1(X,\mathcal{F}(-Y)) = 0$, it follows that $\mathcal{F}$ is globally generated. □

Lemma 2.17. *Let $\pi\colon X \to Z$ be a projective morphism, where X is smooth of dimension n and Z is affine. If $D \in \operatorname{Div}(X)^{\kappa\geq 0}$, $A \in \operatorname{Div}(X)$ is ample and $H \in \operatorname{Div}(X)$ is very ample, then $\mathcal{J}_{\|D\|}(D + K_X + A + nH)$ is globally generated.*

Proof. Pick a positive integer p such that if $pB \subset |pD|$ is a general element, then

$$\mathcal{J}_{\|D\|} = \mathcal{J}_{\frac{1}{p}|pD|} = \mathcal{J}_B.$$

Then by Theorem 2.11, $H^i(X, \mathcal{J}_{\|D\|}(D + K_X + A + mH)) = 0$ for all $i > 0$ and $m \geq 0$, and we may apply Lemma 2.16. □

To end this section, I will state the main technical result of [HM08] which will enable me to derive a version of extension results in Chapter 6, in order to prove that the restricted algebra is finitely generated. The stable base locus $\mathbf{B}(D)$ is defined in Definition 2.19.

Theorem 2.18 ([HM08, Theorem 5.3])**.** *Let $\pi\colon X \to Z$ be a projective morphism to a normal affine variety Z. Suppose that (X, Δ) is log smooth, $S = \lfloor \Delta \rfloor$ is irreducible and let k be a positive integer such that $D = k(K_X + \Delta)$ is Cartier. If $\mathbf{B}(D)$ does not contain any log canonical centre of $(X, \lceil \Delta \rceil)$ and if A is a sufficiently ample Cartier divisor, then*

$$H^0(S, \mathcal{J}_{\|mD_{|S}\|}(mD + A)) \subset \operatorname{Im}\big(H^0(X, \mathcal{O}_X(mD + A)) \to H^0(S, \mathcal{O}_S(mD + A))\big)$$

for all positive integers m.

2.3 Asymptotic Invariants of Linear Systems

Definition 2.19. Let X be a variety and $D \in \operatorname{WDiv}(X)_{\mathbb{R}}$. For $k \in \{\mathbb{Z}, \mathbb{Q}, \mathbb{R}\}$, define

$$|D|_k = \{C \in \operatorname{WDiv}(X)_k : C \geq 0, C \sim_k D\}.$$

The *stable base locus* of D is

$$\mathbf{B}(D) = \bigcap_{C \in |D|_{\mathbb{R}}} \operatorname{Supp} C$$

if $|D|_{\mathbb{R}} \neq \emptyset$, otherwise we define $\mathbf{B}(D) = X$. The *stable fixed locus* of D, denoted $\mathbf{Fix}(D)$, is a divisorial part of $\mathbf{B}(D)$. The *diminished base locus* is $\mathbf{B}_{-}(D) = \bigcup_{\varepsilon>0} \mathbf{B}(D+\varepsilon A)$ for an ample divisor A; this definition does not depend on a choice of A. In particular $\mathbf{B}_{-}(D) \subset \mathbf{B}(D)$.

By [BCHM06, Lemma 3.5.3], $\mathbf{B}(D) = \bigcap_{C \in |D|_{\mathbb{Q}}} \operatorname{Supp} C$ when D is a $\mathbb{Q}$-divisor, which is the standard definition of the stable base locus. It is elementary that $\mathbf{B}(D_1 + D_2) \subset \mathbf{B}(D_1) \cup \mathbf{B}(D_2)$ for $D_1, D_2 \in \operatorname{WDiv}(X)_{\mathbb{R}}$. In other words, the set $\{D \in \operatorname{WDiv}(X)_{\mathbb{R}} : x \notin \mathbf{B}(D)\}$ is convex for every point $x \in X$.

Definition 2.20. Let Z be a closed subvariety of a smooth variety X and let $D \in \operatorname{WDiv}(X)_{\mathbb{Q}}^{\kappa \geq 0}$. The *asymptotic order of vanishing of D along Z* is

$$\operatorname{ord}_Z \|D\| = \inf\{\operatorname{mult}_Z C : C \in |D|_{\mathbb{Q}}\}.$$

More generally, one can consider any discrete valuation ν of $k(X)$ and define

$$\nu\|D\| = \inf\{\nu(C) : C \in |D|_{\mathbb{Q}}\}$$

for $D \in \operatorname{WDiv}(X)_{\mathbb{Q}}^{\kappa \geq 0}$. Then [ELM+06] shows that $\nu\|D\| = \nu\|E\|$ if D and E are numerically equivalent big divisors, and that ν extends to a sublinear function on $\operatorname{Big}(X)_{\mathbb{R}}$.

Remark 2.21. When X is projective, Nakayama in [Nak04] defines a function $\sigma_Z \colon \overline{\operatorname{Big}(X)} \to \mathbb{R}_+$ by

$$\sigma_Z(D) = \lim_{\varepsilon \downarrow 0} \operatorname{ord}_Z \|D + \varepsilon A\|$$

for any ample $\mathbb{R}$-divisor A, and shows that it agrees with $\operatorname{ord}_Z \|\cdot\|$ on big classes. Analytic properties of these invariants were studied in [Bou04].

We can define the restricted version of the invariant introduced.

Definition 2.22. Let S be a smooth divisor on a smooth variety X and let $D \in \operatorname{Div}(X)_{\mathbb{R}}^{\kappa \geq 0}$ be such that $S \not\subset \mathbf{B}(D)$. Let P be a closed subvariety of S. The *restricted asymptotic order of vanishing of $|D|_S$ along P* is

$$\operatorname{ord}_P \|D\|_S = \inf\{\operatorname{mult}_P C_{|S} : C \sim_{\mathbb{R}} D, C \geq 0, S \not\subset \operatorname{Supp} C\}.$$

In the case of rational divisors, the infimum above can be taken over rational divisors:

Lemma 2.23. *Let X be a smooth variety, $D \in \mathrm{Div}(X)_{\mathbb{Q}}^{\kappa\geq 0}$ and let $D' \geq 0$ be an $\mathbb{R}$-divisor such that $D \sim_{\mathbb{R}} D'$. Then for every $\varepsilon > 0$ there is a $\mathbb{Q}$-divisor $D'' \geq 0$ such that $D \sim_{\mathbb{Q}} D''$, $\operatorname{Supp} D' = \operatorname{Supp} D''$ and $\|D' - D''\| < \varepsilon$. In particular, if $S \subset X$ is a smooth divisor such that $S \not\subset \mathbf{B}(D)$, then for every closed subvariety $P \subset S$ we have*

$$\operatorname{ord}_P \|D\|_S = \inf\{\operatorname{mult}_P C_{|S} : C \sim_{\mathbb{Q}} D, C \geq 0, S \not\subset \operatorname{Supp} C\}.$$

Proof. Let $D' = D + \sum_{i=1}^{p} r_i(f_i)$ for $r_i \in \mathbb{R}$ and $f_i \in k(X)$. Let $F_1, \dots, F_N$ be the components of D and of all (f_i), and assume that $\operatorname{mult}_{F_j} D' = 0$ for $j = 1, \dots, \ell$ and $\operatorname{mult}_{F_j} D' > 0$ for $j = \ell + 1, \dots, N$. Let $(f_i) = \sum_{j=1}^{N} \varphi_{ij} F_j$ for all i and $D = \sum_{j=1}^{N} \delta_j F_j$. Then we have $\delta_j + \sum_{i=1}^{p} \varphi_{ij} r_i = 0$ for $j = 1, \dots, \ell$. Let $\mathcal{K} \subset \mathbb{R}^p$ be the space of solutions of the system $\sum_{i=1}^{p} \varphi_{ij} x_i = -\delta_j$ for $j = 1, \dots, \ell$. Then $\mathcal{K}$ is a rational affine subspace and $(r_1, \dots, r_p) \in \mathcal{K}$, thus for $0 < \eta \ll 1$ there is a rational point $(s_1, \dots, s_p) \in \mathcal{K}$ with $\|s_i - r_i\| < \eta$ for all i. Therefore for η sufficiently small, setting $D'' = D + \sum_{i=1}^{p} s_i(f_i)$ we have the desired properties. □

Remark 2.24. Similarly as in Remark 2.21, [Hac08] introduces a function $\sigma_P \| \cdot \|_S \colon \mathcal{C}_- \to \mathbb{R}_+$ by

$$\sigma_P \|D\|_S = \lim_{\varepsilon \downarrow 0} \operatorname{ord}_P \|D + \varepsilon A\|_S$$

for any ample $\mathbb{R}$-divisor A, where $\mathcal{C}_- \subset \overline{\mathrm{Big}(X)}$ is the set of classes of divisors D such that $S \not\subset \mathbf{B}_-(D)$. Then one can define a formal sum $N_\sigma \|D\|_S = \sum \sigma_P \|D\|_S \cdot P$ over all prime divisors P on S. If $S \not\subset \mathbf{B}(D)$, then for every $\varepsilon_0 > 0$ we have $\lim_{\varepsilon \downarrow \varepsilon_0} \sigma_P \|D + \varepsilon A\|_S = \operatorname{ord}_P \|D + \varepsilon_0 A\|_S$ for any ample divisor A on X similarly as in [Nak04, Lemma 2.1.1], cf. [Hac08, Lemma 7.8].

In this thesis I need a few basic properties cf. [Hac08, Lemma 7.14].

Lemma 2.25. *Let S be a smooth divisor on a smooth projective variety X, let $D \in \mathrm{Div}(X)_{\mathbb{R}}^{\kappa\geq 0}$ be such that $S \not\subset \mathbf{B}(D)$ and let P be a closed subvariety of S. If A is an ample $\mathbb{R}$-divisor on X, then $\operatorname{ord}_P \|D + A\|_S \leq \operatorname{ord}_P \|D\|_S$, and in particular*

$\sigma_P\|D\|_S \leq \operatorname{ord}_P \|D\|_S$. *If D and A are $\mathbb{Q}$-divisors and $\sigma_P\|D\|_S = 0$, then there is a positive integer l such that* $\operatorname{mult}_P \operatorname{Fix}|l(D+A)|_S = 0$.

Proof. The first statement is trivial. For the second one, we have $\operatorname{ord}_P \|D+\frac{1}{2}A\|_S = 0$. Set $n = \dim X$, let H be a very ample divisor on X and fix a positive integer l such that $H' = \frac{l}{2}A - (K_X + S) - (n+1)H$ is very ample. Let $\Delta \sim_{\mathbb{Q}} D + \frac{1}{2}A$ be a $\mathbb{Q}$-divisor such that $S \not\subset \operatorname{Supp}\Delta$ and $\operatorname{mult}_P \Delta_{|S} < 1/l$. We have

$$H^i(X, \mathcal{J}_{l\Delta_{|S}}(K_S + H'_{|S} + (n+1)H_{|S} + l\Delta_{|S} + mH_{|S})) = 0$$

for $m \geq -n$ by Nadel vanishing. Since $l(D+A) \sim_{\mathbb{Q}} K_X + S + H' + (n+1)H + l\Delta$, the sheaf $\mathcal{J}_{l\Delta_{|S}}(l(D+A))$ is globally generated by Lemma 2.16 and its sections lift to $H^0(X, l(D+A))$ by Lemma 2.12. Since $\operatorname{mult}_P(l\Delta_{|S}) < 1$, $\mathcal{J}_{l\Delta_{|S}}$ does not vanish along P and so $\operatorname{mult}_P \operatorname{Fix}|l(D+A)|_S = 0$. □

2.4 Diophantine Approximation

Techniques of Diophantine approximation have appeared in almost all recent work in birational geometry after the paper of Shokurov [Sho03], since it became increasingly clear that we have to work with real divisors, as limits of rational divisors. I present several versions of approximation that will be used in different contexts to prove rationality, or polyhedrality, of certain objects.

Lemma 2.26. *Let $\Lambda \subset \mathbb{R}^n$ be a lattice spanned by rational vectors, and let $V = \Lambda \otimes_{\mathbb{Z}} \mathbb{R}$. Fix a vector $v \in V$ and denote $X = \mathbb{N}v + \Lambda$. Then the closure of X is symmetric with respect to the origin. Moreover, if $\pi\colon V \to V/\Lambda$ is the quotient map, then the closure of $\pi(X)$ is a finite disjoint union of connected components. If v is not contained in any proper rational affine subspace of V, then X is dense in V.*

Proof. I am closely following the proof of [BCHM06, Lemma 3.7.6]. Let G be the closure of $\pi(X)$. Since G is infinite and V/Λ is compact, G has an accumulation point. It then follows that zero is also an accumulation point and that G is a closed subgroup. The connected component G_0 of the identity in G is a Lie subgroup of V/Λ and so by [Bum04, Theorem 15.2], G_0 is a torus. Thus $G_0 = V_0/\Lambda_0$, where $V_0 = \Lambda_0 \otimes_{\mathbb{Z}} \mathbb{R}$ is a rational subspace of V. Since G/G_0 is discrete and compact, it is finite, and it is straightforward that X is symmetric with respect to the origin. Therefore a translate of v by a rational vector is contained in V_0, and so if v is not contained in any proper rational affine subspace of V, then $V_0 = V$. □

The next result is [BCHM06, Lemma 3.7.7].

Lemma 2.27. *Let $x \in \mathbb{R}^n$ and let W be the smallest rational affine space containing x. Fix a positive integer k and a positive real number ε. Then there are $w_1, \ldots, w_p \in W \cap \mathbb{Q}^n$ and positive integers $k_1, \ldots, k_p$ divisible by k, such that $x = \sum_{i=1}^p r_i w_i$ with $r_i > 0$ and $\sum r_i = 1$, $\|x - w_i\| < \varepsilon/k_i$ and $k_i w_i/k$ is integral for every i.*

I will need a refinement of this lemma when the smallest rational affine space containing a point is not necessarily of maximal dimension.

Lemma 2.28. *Let $x \in \mathbb{R}^n$, let $0 < \varepsilon, \eta \ll 1$ be rational numbers and let $w_1 \in \mathbb{Q}^n$ and $k_1 \in \mathbb{N}$ be such that $\|x - w_1\| < \varepsilon/k_1$ and $k_1 w_1$ is integral. Then there are $w_2, \ldots, w_m \in \mathbb{Q}^n$, positive integers $k_2, \ldots, k_m$ such that $\|x - w_i\| < \varepsilon/k_i$ and $k_i w_i$ is integral for every i, and positive numbers $r_1, \ldots, r_m$ such that $x = \sum_{i=1}^m r_i w_i$ and $\sum r_i = 1$. Furthermore, we can assume that $w_3, \ldots, w_m$ belong to the smallest rational affine space containing x, and we can write*

$$x = \frac{k_1}{k_1 + k_2} w_1 + \frac{k_2}{k_1 + k_2} w_2 + \xi,$$

with $\|\xi\| < \eta/(k_1 + k_2)$.

Proof. Let W be the minimal rational affine subspace containing x, let $\pi\colon \mathbb{R}^n \to \mathbb{R}^n/\mathbb{Z}^n$ be the quotient map and let G be the closure of the set $\pi(\mathbb{N}x + \mathbb{Z}^n)$. Then by Lemma 2.26 we have $\pi(-k_1 x) \in G$ and there is $k_2 \in \mathbb{N}$ such that $\pi(k_2 x)$ is in the connected component of $\pi(-k_1 x)$ in G and $\|k_2 x - y\| < \eta$ for some $y \in \mathbb{R}^n$ with $\pi(y) = \pi(-k_1 x)$. Thus there is a point $w_2 \in \mathbb{Q}^n$ such that $k_2 w_2 \in \mathbb{Z}^n$, $\|k_2 x - k_2 w_2\| < \varepsilon$ and the open segment (w_1, w_2) intersects W.

Pick $t \in (0, 1)$ such that $w_t = t w_1 + (1 - t) w_2 \in W$, and choose, by Lemma 2.27, rational points $w_3, \ldots, w_m \in W$ and positive integers $k_3, \ldots, k_m$ such that $k_i w_i \in \mathbb{Z}^n$, $\|x - w_i\| < \varepsilon/k_i$ and $x = \sum_{i=3}^m r_i w_i + r_t w_t$ with $r_t > 0$ and all $r_i > 0$, and $r_t + \sum_{i=3}^m r_i = 1$. Thus $x = \sum_{i=1}^m r_i w_i$ with $r_1 = t r_t$ and $r_2 = (1 - t) r_t$.

Finally, observe that the vector $y/k_2 - w_2$ is parallel to the vector $x - w_1$ and $\|y - k_2 w_2\| = \|k_1 x - k_1 w_1\|$. Denote $z = x - y/k_2$. Then

$$\frac{x - w_1}{(w_2 + z) - x} = \frac{x - w_1}{w_2 - y/k_2} = \frac{k_2}{k_1},$$

so

$$x = \frac{k_1}{k_1 + k_2} w_1 + \frac{k_2}{k_1 + k_2}(w_2 + z) = \frac{k_1}{k_1 + k_2} w_1 + \frac{k_2}{k_1 + k_2} w_2 + \xi,$$

where $\|\xi\| = \|k_2 z/(k_1 + k_2)\| < \eta/(k_1 + k_2)$. □

Remark 2.29. Assuming notation from the previous proof, the connected components of G are precisely the connected components of the set $\pi(\bigcup_{k>0} kW)$. Therefore $y/k_2 \in W$.

Remark 2.30. Assume $\lambda \colon V \to W$ is a linear map between vector spaces such that $\lambda(V_{\mathbb{Q}}) \subset W_{\mathbb{Q}}$. Let $x \in V$ and let $H \subset V$ be the smallest rational affine subspace containing x. Then $\lambda(H)$ is the smallest rational affine subspace of W containing $\lambda(x)$. Otherwise, assume $H' \neq \lambda(H)$ is the smallest rational affine subspace containing $\lambda(x)$. Then $\lambda^{-1}(H')$ is a rational affine subspace containing x and $H \not\subset \lambda^{-1}(H')$, a contradiction.

Definition 2.31. For a real number α set $\|\alpha\| := \min\{\alpha - \lfloor \alpha \rfloor, \lceil \alpha \rceil - \alpha\}$.

The following is a slightly weaker version of [Cas57, Chapter I, Theorem VII] which is sufficient for the purposes of this thesis. It can be viewed both as a strengthening and a weakening Lemma 2.26. On one hand, it gives an effective rational approximation of a point in $\mathbb{R}^n$ in terms of denominators of approximation points. On the other hand, it does not give uniformity of the distribution of approximations in the unit cube as in Lemma 2.26.

Theorem 2.32. *Let $\theta_1, \dots, \theta_n$ be real numbers. There are infinitely many positive integers q such that*

$$\max\{\|q\theta_1\|, \dots, \|q\theta_n\|\} < q^{-1/n}.$$

Chapter 3

Finite Generation in the MMP

In this chapter I review a part of the recent progress in settling some of the main conjectures in the Minimal Model Program. I concentrate on finite generation of the canonical ring, in particular on the techniques introduced by Shokurov, Hacon and M^{c}Kernan in order to resolve a special case related to pl flips. Pl flips are important because they give a good candidate for what to restrict our algebra to– it is the unique log canonical centre which is, possibly after shrinking, proportional to an adjoint bundle. This enables one to deal pretty quickly with the issue of finite generation of the kernel of the restriction map, and the focus shifts to finite generation of the image. Some of these methods will be employed in Chapter 6 in the general construction related to the finite generation. In particular, the central role is played by the extension theorem of Hacon and M^{c}Kernan.

3.1 Review of the Minimal Model Program

In this section I review the Minimal Model Program in the case of log canonical singularities. Some of the results have only been established recently in the work by Ambro and Fujino [Amb03, Fuj07b].

The base of the programme is the following fundamental theorem.

Theorem 3.1 (Cone and Contraction Theorem)**.** *Let (X, Δ) be a log canonical pair and let $\pi\colon X \to Z$ be a projective morphism. Denote $R_C = \mathbb{R}_+[C]$ for a rational curve C in X and its class $[C] \in \mathrm{NE}(X)$. Then:*

(1) *there are countably many rational curves* C_i *such that* $\pi(C_i)$ *is a point for every* i *and* $0 < -(K_X + \Delta) \cdot C_i < 2 \dim X$*, and*

$$\overline{\mathrm{NE}}(X/Z) = \overline{\mathrm{NE}}(X/Z)_{K_X+\Delta\geq 0} + \sum_i R_{C_i}.$$

Such R_C *are called* extremal rays*,*

(2) *for any* $\varepsilon > 0$ *and any* π*-ample* $\mathbb{R}$*-divisor* H *we have*

$$\overline{\mathrm{NE}}(X/Z) = \overline{\mathrm{NE}}(X/Z)_{K_X+\Delta+\varepsilon H\geq 0} + \sum_{\text{finite}} R_{C_i},$$

(3) *let* $R \subset \overline{\mathrm{NE}}(X/Z)$ *be a* $(K_X+\Delta)$*-negative extremal ray. Then there is a unique morphism* $\varphi_R \colon X \to Y$ *such that* φ_R *has connected fibres,* Y *is projective over* Z*,* $\rho(Y/Z) = \rho(X/Z) - 1$*, and an irreducible curve* $C \subset X$ *is mapped to a point by* φ_R *if and only if* $[C] \in R$*. Furthermore, if* L *is a line bundle on* X *such that* $L \in R^\perp$*, then there is a line bundle* L_Y *on* Y *such that* $L \simeq \varphi_R^* L_Y$*. The map* φ_R *is called the* contraction *of* R*.*

Remark 3.2. The estimate on the length of rays in Theorem 3.1(1) is obtained using the full force of the MMP for klt pairs [BCHM06]. It is conjectured that there is a sharper estimate $0 < -(K_X+\Delta)\cdot C_i < \dim X + 1$. I will not use these estimates here.

The following result is closely related to Contraction Theorem.

Theorem 3.3 (Basepoint Free Theorem)**.** *Assume* (X, Δ) *is a klt pair,* $f \colon X \to Z$ *a proper morphism and* D *an* f*-nef Cartier divisor such that* $dD - (K_X + \Delta)$ *is* f*-nef and* f*-big for some* $d > 0$*. Then* δD *is* f*-free for all* $\delta \gg 0$*.*

It is predicted that the outcome of the MMP should be the following:

Conjecture 3.4 (Hard Dichotomy)**.** *Let* (X, Δ) *be a log canonical pair.*

(1) *If* $\kappa(X, K_X + \Delta) \geq 0$*, then there is a birational map* $\varphi \colon X \dashrightarrow Y$ *such that* $K_Y + \varphi_*\Delta$ *is nef;* Y *is a* minimal model*.*

(2) If $\kappa(X, K_X + \Delta) = -\infty$, then there exist a birational map $\varphi\colon X \dashrightarrow Y$ and a surjective contraction $Y \to W$ of an $(K_Y + \varphi_*\Delta)$-negative extremal ray to a normal projective variety W with $\dim W < \dim Y$*; Y is a* Mori fibre space.

Let me mention here that if X is a Mori fibre space, then $\kappa(X) = -\infty$ and X is *uniruled*, i.e. there exists a generically finite map $Y \times \mathbb{P}^1 \dashrightarrow X$ with $\dim Y = \dim X - 1$ (or equivalently X is covered by rational curves), see [Mat02, Chapter 3]. The reverse implication is much harder to prove. The greatest contributions in that direction are [BDPP04], which proves that if X is smooth and K_X is not pseudoeffective, then X is uniruled, and [BCHM06], which proves that if $K_X + \Delta$ is klt and not pseudoeffective, then the MMP ends with a Mori fibre space.

Starting from Cone and Contraction Theorem, the standard recursive procedure for the MMP of log canonical pairs goes as in [KM98, 3.31], see Figure 3.1. The main obstacles to completing the programme are the following two conjectures.

Existence of Flips Conjecture. *Let (X, Δ) be a $\mathbb{Q}$-factorial log canonical pair and let $f\colon X \to Z$ be a* flipping contraction, *i.e. assume f is small and projective over Z, $-(K_X + \Delta)$ is an f-ample $\mathbb{R}$-divisor, and $\rho(X/Z) = 1$.*

Then there exists a small projective contraction $f^+\colon X^+ \to Z$ from a normal pair (X^+, Δ^+), called a flip *of f, such that there is a commutative diagram*

$$\begin{array}{ccc} X & \overset{g}{\dashrightarrow} & X^+ \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f^+} & \\ & Z & \end{array}$$

with $\Delta^+ = g_*\Delta$, where $K_{X^+} + \Delta^+$ is f^+-ample.

Remark 3.5. By [KMM87, Lemma 3-2-5], a map $f\colon X \to Z$ is a flipping contraction if and only if it is the contraction of an extremal ray, and f-numerical and f-linear equivalence coincide.

Termination of Flips Conjecture. *There does not exist an infinite sequence of flips in the flowchart in Figure 3.1.*

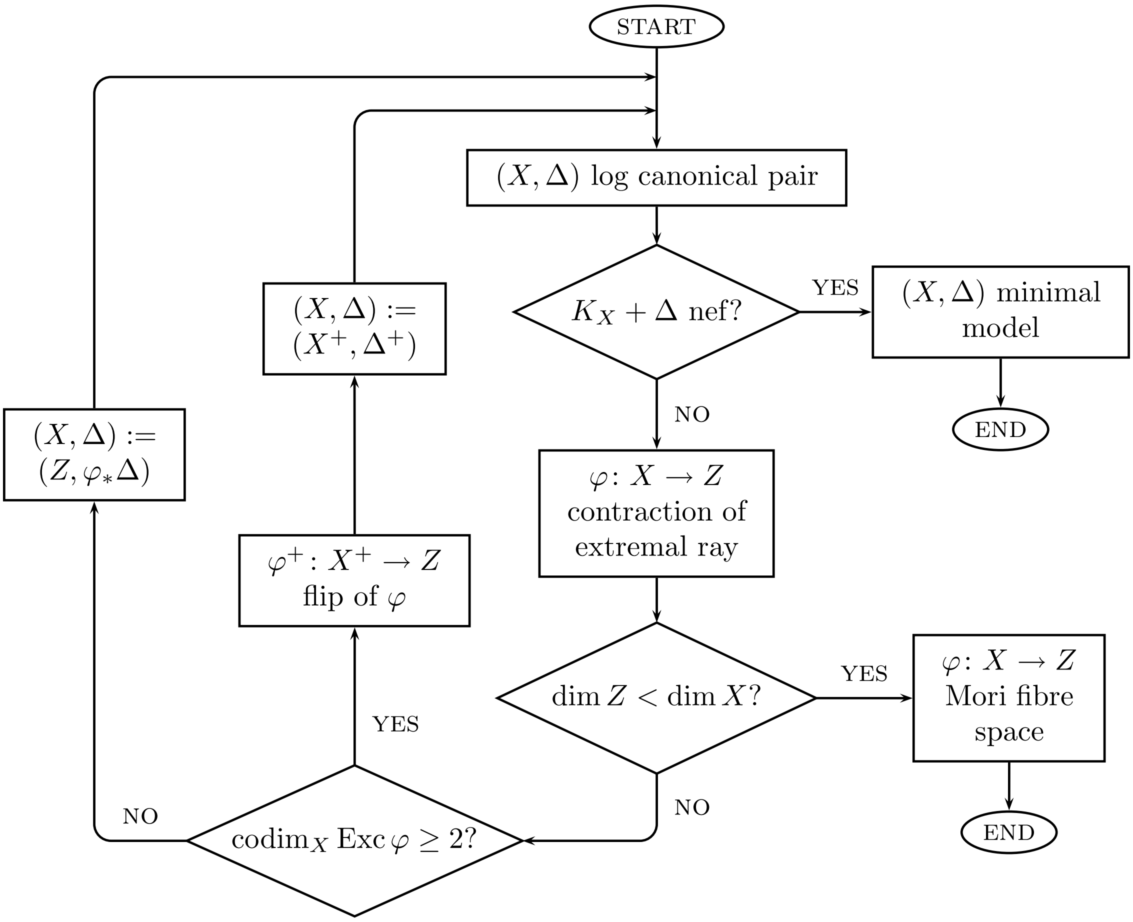


Figure 3.1: Flowchart of the Minimal Model Program

3.2 Finite Generation and Flips

The following result gives a connection between finite generation problems and birational geometry, see [KM98, Proposition 3.37] and [KMM87, Proposition 5-1-11].

Theorem 3.6. *Let (X,Δ) be a $\mathbb{Q}$-factorial log canonical pair over a variety W and let $f\colon X\to Z$ be a flipping contraction. The flip of f exists if and only if the relative canonical ring*

$$R(X/Z,K_X+\Delta)=\bigoplus_{n\geq 0} f_*\mathcal{O}_X(\lfloor n(K_X+\Delta)\rfloor)$$

is a finitely generated $\mathcal{O}_Z$-algebra. Moreover, in that case the flip is unique, $X^+=$

$\operatorname{Proj}_Z R(X/Z, K_X + \Delta)$, *$X^+$ is $\mathbb{Q}$-factorial and $\rho(X^+/W) = \rho(X/W)$.*

We can consider a weaker version of the Finite Generation Conjecture.

Conjecture 3.7. *Let (X, Δ) be a log smooth projective log canonical pair, where $K_X + \Delta$ is big. Then the canonical ring $R(X, K_X + \Delta)$ is a finitely generated $\mathbb{C}$-algebra.*

The following result shows that existence of flips is a consequence of the weaker Finite Generation Conjecture.

Lemma 3.8. *Let (X, Δ) be a $\mathbb{Q}$-factorial log canonical pair and let $f : X \to Z$ be a flipping contraction. Assume Conjecture 3.7. Then the flip of f exists.*

Proof. The proof is along the lines of [Fuj07b, Lemma 2.5]. Since the problem is local, we can assume Z is affine. By compactifying X and Z and by resolving singularities, we can further assume that X and Z are projective, X is smooth, and $\operatorname{Supp}\Delta$ is a simple normal crossing divisor. Let A be a general sufficiently ample divisor on Z and set $\Delta' = \Delta + f^*A$. Then $K_X + \Delta'$ is big, (X, Δ') is log canonical, and $K_X + \Delta$ and $K_X + \Delta'$ are negative on the same curves, so it is enough to prove the existence of the $(K_X + \Delta')$-flip. But this follows from Theorem 3.6 since the ring $R(X, K_X + \Delta')$ is finitely generated by Conjecture 3.7. □

In the case of klt singularities, it is enough to prove the weaker version of the finite generation in order to obtain Finite Generation Conjecture, see Lemma 3.11 below. First we recall the following notion. If $R = \bigoplus_{n\in\mathbb{N}} R_n$ is a graded algebra which is an integral domain and if d is a positive integer, the algebra $R^{(d)} = \bigoplus_{n\in\mathbb{N}} R_{dn}$ is called a *truncation* of R. The following basic result, which will be partially generalised in Lemma 5.5(1), says that we can freely pass to truncations when dealing with finite generation issues.

Lemma 3.9. *Let R be a graded algebra which is an integral domain and let d be a positive integer. Then R is finitely generated if and only if the truncation $R^{(d)}$ is finitely generated.*

Proof. Fix a positive integer d. It is easy to see that there is an action of the cyclic group $\mathbb{Z}_d$ on R such that $R^{(d)}$ is the ring of invariants of R under this action. Thus if R is finitely generated, so is $R^{(d)}$ according to E. Noether's theorem on the finite generation of the ring of invariants by a finite group.

Now assume $R^{(d)}$ is finitely generated. Each $f \in R$ is integral over $R^{(d)}$ since it is a zero of the polynomial $T^d - f^d \in R^{(d)}[T]$. Therefore R is finitely generated by the theorem of E. Noether on finiteness of integral closures. $\square$

The key result is the following theorem of Fujino and Mori [FM00, Theorem 5.2] which is a consequence of their Canonical Bundle Formula.

Theorem 3.10. *Let (X, Δ) be a proper klt pair with $\kappa(X, K_X + \Delta) = l \geq 0$. Then there exist a projective l-dimensional klt pair (X', Δ') with X' smooth and $\kappa(X', K_{X'} + \Delta') = l$, and positive integers d, d' such that*

$$R(X, K_X + \Delta)^{(d)} \simeq R(X', K_{X'} + \Delta')^{(d')}.$$

Now we can prove the result promised.

Lemma 3.11. *Let (X, Δ) be a projective klt pair and assume Conjecture 3.7. Then the canonical ring $R(X, K_X + \Delta)$ is finitely generated.*

Proof. We can assume $\kappa(X, K_X + \Delta) = l \geq 0$ since the finite generation is trivial otherwise. By Theorem 3.10, there exist a pair (X', Δ') with X' smooth and $K_{X'} + \Delta'$ big, and positive integers d, d' such that $R(X, K_X + \Delta)^{(d)} \simeq R(X', K_{X'} + \Delta')^{(d')}$. Let $f \colon Y \to X'$ be a log resolution of the pair (X', Δ'), and let $\Gamma = \mathbf{B}(X', \Delta')_Y$. Let k be a positive integer such that $k(K_{X'} + \Delta')$ is Cartier. Then $K_Y + \Gamma$ is big and $R(X', K_{X'} + \Delta')^{(k)} \simeq R(Y, K_Y + \Gamma)^{(k)}$, and therefore $R(X, K_X + \Delta)$ is finitely generated by Conjecture 3.7 and Lemma 3.9. $\square$

3.3 Pl Flips

In this section I concentrate on the method that was used to prove the existence of pl flips in [Sho03, HM07, HM08]. Much of the presentation is taken from [HM08].

The following theorem is [BCHM06, Theorem F], and it establishes finite generation of the canonical ring and certain asymptotic properties of adjoint divisors using the full force of the MMP. Granting it in dimension $n-1$, we will prove finite generation of certain algebras in dimension n.

Theorem 3.12. *Let $\pi\colon X \to Z$ be a projective morphism to a normal affine variety. Let $(X, \Delta = A + B)$ be a $\mathbb{Q}$-factorial klt pair of dimension n, where A is an ample $\mathbb{Q}$-divisor and $K_X + \Delta$ is pseudo-effective.*

(1) If $K_X + \Delta$ is $\mathbb{Q}$-Cartier, then the canonical ring $R(X, K_X + \Delta)$ is finitely generated.

(2) Let $V \subset \mathrm{WDiv}(X)_{\mathbb{R}}$ be the vector space spanned by the components of B. Then there is a constant $\delta > 0$ such that if $B' \in V$, where $K_X + A + B'$ is log canonical and $\|B' - B\| < \delta$, then $\mathbf{Fix}(K_X + \Delta) \subset \mathbf{Fix}(K_X + A + B')$.

(3) Let $W \subset V$ be the smallest rational affine subspace of $\mathrm{WDiv}(X)_{\mathbb{R}}$ containing B. Then there is a constant $\eta > 0$ and a positive integer $r > 0$ such that if $B' \in W$ is any divisor and k is any positive integer such that $\|B' - B\| < \eta$ and $k(K_X + A + B')/r$ is Cartier, then $\mathrm{Supp}(\mathrm{Fix}\,|k(K_X + A + B')|) \subset \mathbf{Fix}(K_X + \Delta)$.

Let us recall the definition of the main object of this section.

Definition 3.13. Let (X, Δ) be a $\mathbb{Q}$-factorial dlt pair and $f\colon (X, \Delta) \to Z$ a flipping contraction. We say f is a *pre limiting (pl) flipping contraction* if there is an f-negative irreducible component $S \subset \lfloor \Delta \rfloor$.

The following result of Shokurov is fundamental in order to apply finite generation techniques. For an accessible proof see [Fuj07a, Theorem 4.2.1].

Theorem 3.14. *Assume the MMP for $\mathbb{Q}$-factorial dlt pairs in dimension $n - 1$. If flips of pl flipping contractions exist in dimension n, then flips of klt flipping contractions exist in dimension n.*

Assumption on the MMP in dimension $n-1$ can be relaxed, and that is a route undertaken in [BCHM06] to complete the proof of finite generation of the canonical ring of klt pairs using a convoluted induction process heavily involving techniques of the MMP.

Therefore, we can concentrate on proving the existence of pl flips. If $f\colon (X, \Delta) \to Z$ is a pl flipping contraction, where $S \subset \lfloor \Delta \rfloor$ is an f-negative component, then for a small positive rational number ε the pair $(X, S + (1 - \varepsilon)(\Delta - S))$ is plt and $K_X + S + (1 - \varepsilon)(\Delta - S)$ is f-negative, so we can assume that (X, Δ) is plt and $\lfloor \Delta \rfloor$

is irreducible. Since the question of existence of flips is local, we can assume that the base is affine. In particular, we will prove

Theorem 3.15. *Assume Theorem 3.12 in dimension $n-1$. Let (X,Δ) be a plt pair of dimension n, where $S = \lfloor \Delta \rfloor$ is a prime divisor, and consider a pl flipping contraction $f\colon X \to Z$ with Z affine. Then the algebra $R(X/Z, K_X+\Delta)$ is finitely generated. In particular, the flip of f exists.*

The rest of this section is devoted to proving Theorem 3.15.

Remark 3.16. For a Cartier divisor D and a prime Cartier divisor S on a variety X, let $\sigma_S \in H^0(X,S)$ be a section such that $\operatorname{div}\sigma_S = S$. From the exact sequence

$$H^0(X,\mathcal{O}_X(D-S)) \xrightarrow{\cdot\sigma_S} H^0(X,\mathcal{O}_X(D)) \xrightarrow{\rho_{D,S}} H^0(S,\mathcal{O}_S(D))$$

we denote $\operatorname{res}_S H^0(X,\mathcal{O}_X(D)) = \operatorname{Im}(\rho_{D,S})$.

Definition 3.17. Let (X,Δ) be a plt pair of dimension n, where $S = \lfloor \Delta \rfloor$ is a prime divisor, and let $f\colon X \to Z$ be a projective morphism with Z affine. The *restricted algebra* of $R(X,K_X+\Delta)$ is

$$R_S(X,K_X+\Delta) = \bigoplus_{n\geq 0} \operatorname{res}_S H^0(X,\lfloor n(K_X+\Delta)\rfloor).$$

The idea from the proof of the following result will serve as a model in the proof given in Section 6.2.

Lemma 3.18. *Let (X,Δ) be a plt pair of dimension n, where $S = \lfloor \Delta \rfloor$ is a prime divisor, and let $f\colon X \to Z$ be a pl flipping contraction with Z affine. Then $R(X,K_X+\Delta)$ is finitely generated if and only if $R_S(X,K_X+\Delta)$ is finitely generated.*

Proof. We will concentrate on sufficiency, since necessity is obvious.

By Remark 3.5 numerical and linear equivalence over Z coincide. Since $\rho(X/Z) = 1$, and both S and $K_X+\Delta$ are f-negative, there exists a positive rational number r such that $S \sim_{\mathbb{Q},f} r(K_X+\Delta)$. By considering open subvarieties of Z we can assume that $S - r(K_X+\Delta)$ is $\mathbb{Q}$-linearly equivalent to a pullback of a principal divisor.

Therefore $S \sim_{\mathbb{Q}} r(K_X+\Delta)$, and since then $R(X,S)$ and $R(X,K_X+\Delta)$ have isomorphic truncations, it is enough to prove that $R(X,S)$ is finitely generated by

Lemma 3.9. Since a truncation of $\operatorname{res}_S R(X,S)$ is isomorphic to a truncation of $R_S(X, K_X+\Delta)$, we have that $\operatorname{res}_S R(X,S)$ is finitely generated. If $\sigma_S \in H^0(X,S)$ is a section such that $\operatorname{div}\sigma_S = S$ and $\mathcal{H}$ is a finite set of generators of the finite dimensional vector space $\bigoplus_{i=1}^d \operatorname{res}_S H^0(X,iS)$, for some d, such that the set $\{s_{|S} : s \in \mathcal{H}\}$ generates $\operatorname{res}_S R(X,S)$, it is easy to see that $\mathcal{H}\cup\{\sigma_S\}$ is a set of generators of $R(X,S)$, since $\ker(\rho_{kS,S}) = H^0(X,(k-1)S)\cdot\sigma_S$ for all k, in the notation of Remark 3.16. □

The following is the Hacon-McKernan extension theorem [HM08, Theorem 6.3].

Theorem 3.19. *Let $\pi\colon X \to Z$ be a projective morphism to an affine variety Z, where $(X, \Delta = S+A+B)$ is a plt pair, $S = \lfloor\Delta\rfloor$ is irreducible, (X,S) is log smooth, A is a general ample $\mathbb{Q}$-divisor and $(S, \Omega + A_{|S})$ is canonical, where $\Omega = (\Delta - S)_{|S}$. Assume $S \not\subset \mathbf{B}(K_X+\Delta)$, and let*

$$F = \liminf_{m\to\infty} \tfrac{1}{m} \operatorname{Fix}|m(K_X+\Delta)|_S.$$

If $\varepsilon > 0$ is any rational number such that $\varepsilon(K_X+\Delta)+A$ is ample, and if Φ is any $\mathbb{Q}$-divisor on S and $k > 0$ is any integer such that both $k\Delta$ and $k\Phi$ are Cartier and $\Omega \wedge (1-\frac{\varepsilon}{k})F \le \Phi \le \Omega$, then

$$|k(K_S+\Omega-\Phi)| + k\Phi \subset |k(K_X+\Delta)|_S.$$

The next result is a crucial application of the extension of sections, and it will also be used in Chapter 6.

Theorem 3.20. *Assume Theorem 3.12 in dimension $n-1$. Let $\pi\colon X \to Z$ be a projective morphism to an affine variety, and let $(X, \Delta = S+A+B)$ be a plt pair of dimension n, $S = \lfloor\Delta\rfloor$ is irreducible, (X,S) is log smooth, A is a general ample $\mathbb{Q}$-divisor, B is a $\mathbb{Q}$-divisor and $(S,\Omega + A_{|S})$ is canonical, where $\Omega = (\Delta - S)_{|S}$. Assume $S \not\subset \mathbf{B}(K_X+\Delta)$, and let*

$$F = \liminf_{m\to\infty} \tfrac{1}{m} \operatorname{Fix}|m(K_X+\Delta)|_S.$$

Then $\Theta = \Omega - \Omega \wedge F$ *is rational. In particular, if both* $k\Delta$ *and* $k\Theta$ *are Cartier then*

$$|k(K_S + \Theta)| + k(\Omega - \Theta) = |k(K_X + \Delta)|_S,$$

and

$$R_S(X, K_X + \Delta)^{(k)} \simeq R(S, K_S + \Theta)^{(k)}.$$

Proof. Suppose that Θ is not rational and let $V \subset \mathrm{Div}(S)_{\mathbb{R}}$ be the vector space spanned by the components of $\Theta - A_{|S}$. Then there is a constant $\delta > 0$ such that if $\Phi \in V$ is effective and $\|\Phi + A_{|S} - \Theta\| < \delta$, then $\Phi + A_{|S}$ has the same support as Θ and $\mathbf{Fix}(K_S + \Theta) \subset \mathbf{Fix}(K_S + \Phi + A_{|S})$ by Theorem 3.12(2). Pick $l \gg 0$ so that $l(K_X + \Delta)$ is Cartier, $\Theta_l = \Omega - \Omega \wedge F_l \in V$ and $\|\Theta_l - \Theta\| < \delta$. Then

$$|l(K_X + \Delta)|_S \subset |l(K_S + \Theta_l)| + l(\Omega \wedge F_l),$$

hence $\mathrm{Fix}\,|l(K_S + \Theta_l)|$ does not contain any components of Θ_l. It follows that no component of Θ is in $\mathbf{B}(K_S + \Theta)$.

Let $W \subset V$ be the smallest rational affine space which contains $\Theta - A_{|S}$. By Theorem 3.12(3), take a positive integer $r > 0$ and a positive constant $\eta > 0$ such that if $\Phi \in W$, $k(\Phi + A_{|S})/r$ is Cartier and $\|\Phi + A_{|S} - \Theta\| < \eta$, then $\mathrm{Supp}(\mathrm{Fix}\,|k(K_S + \Phi + A_{|S})|) \subset \mathbf{Fix}(K_S + \Theta)$.

Pick a rational number $\varepsilon > 0$ such that $\varepsilon(K_X + \Delta) + A$ is ample, and let f be the smallest non-zero coefficient of F. By Lemma 2.26, we may find an effective divisor $\Phi \in W$, a prime divisor G on S and a positive integer k such that both $k(\Phi + A_{|S})/r$ and $k\Delta/r$ are Cartier, $\|\Phi + A_{|S} - \Theta\| < \min(\delta, \eta, f\varepsilon/k)$, and $\mathrm{mult}_G \Phi > \mathrm{mult}_G \Theta$. Then it is easy to check that $\Omega \wedge (1 - \frac{\varepsilon}{k})F \le \Omega - \Phi$, so Theorem 3.19 implies that

$$|k(K_S + \Phi)| + k(\Omega - \Phi) \subset |k(K_X + \Delta)|_S.$$

Since $\mathrm{mult}_G \Phi > \mathrm{mult}_G \Theta$, we have that G is a component of $\mathrm{Fix}\,|k(K_S + \Phi)|$, and therefore a component of $\mathbf{Fix}(K_S + \Theta)$ because $\|\Phi - \Theta\| < \eta$, a contradiction.

Thus Θ is rational, and we are done by Theorem 3.19. □

Corollary 3.21. *Assume Theorem 3.12 in dimension* $n - 1$*. Let* $\pi\colon X \to Z$ *be a projective morphism to an affine variety* Z*, where* $(X, \Delta = S + A + B)$ *is a plt pair* *of dimension n, $S = \lfloor \Delta \rfloor$ is irreducible with $S \not\subset \mathbf{B}(K_X + \Delta)$, (X, S) is log smooth, A is a general ample $\mathbb{Q}$-divisor and B is a $\mathbb{Q}$-divisor.*

Then there exist a birational morphism $g\colon T \to S$, a positive integer l and a klt pair (T, Θ) such that $K_T + \Theta$ is $\mathbb{Q}$-Cartier and

$$R_S(X, K_X + \Delta)^{(l)} \simeq R(T, K_T + \Theta)^{(l)}.$$

Proof. By Lemma 2.7, there is a log resolution $f\colon Y \to X$ such that the components of $\Gamma' - T$ are disjoint, where $T = f_*^{-1}S$ and $\Gamma' = \mathbf{B}(X, \Delta)_Y$. In particular, the pair $(T, (\Gamma' - T)_{|T})$ is terminal and note that $T \not\subset \mathbf{B}(K_Y + \Gamma')$ as $S \not\subset \mathbf{B}(K_X + \Delta)$.

Since A is general, we have $f^*A = f_*^{-1}A$. Let H be a small effective f-exceptional $\mathbb{Q}$-divisor such that $f^*A - H$ is ample and $(Y, \Gamma' + H)$ is plt. Let $C \sim_{\mathbb{Q}} f^*A - H$ be a general ample divisor, and set $\Gamma = \Gamma' - f^*A + H + C$ and $\Psi = (\Gamma - T)_{|T}$. Observe that $\Gamma \geq 0$, $\Gamma \sim_{\mathbb{Q}} \Gamma'$ and the pair (Y, Γ) is plt. Then for any k sufficiently divisible we have $R(X, K_X + \Delta)^{(k)} \simeq R(Y, K_Y + \Gamma)^{(k)}$ and $R_S(X, K_X + \Delta)^{(k)} \simeq R_T(Y, K_Y + \Gamma)^{(k)}$. Since $(T, \Psi + C_{|T})$ is terminal, we can apply Theorem 3.20 to (Y, Γ). □

Finally we have:

Proof of Theorem 3.15. We may assume that Z is affine and it suffices to prove that the restricted algebra is finitely generated by Lemma 3.18. Since S is mobile and $\Delta - S$ is big over Z, we can write $\Delta - S \sim_{\mathbb{Q}} A + B$, where A is a general ample $\mathbb{Q}$-divisor, $B \geq 0$ and $S \not\subset \operatorname{Supp} B$. Set $\Delta' = S + (1 - \varepsilon)(\Delta - S) + \varepsilon A + \varepsilon B$ for a sufficiently small positive rational number ε. Then the pair (X, Δ') is plt and $K_X + \Delta' \sim_{\mathbb{Q}} K_X + \Delta$, so we may replace Δ by Δ' by Lemma 3.9. Therefore we can assume that $\Delta = S + A + B$, where A is a general ample $\mathbb{Q}$-divisor and $B \geq 0$, and the result follows from Corollary 3.21 and Theorem 3.12. □

Chapter 4

Convex Geometry

In this chapter I build techniques in order to prove that superlinear functions satisfying suitable conditions are piecewise linear. I exhibit general properties of such maps, concentrating on the central role of Lipschitz continuity. The results obtained below will be used in Chapters 5 and 6. I use without explicit mention basic properties of closed cones, see [Deb01, Section 6.3].

4.1 Functions on Monoids and Cones

Firstly we recall a definition.

Definition 4.1. Let $\mathcal{C}$ be a cone in $\mathbb{R}^n$ and let $\|\cdot\|$ be any norm on $\mathbb{R}^n$. A function $f\colon \mathcal{C} \to \mathbb{R}$ is *locally Lipschitz* if for every point $x \in \operatorname{int}\mathcal{C}$ there are a closed ball $B_x \subset \mathcal{C}$ centred at x and a constant λ_x such that $|f(y) - f(z)| \leq \lambda_x \|y - z\|$ for all $y, z \in B_x$.

Every locally Lipschitz function is continuous on $\operatorname{int}\mathcal{C}$. Therefore if a function is locally Lipschitz, we say it is *locally Lipschitz continuous*. The next result can be found in [HUL93].

Proposition 4.2. *Let $\mathcal{C}$ be a cone in $\mathbb{R}^n$ and let $f\colon \mathcal{C} \to \mathbb{R}$ be a concave function. Then f is locally Lipschitz continuous on the topological interior of $\mathcal{C}$ with respect to any norm $\|\cdot\|$ on $\mathbb{R}^n$.*

In particular, let $\mathcal{C}$ be a rational polyhedral cone and assume a function $g\colon \mathcal{C}_{\mathbb{Q}} \to \mathbb{Q}$ is $\mathbb{Q}$-superadditive. Then g extends to a unique superlinear function on $\mathcal{C}$.

Proof. Since f is locally Lipschitz if and only if $-f$ is locally Lipschitz, we can assume f is convex. Fix $x = (x_1, \dots, x_n) \in \operatorname{int} \mathcal{C}$, and let $\Delta = \{(y_1, \dots, y_n) \in \mathbb{R}^n_+ : \sum y_i \le 1\}$. It is easy to check that translations of the domain do not affect the result, so we may assume $x \in \operatorname{int} \Delta \subset \operatorname{int} \mathcal{C}$.

Firstly let us prove that f is bounded above on Δ. Let $\{e_i\}$ be the standard basis in $\mathbb{R}^n$, $y = (y_1, \dots, y_n) \in \Delta$ and let $y_0 = 1 - \sum y_i \ge 0$. Then

$$f(y) = f\Big(\sum y_i e_i + y_0 \cdot 0\Big) \le \sum y_i f(e_i) + y_0 f(0)$$
$$\le \max\{f(0), f(e_1), \dots, f(e_n)\} =: M.$$

For each $\gamma > 0$ denote $B_x(\gamma) = \{z \in \mathbb{R}^n : \|z - x\| \le \gamma\}$. Choose δ such that $B_x(2\delta) \subset \operatorname{int} \Delta$. Again by translating the domain and composing f with a linear function we may assume that $x = 0$ and $f(0) = 0$. Then for all $y \in B_0(2\delta)$ we have

$$-f(y) = -f(y) + 2f(0) \le -f(y) + f(y) + f(-y) = f(-y) \le M,$$

so $|f| \le M$ on $B_0(2\delta)$.

Fix $u, v \in B_0(\delta)$. Set $\alpha = \|v - u\|/\delta$ and $w = v + \alpha^{-1}(v - u) \in B_0(2\delta)$ so that $v = \frac{\alpha}{\alpha+1} w + \frac{1}{\alpha+1} u$. Then convexity of f gives

$$\begin{aligned} f(v) - f(u) &\le \frac{\alpha}{\alpha+1} f(w) + \frac{1}{\alpha+1} f(u) - f(u) \\ &= \frac{\alpha}{\alpha+1}\big(f(w) - f(u)\big) \le 2M\alpha = \frac{2M}{\delta}\|v - u\|. \end{aligned}$$

Similarly $f(u) - f(v) \le 2M\|u - v\|/\delta$, giving

$$|f(v) - f(u)| \le L\|v - u\|$$

for all $u, v \in B_0(\delta)$ and $L = 2M/\delta$.

For the second claim, it is enough to apply the proof of the first part of the lemma with respect to the sup-norm $\|\cdot\|_\infty$; observe that $\|\cdot\|_\infty$ takes values in $\mathbb{Q}$ on $\mathcal{C}_\mathbb{Q}$. Applied to the interior of $\mathcal{C}$ and to the relative interiors of the faces of $\mathcal{C}$ shows g is locally Lipschitz, and therefore extends to a unique superlinear function on the whole $\mathcal{C}$. □

I will use the following result, classically referred to as Gordan's lemma, often without explicit mention.

Lemma 4.3. *Let $\mathcal{S} \subset \mathbb{N}^r$ be a finitely generated monoid and let $\mathcal{C} \subset \mathbb{R}^r$ be a rational polyhedral cone. Then the monoid $\mathcal{S} \cap \mathcal{C}$ is finitely generated.*

Proof. Assume first that $\dim \mathcal{C} = r$. Let $\ell_1, \ldots, \ell_m$ be linear functions on $\mathbb{R}^r$ with integral coefficients such that $\mathcal{C} = \bigcap_{i=1}^m \{z \in \mathbb{R}^r : \ell_i(z) \geq 0\}$ and define $\mathcal{S}_0 = \mathcal{S}$ and $\mathcal{S}_i = \mathcal{S}_{i-1} \cap \{z \in \mathbb{R}^r : \ell_i(z) \geq 0\}$ for $i = 1, \ldots, m$; observe that $\mathcal{S} \cap \mathcal{C} = \mathcal{S}_m$. Assuming by induction that $\mathcal{S}_{i-1}$ is finitely generated, by [Swa92, Theorem 4.4] we have that $\mathcal{S}_i$ is finitely generated.

Now assume $\dim \mathcal{C} < r$ and let $\mathcal{H}$ be a rational hyperplane containing $\mathcal{C}$. Let ℓ be the linear function with rational coefficients such that $\mathcal{H} = \ker(\ell)$. From the first part of the proof applied to the functions ℓ and $-\ell$ we have that the monoid $\mathcal{S} \cap \mathcal{H}$ is finitely generated. Now we proceed by descending induction on r. □

The following simple lemmas will turn out to be indispensable and they show that in the context of our assumptions it is enough to check additivity (respectively linearity) of the map at one point only.

Lemma 4.4. *Let $\mathcal{S} = \sum_{i=1}^n \mathbb{N} e_i$ be a monoid and let $f : \mathcal{S} \to G$ be a superadditive map to a monoid G. Assume that there is a point $s_0 = \sum s_i e_i \in \mathcal{S}$ with all $s_i > 0$ such that $f(s_0) = \sum s_i f(e_i)$ and that $f(\kappa s_0) = \kappa f(s_0)$ for every positive integer κ. Then the map f is additive.*

Proof. For $p = \sum p_i e_i \in \mathcal{S}$, let κ_0 be a big enough positive integer such that $\kappa_0 s_i \geq p_i$ for all i. Then we have

$$\begin{aligned}\sum_{i=1}^n \kappa_0 s_i f(e_i) = \kappa_0 f(s_0) = f(\kappa_0 s_0) &\geq f(p) + \sum_{i=1}^n f\big((\kappa_0 s_i - p_i)e_i\big) \\ &\geq \sum_{i=1}^n f(p_i e_i) + \sum_{i=1}^n f\big((\kappa_0 s_i - p_i)e_i\big) \\ &\geq \sum_{i=1}^n p_i f(e_i) + \sum_{i=1}^n (\kappa_0 s_i - p_i) f(e_i) = \sum_{i=1}^n \kappa_0 s_i f(e_i).\end{aligned}$$

Therefore all inequalities are equalities and $f(p) = \sum p_i f(e_i)$. □

Analogously we can prove a continuous counterpart of the previous result.

Lemma 4.5. *Let $\mathcal{C} = \sum_{i=1}^{n} \mathbb{R}_+ e_i$ be a cone in $\mathbb{R}^r$ and let $f\colon \mathcal{C} \to V$ be a superlinear map to a cone V. Assume that there is a point $s_0 = \sum s_i e_i \in \mathcal{C}$ with all $s_i > 0$ such that $f(s_0) = \sum s_i f(e_i)$. Then the map f is linear.*

4.2 Forcing Diophantine Approximation

In this section I will prove the following.

Theorem 4.6. *Let $\mathcal{S} \subset \mathbb{N}^r$ be a finitely generated monoid and let $f\colon \mathcal{S}_\mathbb{R} \to \mathbb{R}$ be a superlinear map. Assume that there is a real number $c > 0$ such that for every $s_1, s_2 \in \mathcal{S}$, either $f(s_1 + s_2) = f(s_1) + f(s_2)$ or $f(s_1 + s_2) \geq f(s_1) + f(s_2) + c$. Let $\mathcal{C}$ be a rational polyhedral cone in $\operatorname{int} \mathcal{S}_\mathbb{R}$. Then $f_{|\mathcal{C}}$ is rationally piecewise linear.*

Corollary 4.7. *Let $\mathcal{S} \subset \mathbb{N}^r$ be a finitely generated monoid and let $f\colon \mathcal{S}_\mathbb{R} \to \mathbb{R}$ be a superlinear map such that $f(\mathcal{S}) \subset \mathbb{Z}$. Let $\mathcal{C}$ be a rational polyhedral cone in $\operatorname{int} \mathcal{S}_\mathbb{R}$. Then $f_{|\mathcal{C}}$ is rationally piecewise linear.*

Remark 4.8. In Theorem 4.6 and Corollary 4.7, instead of $\mathcal{S} \subset \mathbb{N}^r$ we can assume that $\mathcal{S} \subset \mathbb{Q}^r$ and that $\mathcal{S}_\mathbb{R}$ is strongly convex.

Example 4.9. The condition $f(\mathcal{S}) \subset \mathbb{Z}$ in Corollary 4.7 is crucial. Let $\mathcal{S} = \mathbb{N}(0,1) + \mathbb{N}(1,2) \subset \mathbb{R}^2$ and let $f\colon [0,1] \to \mathbb{R}$ be a function given by $f(x) = -x^2 + 2x + 1$. Let $x_n = (\frac{1}{n}, f(\frac{1}{n}))$ for positive integers n, set $\mathcal{C}_n = \mathbb{R}_+ x_n + \mathbb{R}_+ x_{n+1}$ and define $g(\alpha x_n + \beta x_{n+1}) = \alpha f(\frac{1}{n}) + \beta f(\frac{1}{n+1})$ for $\alpha, \beta \geq 0$. We obviously have $g(\mathcal{S}) \subset \mathbb{Q}$ and that g is superlinear and continuous, but it is not PL on the cone $\mathcal{S}_\mathbb{R}$.

The first step in the proof of Theorem 4.6 is the following.

Lemma 4.10. *Let $\mathcal{S} = \mathbb{N}^{r+1}$ and let $f\colon \mathcal{S}_\mathbb{R} \to \mathbb{R}$ be a superlinear map. Assume that there is a real number $c > 0$ such that for every $s_1, s_2 \in \mathcal{S}$, either $f(s_1 + s_2) = f(s_1) + f(s_2)$ or $f(s_1 + s_2) \geq f(s_1) + f(s_2) + c$. Let $x = (1, x_1, \ldots, x_r) \in \operatorname{int} \mathcal{S}_\mathbb{R}$ and let R be a ray in $\mathcal{S}_\mathbb{R}$ not containing x.*

Then there exists a ray $R' \subset \mathbb{R}_+ x + R$ not containing x such that the map $f|_{\mathbb{R}_+ x + R'}$ is linear.

Proof. By induction, I assume Theorem 4.6 when $\dim \mathcal{S}_\mathbb{R} = r$.

The proof consists of three parts. In Steps 2-8 I assume the components of x are linearly independent over $\mathbb{Q}$. In Step 9 I assume that x is a rational point while the remaining case when x is a non-rational point which belongs to a rational hyperplane is settled in Step 10.

Step 1: Let H be any 2-plane not contained in a rational hyperplane. Points of the form $(1, z_1, \dots, z_r)$, where $1, z_1, \dots, z_r$ are linearly independent over $\mathbb{Q}$, are dense on the line $L = H \cap (z_0 = 1)$. Otherwise there would exist an open neighbourhood U on L such that for each point $z \in U$ there is a rational hyperplane $H_z \supset \mathbb{R}z$. But the set of rational hyperplanes is countable.

On the other hand, fix a rational point $t \in \mathbb{R}^{r+1} \backslash H$ and observe rational hyperplanes containing $\mathbb{R}_+ t$. I claim that the set of points which are intersections of those hyperplanes and the line L are dense on L. To see this, let $y = (1, y_1, \dots, y_r)$ be a point in H and let $\mathcal{A} = (\alpha_0 z_0 + \cdots + \alpha_r z_r = 0)$ be any hyperplane containing y and t. H is given as a solution of a system of $r-1$ linear equations in $z_0, \dots, z_r$, thus y is a solution of a system of r linear equations and the components of y are linear functions in $\alpha_0, \dots, \alpha_r$, where α_i are linearly dependent over $\mathbb{Q}$ (since $t \in \mathcal{A}$). Therefore, without loss of generality, wiggling α_i for $i < r$ we can obtain a point $y' \in L$ arbitrarily close to y which belongs to a rational hyperplane $\mathcal{A}' = (\alpha'_0 z_0 + \cdots + \alpha'_r z_r = 0)$ containing t. Furthermore, if H contains a rational point t_0, then y' cannot belong to a rational plane $\widetilde{\mathcal{A}}$ of dimension $< n-1$ since otherwise H would be contained in a rational hyperplane generated by $\widetilde{\mathcal{A}}$ and t_0.

Step 2: In Steps 2-8 I assume that the real numbers $1, x_1, \dots, x_r$ are linearly independent over $\mathbb{Q}$.

By Theorem 2.32, there are infinitely many positive integers q such that

$$\|q x_i\| < q^{-1/r} \tag{4.1}$$

for all i. Fix such a q big enough so that the ball of radius $1/q$ centred at x is contained in $\operatorname{int} \mathcal{S}_\mathbb{R}$ and so that $q^{1/r} > r$, and in particular $\sum \|q x_i\| < 1$. Let p_i be positive integers with $|q x_i - p_i| < q^{-1/r}$.

Let

$$\widehat{p}_i = \begin{cases} \lfloor qx_i \rfloor & \text{if } p_i = \lceil qx_i \rceil \\ \lceil qx_i \rceil & \text{if } p_i = \lfloor qx_i \rfloor. \end{cases}$$

Let $e_0, e_1, \dots, e_r$ be the standard basis of $\mathbb{R}^{r+1}$. Set

$$u_0 = qe_0 + \sum p_i e_i \quad \text{and} \quad u_i = qe_0 + \sum\nolimits_{j \neq i} p_j e_j + \widehat{p}_i e_i$$

for $i = 1, \dots, r$. From (4.1) we have

$$\|x - u_0/q\|_\infty < q^{-1-1/r}. \tag{4.2}$$

It is easy to see that $u_0, \dots, u_r$ are linearly independent and that

$$\Big(1 - \sum \|qx_i\|\Big) u_0 + \sum \|qx_i\| u_i = qx. \tag{4.3}$$

Assume that for every open cone U containing x the map $f_{|U}$ *is not linear*. Then in Steps 3-7 I will prove that for all $q \gg 0$ satisfying (4.1) we have

$$f(x) = \Big(1 - \sum \|qx_i\|\Big) f(u_0/q) + \sum \|qx_i\| f(u_i/q) + e_q, \tag{4.4}$$

where $e_q \geq c(1 - \sum \|qx_i\|)/q$. I will then derive a contradiction in Step 8.

Step 3: Let $\mathcal{K} = \sum_{i \geq 0} \mathbb{R}_+ u_i$ and $\mathcal{K}_i = \mathbb{R}_+ x + \sum_{j \neq i} \mathbb{R}_+ u_j$ for $i = 0, \dots, r$; observe that $\mathcal{K} = \bigcup_{i \geq 0} \mathcal{K}_i$. Define the sequences $v_n \in \mathbb{N}^{r+1}$ and $j_n \in \mathbb{N}$ as follows: set $v_0 = \sum_{i \geq 0} u_i$. If v_n is defined then, since the components of x are linearly independent over $\mathbb{Q}$, there is a unique $j_n \in \{0, \dots, r\}$ such that v_n belongs to the *interior* of $\mathcal{K}_{j_n}$. Set $v_{n+1} = v_n + u_{j_n}$. Define the sequence of non-negative real numbers e_n by

$$f(v_{n+1}) = f(v_n) + f(u_{j_n}) + e_n.$$

Step 4: In this step I assume that for all $n \geq n_0$ with $j_n = 0$ we have $e_n \geq c$. Then we have

$$f(v_n) = \sum_{i=0}^{r} \alpha_i^{(n)} f(u_i) + e^{(n)}, \tag{4.5}$$

where $\alpha_i^{(n)} \in \mathbb{N}$ and $e^{(n)} \geq c(\alpha_0^{(n)} - n_0)$. Observe that $v_n = \sum_{i=0}^r \alpha_i^{(n)} u_i$, and therefore from Lemma 4.11 we have

$$qx = \lim_{n\to\infty} \frac{v_n}{n} = \sum_{i=0}^r \lim_{n\to\infty} \frac{\alpha_i^{(n)}}{n} u_i.$$

Since u_i are linearly independent, from (4.3) we obtain

$$\lim_{n\to\infty} \frac{\alpha_0^{(n)}}{n} = 1 - \sum \|qx_i\|$$

and

$$\lim_{n\to\infty} \frac{\alpha_i^{(n)}}{n} = \|qx_i\| \quad \text{for} \quad i > 0.$$

Dividing (4.5) by n, taking a limit when $n \to \infty$ and using continuity of f and Lemma 4.11 we obtain

$$f(qx) = \Big(1 - \sum \|qx_i\|\Big) f(u_0) + \sum \|qx_i\| f(u_i) + \widehat{e}_q,$$

where $\widehat{e}_q \geq c(1 - \sum \|qx_i\|)$. Dividing now by q we get (4.4).

Step 5: In Steps 5-7 I assume there are infinitely many n with $j_n = 0$ and $e_n = 0$. Then by Lemma 4.5 the map $f|_{\mathbb{R}_+ v_n + \mathbb{R}_+ u_0}$ is linear for each such n (observe that when $r = 1$ this finishes the proof since then $x \in \operatorname{int}(\mathbb{R}_+ v_n + \mathbb{R}_+ u_0)$). But then we have

$$f(v_n/n + u_0) = f(v_n/n) + f(u_0),$$

so letting $n \to \infty$ and using Lemma 4.11 we get

$$f(qx + u_0) = f(qx) + f(u_0),$$

thus the map $f|_{\mathbb{R}_+ x + \mathbb{R}_+ u_0}$ is linear by Lemma 4.5.

Let us first prove that there is an $(r+1)$-dimensional polyhedral cone $\mathcal{C}_{r+1}$ such that $\mathbb{R}_+ x + \mathbb{R}_+ u_0 \subset \mathcal{C}_{r+1}$, $(\mathbb{R}_+ x + \mathbb{R}_+ u_0) \cap \operatorname{int} \mathcal{C}_{r+1} \neq \emptyset$ and $f_{|\mathcal{C}_{r+1}}$ is linear. Let $t \in \operatorname{int} \mathcal{S}_{\mathbb{R}} \backslash (\mathbb{R}x + \mathbb{R}u_0)$ be a rational point. By Step 1 there is a rational hyperplane

$\mathcal{H} \ni t$ such that there is a nonzero $w \in \mathcal{H} \cap \operatorname{relint}(\mathbb{R}_+x + \mathbb{R}_+u_0)$, and there does not exist a rational plane of dimension $< n-1$ containing w. By Theorem 4.6 applied to $\mathcal{H} \cap \mathcal{S}_\mathbb{R}$ there is an r-dimensional cone $\mathcal{C}_r = \sum_{i=1}^r \mathbb{R}_+h_i \subset \mathcal{H} \cap \mathcal{S}_\mathbb{R}$ such that $w \in \operatorname{relint} \mathcal{C}_r$ and $f_{|\mathcal{C}_r}$ is linear. Set $\mathcal{C}_{r+1} = \mathcal{C}_r + \mathbb{R}_+x + \mathbb{R}_+u_0$. Now if $w = \sum \mu_i h_i$ with all $\mu_i > 0$, since f is linear on $\mathcal{C}_r$ we have

$$\begin{aligned} f\Big(x + u_0 + \sum \mu_i h_i\Big) &= f(x + u_0 + w) \\ &= f(x) + f(u_0) + f(w) = f(x) + f(u_0) + \sum \mu_i f(h_i), \end{aligned}$$

so the map $f_{|\mathcal{C}_{r+1}}$ is linear by Lemma 4.5.

Step 6: Let $\mathcal{C} = \mathbb{R}_+g_1 + \cdots + \mathbb{R}_+g_m$ be any $(r+1)$-dimensional polyhedral cone containing x such that $f_{|\mathcal{C}}$ is linear and let ℓ be the linear extension of $f_{|\mathcal{C}}$ to $\mathbb{R}^{r+1}$. Assume that for a point $h \in \mathcal{S}_\mathbb{R}$ we have $f|_{\mathbb{R}_+h} = \ell|_{\mathbb{R}_+h}$. There are real numbers λ_i such that

$$h = \sum\nolimits_i \lambda_i g_i.$$

Then setting $e := \sum(1 + |\lambda_i|)g_i + h = \sum(1 + |\lambda_i| + \lambda_i)g_i \in \mathcal{C}$ we have

$$\begin{aligned} f(e) = \ell\Big(\sum(1 + |\lambda_i| + \lambda_i)g_i\Big) &= \sum(1 + |\lambda_i| + \lambda_i)\ell(g_i) \\ = \sum(1 + |\lambda_i|)\ell(g_i) + \ell(h) &= \sum(1 + |\lambda_i|)f(g_i) + f(h), \end{aligned}$$

so f is linear on the cone $\mathcal{C} + \mathbb{R}_+h$ by Lemma 4.5. Therefore the set $\widehat{\mathcal{C}} = \{z \in \mathcal{S}_\mathbb{R} : f(z) = \ell(z)\}$ is an $(r+1)$-dimensional closed cone.

Step 7: Since f is not linear in any open neighbourhood of x we have $x \notin \operatorname{int} \widehat{\mathcal{C}}$. Therefore there is a tangent hyperplane T to $\widehat{\mathcal{C}}$ containing x. Let W_1 and W_2 be the half-spaces such that $W_1 \cap W_2 = T$, $W_1 \cup W_2 = \mathbb{R}^{r+1}$ and $\widehat{\mathcal{C}} \subset W_1$. Since $(\mathbb{R}_+x + \mathbb{R}_+u_0) \cap \operatorname{int} \widehat{\mathcal{C}} \neq \emptyset$ we must have $(\mathbb{R}x + \mathbb{R}u_0) \cap W_2 \neq \emptyset$.

By Step 1 applied to the 2-plane $\mathbb{R}x + \mathbb{R}u_0$, for every non-negative $\varepsilon < q^{-1-1/r} - \max\{\|qx_i\|/q\}$ let

$$x_\varepsilon = (1, x_{\varepsilon,1}, \ldots, x_{\varepsilon,r}) \in (\mathbb{R}x + \mathbb{R}u_0) \cap W_2$$

be such that $0 < \|x - x_\varepsilon\|_\infty \leq \varepsilon$ and the components of x_ε are linearly independent over $\mathbb{Q}$. The map $f|_{\mathbb{R}_+ u_0 + \mathbb{R}_+ x_\varepsilon}$ is not linear since otherwise we would have $f(x_\varepsilon) = \ell(x_\varepsilon)$. Observe that $|qx_{\varepsilon,i} - p_i| < q^{-1/r}$ for every i. Then as in Step 4 we have

$$f(qx_\varepsilon) = \Big(1 - \sum \|qx_{\varepsilon,i}\|\Big) f(u_0) + \sum \|qx_{\varepsilon,i}\| f(u_i) + \widehat{e}_q,$$

where $\widehat{e}_q \geq c(1 - \sum \|qx_{\varepsilon,i}\|)$. Finally dividing by q and letting $\varepsilon \to 0$ we obtain (4.4).

Step 8: Therefore for all $q \gg 0$ satisfying (4.1) we have (4.4). Then since f is locally Lipschitz around x there is a constant $L > 0$ such that

$$\begin{aligned} c(q^{1/r} - r)q^{-1-1/r} &< c\Big(1 - \sum \|qx_i\|\Big)/q \leq e_q \\ &= f(x) - \Big(1 - \sum \|qx_i\|\Big) f(u_0/q) - \sum \|qx_i\| f(u_i/q) \\ &= \big(f(x) - f(u_0/q)\big) + \sum \|qx_i\| \big(f(u_0/q) - f(u_i/q)\big) \\ &\leq L\|x - u_0/q\|_\infty + \sum \|qx_i\| L \|u_0/q - u_i/q\|_\infty \\ &< Lq^{-1-1/r} + \sum_{i=1}^{r} q^{-1/r} L q^{-1} = L(r+1)q^{-1-1/r}, \end{aligned}$$

where I used (4.1) and (4.2). Hence $L > \frac{c}{r+1}(q^{1/r} - r)$ for $q \gg 0$, a contradiction.

Thus if $1, x_1, \dots, x_r$ are linearly independent over $\mathbb{Q}$ then there is an open cone containing x where f is linear, so the lemma follows. In particular there are linearly independent rational rays $R_1, \dots, R_{r+1} \subset \operatorname{int} \mathcal{S}_\mathbb{R}$ such that $R \subset \operatorname{int}(R_1 + \cdots + R_{r+1})$ and the map $f|_{R_1 + \cdots + R_{r+1}}$ is linear.

Step 9: Assume now that x is a rational point. By induction I assume there does not exist a rational hyperplane containing $\mathbb{R}_+ x$ and R. By clearing denominators I can assume $x = (\kappa, x_1, \dots, x_r)$ where $\kappa, x_i \in \mathbb{N}$.

Fix q big enough so that the ball of radius $1/q$ centred at x is contained in $\operatorname{int} \mathcal{S}_\mathbb{R}$ and so that $q^{1/r} > r$. Fix a positive $\varepsilon < q^{-1-1/r}$. By Step 1 there is a point

$$x_\varepsilon = (\kappa, x_{\varepsilon,1}, \dots, x_{\varepsilon,r}) \in \mathbb{R}_+ x + R$$

such that $\|x - x_\varepsilon\|_\infty \leq \varepsilon$ and the components of x_ε are linearly independent over $\mathbb{Q}$.

Set $u_0 = qx$, define integers p_i and $\widehat{p}_i$ with respect to x_ε as in Step 2 and set

$$u_i = q\kappa e_0 + \sum_{j\neq i} p_j e_j + \widehat{p}_i e_i$$

for $i = 1, \dots, r$. Then $u_0, \dots, u_r$ are linearly independent and we have

$$\Big(1 - \sum \|qx_{\varepsilon,i}\|\Big)u_0 + \sum \|qx_{\varepsilon,i}\| u_i = qx_\varepsilon.$$

With respect to x_ε define the sequences $v_n \in \mathbb{N}^{r+1}$ and $(j_n, e_n) \in \mathbb{N} \times \mathbb{R}_+$ as in Step 3. Assume that for all $n \geq n_0$ with $j_n = 0$ we have $e_n \geq c$. Then as in Step 4 we obtain

$$f(qx_\varepsilon) = \Big(1 - \sum \|qx_{\varepsilon,i}\|\Big)f(u_0) + \sum \|qx_{\varepsilon,i}\| f(u_i) + \widehat{e}_q, \tag{4.6}$$

where $\widehat{e}_q \geq c(1 - \sum \|qx_{\varepsilon,i}\|)$. If (4.6) stands for every $\varepsilon < q^{-1-1/r}$ then dividing (4.6) by q and letting $\varepsilon \to 0$ we get

$$f(x) = f(x) + e_q$$

where $e_q \geq c/q$, a contradiction. Therefore there is a positive $\varepsilon < q^{-1-1/r}$ such that there are infinitely many n with $j_n = 0$ and $e_n = 0$. But then as in Step 5 we have that the map $f|_{\mathbb{R}_+x+\mathbb{R}_+x_\varepsilon}$ is linear and we are done.

Step 10: Assume finally that x is a non-rational point contained in a rational hyperplane; let H be a rational plane of the smallest dimension containing x and set $k = \dim H$. Let $R = \mathbb{R}_+v$.

By Theorem 4.6 there is a rational cone $\mathcal{C} = \sum_{i=1}^k \mathbb{R}_+g_i \subset H$ with g_i being rational points such that $f_{|\mathcal{C}}$ is linear and $x \in \operatorname{relint}\mathcal{C}$, or equivalently $x = \sum \lambda_i g_i$ with all $\lambda_i > 0$. Take a rational point $y = \sum_{i=1}^k g_i$. Then by Step 9 there is a point $x' = \alpha y + \beta v$ with $\alpha, \beta > 0$ such that the map $f|_{\mathbb{R}_+y+\mathbb{R}_+x'}$ is linear. Now we have

$$f\Big(\sum g_i + x'\Big) = f(y + x') = f(y) + f(x') = \sum f(g_i) + f(x'),$$

so the map $f|_{\mathcal{C}+\mathbb{R}_+x'}$ is linear by Lemma 4.5. Taking $\mu = \max_i\{\frac{\alpha}{\lambda_i\beta}\}$ and setting

$\hat{v} = \mu x + v \in \operatorname{relint}(\mathbb{R}_+ x + R)$, it is easy to check that

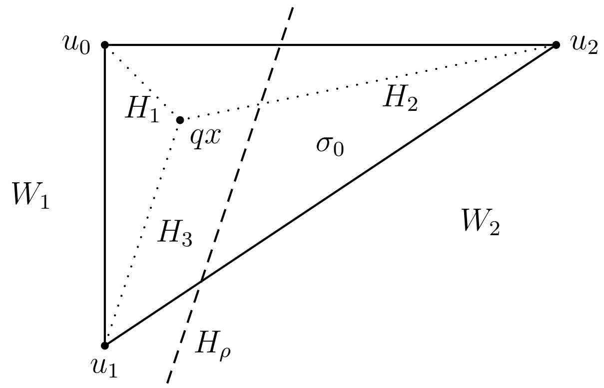
$$\hat{v} = \sum(\mu\lambda_i - \tfrac{\alpha}{\beta})g_i + \tfrac{1}{\beta}x' \in \mathcal{C} + \mathbb{R}_+ x',$$

so the map $f|_{\mathbb{R}_+ x + \mathbb{R}_+ \hat{v}}$ is linear. □

Lemma 4.11. *Assume the notation from Lemma 4.10. Then*

$$\lim_{n\to\infty} \frac{v_n}{n} = qx.$$

Proof. I work with the standard scalar product $\langle \cdot, \cdot \rangle$ and the induced Euclidean norm $\| \cdot \|$; denote $w_n = \frac{v_n}{n+r+1}$. It is enough to prove $\lim_{n\to\infty} w_n = qx$. By restricting to the hyperplane $(z_0 = q)$ in $\mathbb{R}^{r+1}$ I assume the ambient space is $\mathbb{R}^r$.



Step 1: Let σ denote the simplex with vertices $u_0, \dots, u_r$ and let $d = \sqrt{2}$ be the diameter of σ. For each i, let σ_i be the simplex with vertices qx and u_j for $j \neq i$. The points w_n belong to σ and

$$w_{n+1} = \frac{1}{n+r+2}\big((n+r+1)w_n + u_{j_n}\big),$$

so we immediately get

$$\|w_n - w_{n+1}\| \leq \frac{d}{n+r+2}. \tag{4.7}$$

For $\alpha = 1, \dots, \binom{r+1}{2}$ let H_α be all hyperplanes containing the faces of the simplices σ_i which contain qx.

Step 2: Let us prove that for each α and for each n,

$$\operatorname{dist}\{w_{n+1}, H_\alpha\} < \operatorname{dist}\{w_n, H_\alpha\} \tag{4.8}$$

if the segment $[w_n, w_{n+1}]$ does not intersect H_α, and otherwise

$$\operatorname{dist}\{w_{n+1}, H_\alpha\} < \frac{d}{n+r+2}. \tag{4.9}$$

To this end, if H_α contains u_{j_n}, then obviously $\operatorname{dist}\{w_{n+1}, H_\alpha\} < \operatorname{dist}\{w_n, H_\alpha\}$. If H_α does not contain u_{j_n}, then u_{j_n} and w_n are on different sides of H_α. Now if the segment $[w_n, w_{n+1}]$ does not intersect H_α then (4.8) is obvious, whereas otherwise (4.9) follows from (4.7).

Step 3: Now assume that for each α, there are infinitely many segments $[w_n, w_{n+1}]$ intersecting H_α. Then from (4.8) and (4.9) we get

$$\lim_{n\to\infty} \operatorname{dist}\{w_n, H_\alpha\} = 0$$

and thus the sequence w_n accumulates on each of the hyperplanes H_α. But $\bigcap_\alpha H_\alpha = \{qx\}$, so $\lim\limits_{n\to\infty} w_n = qx$.

Step 4: Finally let α_0 be such that no segment $[w_n, w_{n+1}]$ intersects H_{α_0} for all $n \geq n_0$ and $\lim\limits_{n\to\infty} \operatorname{dist}\{w_n, H_{\alpha_0}\} = \rho > 0$ (the sequence $\operatorname{dist}\{w_n, H_{\alpha_0}\}$ converges by (4.8)). Therefore there is a hyperplane H_ρ parallel to H_{α_0} such that $\operatorname{dist}\{H_\rho, H_{\alpha_0}\} = \rho$ and the sequence w_n accumulates on H_ρ; let W_1 and W_2 be the two half-spaces such that $W_1 \cup W_2 = \mathbb{R}^r$ and $W_1 \cap W_2 = H_\rho$. Relabelling we can assume $u_0, \ldots, u_{r-1}, qx \in W_1$ and $w_n, u_r \in W_2$ for all $n \geq n_0$; observe that then $u_{j_n} \in \{u_0, \ldots, u_{r-1}\}$ for all $n \geq n_0$.

By change of coordinates I may assume that H_{α_0} contains the origin. Fix a nonzero vector a perpendicular to H_{α_0} such that $W_2 \subset \{z \in \mathbb{R}^r : \langle a, z\rangle \geq 0\}$. Since $W_2 \cap H_{\alpha_0} = \emptyset$ the linear function $\langle a, \cdot\rangle$ attains its minimum $m > 0$ on the compact set $W_2 \cap \sigma$. Then since $\langle a, u_{j_n} \rangle \leq 0$ for $n \geq n_0$ we have

$$\begin{aligned}\operatorname{dist}\{w_n, H_{\alpha_0}\} - \operatorname{dist}\{w_{n+1}, H_{\alpha_0}\} &= \frac{\langle a, w_n - w_{n+1} \rangle}{\|a\|} \\ &= \frac{\langle a, w_n - u_{j_n} \rangle}{(n+r+2)\|a\|} \geq \frac{m}{(n+r+2)\|a\|},\end{aligned}$$

and therefore

$$\operatorname{dist}\{w_{n_0}, H_{\alpha_0}\} \geq \frac{m}{\|a\|} \sum_{n \geq n_0} \frac{1}{n+r+2} = +\infty,$$

a contradiction. □

Corollary 4.12. *Let $\mathcal{S} \subset \mathbb{N}^{r+1}$ be a finitely generated monoid and let $f \colon \mathcal{S}_{\mathbb{R}} \to \mathbb{R}$ be a superlinear map. Assume there is a real number $c > 0$ such that for every $s_1, s_2 \in \mathcal{S}$, either $f(s_1 + s_2) = f(s_1) + f(s_2)$ or $f(s_1 + s_2) \geq f(s_1) + f(s_2) + c$. Let $\mathcal{C}$ be a polyhedral cone in* $\operatorname{int} \mathcal{S}_{\mathbb{R}}$.

Then for every 2-plane H the map $f_{|\mathcal{C} \cap H}$ is piecewise linear.

Proof. If $\mathcal{C} = \bigcup \mathcal{C}_i$ is a finite subdivision of $\mathcal{C}$ into rational simplicial cones, then $f_{|\mathcal{C} \cap H}$ is PL if and only if $f_{|\mathcal{C}_i \cap H}$ is PL for every i, so I assume $\mathcal{C}$ is simplicial. Take a basis $g_1, \ldots, g_{r+1} \in \mathcal{S}$ of $\mathcal{C}$, set $s := \sum g_i$ and let $0 < \alpha \ll 1$ be a rational number such that $g_i + \alpha(g_i - s) \in \operatorname{int} \mathcal{S}_{\mathbb{R}}$ for all i. Take $g_i' \in \mathcal{S} \cap \mathbb{R}_+\big(g_i + \alpha(g_i - s)\big)$. It is easy to check that g_i' are linearly independent and that $\mathcal{C} \subset \operatorname{int} \sum \mathbb{R}_+ g_i'$. Therefore I can assume $\mathcal{S} = \mathbb{N}^{r+1}$.

By Lemma 4.10, for every ray $R \subset \mathcal{C} \cap H$ there is a polyhedral cone $\mathcal{C}_R$ with $R \subset \mathcal{C}_R \subset \mathcal{C} \cap H$ such that there is a polyhedral decomposition $\mathcal{C}_R = \mathcal{C}_{R,1} \cup \mathcal{C}_{R,2}$ with $f_{|\mathcal{C}_{R,1}}$ and $f_{|\mathcal{C}_{R,2}}$ being linear maps, and if $R \subset \operatorname{relint}(\mathcal{C} \cap H)$, then $R \subset \operatorname{relint} \mathcal{C}_R$.

Let $\| \cdot \|$ be the standard Euclidean norm and let $S^r = \{z \in \mathbb{R}^{r+1} : \|z\| = 1\}$ be the unit sphere. Restricting to the compact set $S^r \cap \mathcal{C} \cap H$ we can choose finitely many polyhedral cones $\mathcal{C}_i$ with $\mathcal{C} \cap H = \bigcup \mathcal{C}_i$ such that each $f_{|\mathcal{C}_i}$ is PL. But then $f_{|\mathcal{C} \cap H}$ is PL. □

Lemma 4.13. *Let f be a superlinear function on a polyhedral cone $\mathcal{C} \subset \mathbb{R}^{r+1}$ with* $\dim \mathcal{C} = r+1$ *such that for every 2-plane H the function $f_{|\mathcal{C} \cap H}$ is piecewise linear. Then f is piecewise linear.*

Proof. I will prove the lemma by induction on the dimension.

Step 1: Fix a ray $R \subset \mathcal{C}$. In this step I prove that for any ray $R' \subset \mathcal{C}$ there is an $(r+1)$-dimensional cone $\mathcal{C}_{(r+1)} \subset \mathcal{C}$ containing R such that the map $f_{|\mathcal{C}_{(r+1)}}$ is linear and $\mathcal{C}_{(r+1)} \cap (R + R') \neq R$.

Let $H_r \supset (R + R')$ be any hyperplane. By induction there is an r-dimensional polyhedral cone $\mathcal{C}_{(r)} = \sum_{i=1}^{r} \mathbb{R}_+ e_i \subset H_r \cap \mathcal{C}$ containing R such that $f_{|\mathcal{C}_{(r)}}$ is linear and $\mathcal{C}_{(r)} \cap (R + R') \neq R$. Set $e_0 = e_1 + \cdots + e_r$. Let H_2 be a 2-plane such that $H_2 \cap H_r = \mathbb{R}_+ e_0$. Since $f_{|H_2 \cap \mathcal{C}}$ is PL, there is a point $e_{r+1} \in H_2 \cap \mathcal{C}$ such that $f|_{\mathbb{R}_+ e_0 + \mathbb{R}_+ e_{r+1}}$ is linear. Set $\mathcal{C}_{(r+1)} = \mathbb{R}_+ e_1 + \cdots + \mathbb{R}_+ e_{r+1}$. Then we have

$$f\Big(\sum e_i\Big) = f(e_0 + e_{r+1}) = f(e_0) + f(e_{r+1}) = \sum f(e_i),$$

so the map $f_{|\mathcal{C}_{(r+1)}}$ is linear by Lemma 4.5. Observe that choosing e_{r+1} appropriately we can ensure that the cone $\mathcal{C}_{(r+1)}$ is contained in either of the half-spaces into which H_r divides $\mathbb{R}^{r+1}$.

Step 2: Fix a ray $R \subset \mathcal{C}$ and let $\mathcal{C}_{(r+1)}$ be any $(r+1)$-dimensional cone such that $f|_{\mathcal{C}_{(r+1)}}$ is linear. Let ℓ be the linear extension of $f_{|\mathcal{C}_{r+1}}$ to $\mathbb{R}^{r+1}$. Let $\widehat{\mathcal{C}} = \{z \in \mathcal{C} : f(z) = \ell(z)\}$; it is a closed cone by Step 6 of the proof of Lemma 4.10.

I claim $\widehat{\mathcal{C}}$ is a locally polyhedral cone (and thus polyhedral). Otherwise, fix a boundary ray R_∞ and let H be any hyperplane containing R_∞ such that $H \cap \operatorname{int} \widehat{\mathcal{C}} \neq \emptyset$. Let R_n be a sequence of boundary rays which converge to R_∞ and they are all on the same side of H.

Let $T \supset R_\infty$ be any hyperplane tangent to $\widehat{\mathcal{C}}$. Fix an $(r-1)$-plane $H_{r-1} \subset T$ containing R_∞ and let $H_{r-1}^{\perp}$ be the unique 2-plane orthogonal to H_{r-1}. For each n consider a hyperplane $H_r^{(n)}$ generated by H_{r-1} and R_n (if $R_n \subset H_{r-1}$ the we can finish by induction on the dimension). Let $\|\cdot\|$ be the standard Euclidean norm and let $S^r = \{z \in \mathbb{R}^{r+1} : \|z\| = 1\}$ be the unit sphere. The set of points $\bigcup_{n\in\mathbb{N}} \big(S^r \cap H_{r-1}^{\perp} \cap H_r^{(n)}\big)$ has a limit P_∞ on the circle $S^r \cap H_{r-1}^{\perp}$ and let $H_r^{(\infty)}$ be the hyperplane generated by H_{r-1} and P_∞; without loss of generality I can assume all R_n are on the same side of $H_r^{(\infty)}$.

Now by the construction in Step 1, there is an $(r+1)$-dimensional cone $\mathcal{C}_\infty$ such that $\mathcal{C}_\infty \cap H_r^{(\infty)}$ is a face of $\mathcal{C}_\infty$, $f_{|\mathcal{C}_\infty}$ is linear and $\mathcal{C}_\infty$ intersects hyperplanes $H_r^{(n)}$

for all $n \gg 0$. In particular $R_n \subset \mathcal{C}_\infty$ for all $n \gg 0$ and $\operatorname{int}\mathcal{C}_\infty \cap \widehat{\mathcal{C}} \neq \emptyset$. Let $w \in \operatorname{int}\mathcal{C}_\infty \cap \widehat{\mathcal{C}}$ and let $B \subset \operatorname{int}\mathcal{C}_\infty$ be a small ball centred at w. Then the cone $B \cap \widehat{\mathcal{C}}$ is $(r+1)$-dimensional (otherwise the cone $\widehat{\mathcal{C}}$ would be contained in a hyperplane) and thus $\mathcal{C}_\infty \cap \widehat{\mathcal{C}}$ is an $(r+1)$-dimensional cone. Therefore the linear extension of $f_{|\mathcal{C}_\infty}$ coincides with ℓ and thus $\mathcal{C}_\infty \subset \widehat{\mathcal{C}}$. Since $R_n \not\subset \operatorname{int}\widehat{\mathcal{C}}$ we must have $R_n \subset \mathcal{C}_\infty \cap H_r^{(\infty)}$, and we finish by induction on the dimension.

Step 3: Again fix a ray $R \subset \mathcal{C}$. By Steps 1 and 2 there is a collection of $(r+1)$-dimensional polyhedral cones $\{\mathcal{C}_\alpha\}_{\alpha \in I_R}$ such that $R \subset \mathcal{C}_\alpha \subset \mathcal{C}$ for every $\alpha \in I_R$, for every ray $R' \subset \mathcal{C}$ there is $\alpha \in I_R$ such that $\mathcal{C}_\alpha \cap (R + R') \neq R$ and for every two distinct $\alpha, \beta \in I_R$ the linear extensions of $f_{|\mathcal{C}_\alpha}$ and $f_{|\mathcal{C}_\beta}$ to $\mathbb{R}^{r+1}$ are not the same function. I will prove that I_R is a finite set.

For each $\alpha \in I_R$ let x_α be a point in $\operatorname{int}\mathcal{C}_\alpha$ and let $H_\alpha = (R + \mathbb{R}_+ x_\alpha) \cup (-R + \mathbb{R}_+ x_\alpha)$. Let $R_\alpha \subset H_\alpha$ be the unique ray orthogonal to R. Let $R^\perp$ be the hyperplane orthogonal to R. For each α let $S^r \cap R^\perp \cap H_\alpha = \{Q_\alpha\}$. If there are infinitely many cones $\mathcal{C}_\alpha$, then the set $\{Q_\alpha : \alpha \in I_R\}$ has an accumulation point Q_∞. Let $H_\infty = (R + \mathbb{R}_+ Q_\infty) \cup (-R + \mathbb{R}_+ Q_\infty)$, let H_n be a sequence in the set $\{H_\alpha\}$ such that $\lim\limits_{n\to\infty} Q_n = Q_\infty$ where $S^r \cap R^\perp \cap H_n = \{Q_n\}$, and let $\mathcal{C}_n$ be the corresponding cones in $\{\mathcal{C}_\alpha\}$.

By assumptions of the lemma there is a point $y \in H_\infty$ such that $f|_{R+\mathbb{R}_+ y}$ is linear. Let x be a point on R and let $\mathcal{H}$ be any hyperplane such that $\mathcal{H} \cap (\mathbb{R}x + \mathbb{R}y) = \mathbb{R}(x+y)$. By induction there are r-dimensional polyhedral cones $\mathcal{C}_1, \dots, \mathcal{C}_k$ in $\mathcal{H} \cap \mathcal{C}$ such that $x + y \in \mathcal{C}_i$ for all i, there is a small r-dimensional ball $B_{(r)} \subset \mathcal{H}$ centred at $x + y$ such that $B_{(r)} \cap \mathcal{C} = B_{(r)} \cap (\mathcal{C}_1 \cup \cdots \cup \mathcal{C}_k)$ and the map $f_{|\mathcal{C}_i}$ is linear for every i. Fix i and let g_{ij} be generators of $\mathcal{C}_i$. Then

$$f\Big(\sum\nolimits_j g_{ij} + x + y\Big) = \sum\nolimits_j f(g_{ij}) + f(x+y) = \sum\nolimits_j f(g_{ij}) + f(x) + f(y),$$

so f is linear on the cone $\widetilde{\mathcal{C}_i} = \mathcal{C}_i + \mathbb{R}_+ x + \mathbb{R}_+ y$ by Lemma 4.5. Therefore if we denote $\widetilde{\mathcal{C}} = \mathcal{C}_1 + \cdots + \mathcal{C}_k + \mathbb{R}_+ x + \mathbb{R}_+ y$, then $f_{|\widetilde{\mathcal{C}}}$ is PL and there is a small ball $B_{(r+1)}$ centred at $x + y$ such that $B_{(r+1)} \cap \mathcal{C} = B_{(r+1)} \cap \widetilde{\mathcal{C}}$.

Take a ball B_ε of radius $\varepsilon \ll 1$ centred at $x + y$ such that $x \notin B_\varepsilon$ and $B_\varepsilon \cap \mathcal{C} = B_\varepsilon \cap \widetilde{\mathcal{C}}$. Since $\|Q_n - Q_\infty\| < \varepsilon$ for $n \gg 0$, then considering the subspace generated by R, Q_n and Q_∞ we obtain that H_n intersects $\operatorname{int} B_\varepsilon$ for $n \gg 0$. Since $\widetilde{\mathcal{C}} = \bigcup \widetilde{\mathcal{C}_i}$, there is an index i_0 such that $\widetilde{\mathcal{C}}_{i_0} \cap \operatorname{int} B_\varepsilon$ intersects infinitely many H_n. In particular, $\widetilde{\mathcal{C}}_{i_0} \cap \operatorname{int} \mathcal{C}_n \neq \emptyset$ for infinitely many n and therefore $\widetilde{\mathcal{C}}_{i_0} \cap \mathcal{C}_n$ is an $(r+1)$-dimensional cone as in Step 2. Thus for every such n the linear extensions of $f_{|\widetilde{\mathcal{C}}_{i_0}}$ and $f_{|\mathcal{C}_n}$ to $\mathbb{R}^{r+1}$ are the same since they coincide with the linear extension of $f_{|\widetilde{\mathcal{C}}_{i_0} \cap \mathcal{C}_n}$, which is a contradiction and I_R is finite.

Step 4: Finally, we have that for every ray $R \subset \mathcal{C}$ the map $f_{|\bigcup_{\alpha \in I_R} \mathcal{C}_\alpha}$ is PL and there is small ball B_R centred at $R \cap S^r$ such that $B_R \cap \mathcal{C} = B_R \cap \bigcup_{\alpha \in I_R} \mathcal{C}_\alpha$. There are finitely many open sets $\operatorname{int} B_R$ which cover the compact set $S^r \cap \mathcal{C}$ and therefore we can choose finitely many polyhedral cones $\mathcal{C}_i$ with $\mathcal{C} = \bigcup \mathcal{C}_i$ such that $f_{|\mathcal{C}_i}$ is PL for every i. Thus f is PL. □

Now I can prove the main result of this section.

Proof of Theorem 4.6. By Corollary 4.12 and Lemma 4.13 the map $f_{|\mathcal{C}}$ is PL; in other words we can choose finitely many polyhedral cones $\mathcal{C}_i$ with $\mathcal{C} = \bigcup \mathcal{C}_i$ such that $f_{|\mathcal{C}_i}$ is linear for each i. We can assume the linear extensions of the maps $f_{|\mathcal{C}_i}$ and $f_{|\mathcal{C}_j}$ to $\mathbb{R}^r$ are not the same by Step 6 of the proof of Lemma 4.10.

Let H be a hyperplane which contains a common $(r-1)$-dimensional face of cones $\mathcal{C}_i$ and $\mathcal{C}_j$ and assume H is not rational. Then similarly as in Step 1 of the proof of Lemma 4.10 there is a point $x \in \mathcal{C}_i \cap \mathcal{C}_j$ whose components are linearly independent over $\mathbb{Q}$. By the proof of Lemma 4.10 there is an r-dimensional cone $\widetilde{\mathcal{C}}$ such that $x \in \operatorname{int} \widetilde{\mathcal{C}}$ and the map $f_{|\widetilde{\mathcal{C}}}$ is linear. But then as in Step 2 of the proof of Lemma 4.13 the cones $\widetilde{\mathcal{C}} \cap \mathcal{C}_i$ and $\widetilde{\mathcal{C}} \cap \mathcal{C}_i$ are r-dimensional and linear extensions of $f_{|\mathcal{C}_i}$ and $f_{|\mathcal{C}_j}$ coincide since they are equal to the linear extension of $f_{|\widetilde{\mathcal{C}}}$, a contradiction. Therefore all $(r-1)$-dimensional faces of the cones $\mathcal{C}_i$ belong to rational hyperplanes and thus $\mathcal{C}_i$ are rational cones. Thus the map $f_{|\mathcal{C}}$ is $\mathbb{Q}$-PL. □

Chapter 5

Higher Rank Algebras

In this chapter I develop the theory of higher rank divisorial and b-divisorial algebras that will be useful in the approach to finite generation in Chapter 6. Mobile b-divisors give useful criteria for assessing whether algebras in question are finitely generated, and a relation to convex geometry techniques from Chapter 4 is established. In the second part of the chapter I formulate natural conjectures that extend rank 1 conjectures of Shokurov [Sho03], and I prove them on curves.

5.1 Algebras Attached to Monoids

I start with the following definitions.

Definition 5.1. Let X be a variety and let $\mathcal{S}$ be a finitely submonoid of $\mathbb{N}^r$. If $\mu\colon \mathcal{S} \to \mathrm{WDiv}(X)^{\kappa \geq 0}$ is an additive map, the algebra

$$R(X, \mu(\mathcal{S})) = \bigoplus_{s \in \mathcal{S}} H^0(X, \mathcal{O}_X(\mu(s)))$$

is called the *divisorial $\mathcal{S}$-graded algebra associated to μ.* When $\mathcal{S} = \bigoplus_{i=1}^{\ell} \mathbb{N}e_i$ is a simplicial cone, the algebra $R(X, \mu(\mathcal{S}))$ is called the *Cox ring associated to* μ, and is denoted also by $R(X; \mu(e_1), \ldots, \mu(e_\ell))$.

Definition 5.2. Let X be a variety, $\mathcal{S}$ a finitely generated submonoid of $\mathbb{N}^r$, and $\mathbf{m}\colon \mathcal{S} \to \mathbf{Div}(X)$ a superadditive map. The system $\mathbf{m}(\mathcal{S}) = \{\mathbf{m}(s)\}_{s \in \mathcal{S}}$ (respectively the map $\mathbf{m}$) is *bounded* if the following two conditions are satisfied:

- there is a reduced divisor F on X such that $\operatorname{Supp}\mathbf{m}(s)_X \subset F$ for every $s \in \mathcal{S}$, i.e. $\mathbf{m}$ has *bounded support on* X,
- for every $s \in \mathcal{S}$, the limit $\lim_{\kappa\to\infty} \frac{1}{\kappa}\mathbf{m}(\kappa s)$ exists in $\mathbf{Div}(X)_{\mathbb{R}}$.

Let $\pi\colon X \to Z$ be a projective morphism of normal varieties and let $\mathbf{m}\colon \mathcal{S} \to \mathbf{Div}(X)$ be a bounded superadditive map such that $\mathcal{O}_X(\mathbf{m}(s))$ is a coherent sheaf for all $s \in \mathcal{S}$. The algebra

$$R(X, \mathbf{m}(\mathcal{S})) = \bigoplus_{s\in\mathcal{S}} \pi_*\mathcal{O}_X(\mathbf{m}(s))$$

is a *b-divisorial* $\mathcal{S}$*-graded* $\mathcal{O}_Z$*-algebra*.

Remark 5.3. Divisorial algebras considered in this thesis are *algebras of sections*. I will occasionally, and without explicit mention, view them as algebras of rational functions, in particular to be able to write $H^0(X, D) \simeq H^0(X, \operatorname{Mob}(D)) \subset k(X)$.

Assume now that X is smooth, $D \in \operatorname{Div}(X)$ and that Γ is a prime divisor on X. If σ_Γ is the global section of $\mathcal{O}_X(\Gamma)$ such that $\operatorname{div}\sigma_\Gamma = \Gamma$, from the exact sequence

$$0 \to H^0(X, \mathcal{O}_X(D - \Gamma)) \xrightarrow{\cdot\sigma_\Gamma} H^0(X, \mathcal{O}_X(D)) \xrightarrow{\rho_{D,\Gamma}} H^0(\Gamma, \mathcal{O}_\Gamma(D))$$

we define $\operatorname{res}_\Gamma H^0(X, \mathcal{O}_X(D)) = \operatorname{Im}(\rho_{D,\Gamma})$. For $\sigma \in H^0(X, \mathcal{O}_X(D))$, I denote $\sigma_{|\Gamma} := \rho_{D,\Gamma}(\sigma)$. Observe that

$$\ker(\rho_{D,\Gamma}) = H^0(X, \mathcal{O}_X(D - \Gamma)) \cdot \sigma_\Gamma, \tag{5.1}$$

and that $\operatorname{res}_\Gamma H^0(X, \mathcal{O}_X(D)) = 0$ if $\Gamma \subset \operatorname{Bs}|D|$. If $D \sim D'$ such that the restriction $D'_{|\Gamma}$ is defined, then

$$\operatorname{res}_\Gamma H^0(X, \mathcal{O}_X(D)) \simeq \operatorname{res}_\Gamma H^0(X, \mathcal{O}_X(D')) \subset H^0(\Gamma, \mathcal{O}_\Gamma(D'_{|\Gamma})).$$

The *restriction of* $R(X, \mu(\mathcal{S}))$ *to* Γ is defined as

$$\operatorname{res}_\Gamma R(X, \mu(\mathcal{S})) = \bigoplus_{s\in\mathcal{S}} \operatorname{res}_\Gamma H^0(X, \mathcal{O}_X(\mu(s))).$$

This is an $\mathcal{S}$-graded, not necessarily divisorial algebra.

Remark 5.4. Under assumptions from Definition 5.1, define the map $\mathbf{Mob}_\mu\colon \mathcal{S} \to \mathbf{Mob}(X)$ by $\mathbf{Mob}_\mu(s) = \mathbf{Mob}(\mu(s))$ for every $s \in \mathcal{S}$. Then we have the b-divisorial algebra

$$R(X, \mathbf{Mob}_\mu(\mathcal{S})) \simeq R(X, \mu(\mathcal{S})).$$

If $\mathcal{S}'$ is a finitely generated submonoid of $\mathcal{S}$, I use $R(X, \mu(\mathcal{S}'))$ to denote the algebra $R(X, \mu_{|\mathcal{S}'}(\mathcal{S}'))$. If $\mathcal{S}$ is a submonoid of $\mathrm{WDiv}(X)^{\kappa\geq 0}$ and $\iota\colon \mathcal{S} \to \mathcal{S}$ is the identity map, I use $R(X, \mathcal{S})$ to denote $R(X, \iota(\mathcal{S}))$.

The following lemma summarises the basic properties of higher rank finite generation.

Lemma 5.5. *Let $\mathcal{S} \subset \mathbb{N}^n$ be a finitely generated monoid and let $R = \bigoplus_{s\in\mathcal{S}} R_s$ be an $\mathcal{S}$-graded algebra.*

1. *(1) Let $\mathcal{S}'$ be a truncation of $\mathcal{S}$. If the $\mathcal{S}'$-graded algebra $R' = \bigoplus_{s\in\mathcal{S}'} R_s$ is finitely generated over R_0, then R is finitely generated over R_0.*
2. *(2) Assume furthermore that $\mathcal{S}$ is saturated and let $\mathcal{S}'' \subset \mathcal{S}$ be a finitely generated saturated submonoid. If R is finitely generated over R_0, then the $\mathcal{S}''$-graded algebra $R'' = \bigoplus_{s\in\mathcal{S}''} R_s$ is finitely generated over R_0.*
3. *(3) Let X be a variety and let $\mathbf{m}\colon \mathcal{S} \to \mathbf{Mob}(X)$ be a superadditive map. If there exists a rational polyhedral subdivision $\mathcal{S}_\mathbb{R} = \bigcup_{i=1}^k \Delta_i$ such that, for each i, $\mathbf{m}_{|\Delta_i\cap\mathcal{S}}$ is an additive map up to truncation, then the algebra $R(X, \mathbf{m}(\mathcal{S}))$ is finitely generated.*

Proof. For (1), let $\mathcal{S} = \sum_{i=1}^n \mathbb{N}e_i$ and $\mathcal{S}' = \sum_{i=1}^n \mathbb{N}\kappa_i e_i$ for positive integers κ_i. It is enough to observe that R is an integral extension of R': for any $\varphi \in R$ we have $\varphi^{\kappa_1\cdots\kappa_n} \in R'$.

For (3), let $\{e_{ij} : j \in I_i\}$ be a finite set of generators of $\Delta_i \cap \mathcal{S}$ by Lemma 4.3 and let κ_{ij} be positive integers such that $\mathbf{m}|_{\sum_{j\in I_i}\mathbb{N}\kappa_{ij}e_{ij}}$ is additive for each i. Set $\kappa := \prod_{i,j}\kappa_{ij}$ and let $\mathcal{S}' = \sum_{i,j}\mathbb{N}\kappa e_{ij}$ be a truncation of $\mathcal{S}$.

Let $\tilde{e} = \sum_{i,j}\lambda_{ij}\kappa e_{ij} \in \Delta_i \cap \mathcal{S}'$ for some $\lambda_{ij} \in \mathbb{N}$. Then $\sum_{i,j}\lambda_{ij}e_{ij} \in \Delta_i \cap \mathcal{S}$ and thus there are $\mu_j \in \mathbb{N}$ such that $\sum_{i,j}\lambda_{ij}e_{ij} = \sum_{j\in I_i}\mu_j e_{ij}$. From here we have

$$\tilde{e} = \kappa \sum\nolimits_{j\in I_i} \mu_j e_{ij} \in \sum\nolimits_{j\in I_i} \mathbb{N}\kappa e_{ij}$$

and therefore $\Delta_i \cap \mathcal{S}' = \sum_{j\in I_i} \mathbb{N}\kappa e_{ij}$ is a truncation of $\sum_{j\in I_i} \mathbb{N}\kappa_{ij} e_{ij}$; in particular $\mathbf{m}_{|\Delta_i\cap\mathcal{S}'}$ is additive for each i.

I claim the algebra $R(X, \mathbf{m}(\mathcal{S}'))$ is finitely generated, and thus the algebra $R(X, \mathbf{m}(\mathcal{S}))$ is finitely generated by part (1). To that end, let $Y \to X$ be a model such that $\mathbf{m}(\kappa e_{ij})$ descend to Y for all i, j, and let $\mathbf{m}_Y : \mathcal{S}' \to \mathrm{WDiv}(Y)$ be the map given by $\mathbf{m}_Y(s) = \mathbf{m}(s)_Y$. Let $s = \sum_{j\in I_i} \nu_{ij}\kappa e_{ij} \in \Delta_i \cap \mathcal{S}'$ for some i and some $\nu_{ij} \in \mathbb{N}$. Then

$$\mathbf{m}(s) = \sum\nolimits_{j\in I_i} \nu_{ij}\mathbf{m}(\kappa e_{ij}) = \sum\nolimits_{j\in I_i} \nu_{ij}\overline{\mathbf{m}(\kappa e_{ij})_Y} = \overline{\sum\nolimits_{j\in I_i} \nu_{ij}\mathbf{m}(\kappa e_{ij})_Y} = \overline{\mathbf{m}(s)_Y},$$

and thus $\mathbf{m}(s)$ descends to Y and $R(X, \mathbf{m}(\mathcal{S}')) \simeq R(Y, \mathbf{m}_Y(\mathcal{S}'))$. Fix i, and consider the free monoid $\widehat{\mathcal{S}_i} = \bigoplus_{j\in I_i} \mathbb{N}\kappa e_{ij}$ and the natural projection $\pi\colon \widehat{\mathcal{S}_i} \to \Delta_i \cap \mathcal{S}_i'$. The Cox ring $R(Y, (\mathbf{m}_Y \circ \pi)(\widehat{\mathcal{S}_i}))$ is finitely generated by [HK00, Lemma 2.8], thus the algebra $R(X, \mathbf{m}(\Delta_i \cap \mathcal{S}'))$ is finitely generated for each i by projection. The set of generators of $R(X, \mathbf{m}(\Delta_i\cap\mathcal{S}'))$ for all i generates $R(X, \mathbf{m}(\mathcal{S}'))$ and the claim follows.

Finally, statement (2) is [ELM+06, Lemma 4.8]. □

Following [Cor07, Lemma 2.3.53], in the rank 1 case we have the converse of Lemma 5.5(3).

Lemma 5.6. *Let X be a variety and $\mathbf{m}\colon \mathbb{N} \to \mathbf{Mob}(X)$ be a superadditive map. The algebra $R(X, \mathbf{m}(\mathbb{N}))$ is finitely generated if and only if there exists an integer i such that $\mathbf{m}(ik) = k\mathbf{m}(i)$ for all $k \geq 0$.*

Proof. We only need to prove necessity as sufficiency was proved in Lemma 5.5(3). Up to truncation, we may assume that $R(X, \mathbf{m}(\mathbb{N}))$ is generated by $H^0(X, \mathbf{m}(1))$. For each j, take a resolution $Y_j \to X$ such that both $\mathbf{m}(1)$ and $\mathbf{m}(j)$ descend to Y_j. Superadditivity and the finite generation imply

$$H^0(Y_j, j\mathbf{m}(1)_{Y_j}) \subset H^0(Y_j, \mathbf{m}(j)_{Y_j}) = H^0(Y_j, \mathbf{m}(1)_{Y_j})^j \subset H^0(Y_j, j\mathbf{m}(1)_{Y_j}).$$

Therefore $j\mathbf{m}(1)_{Y_j} = \mathbf{m}(j)_{Y_j}$ and thus $j\mathbf{m}(1) = \mathbf{m}(j)$. □

Definition 5.7. Let $\mathcal{S}$ be a monoid and let $f\colon \mathcal{S} \to G$ be a superadditive map to a monoid G. For every $s \in \mathcal{S}$, the smallest positive integer ι_s, if it exists, such that $f(\mathbb{N}\iota_s s)$ is an additive system is called the *index* of s (otherwise we set $\iota_s = \infty$).

The following result gives the connection to superlinear functions.

Lemma 5.8. *Let X be a variety, $\mathcal{S} \subset \mathbb{N}^r$ a finitely generated monoid and let $f\colon \mathcal{S} \to G$ be a superadditive map to a monoid G which is a subset of* $\mathrm{WDiv}(X)$ *or* $\mathbf{Div}(X)$*, such that for every $s \in \mathcal{S}$ the index ι_s is finite.*

Then there is a unique superlinear function $f^\sharp\colon \mathcal{S}_\mathbb{R} \to G_\mathbb{R}$ such that for every $s \in \mathcal{S}$ there is a positive integer λ_s with $f(\lambda_s s) = f^\sharp(\lambda_s s)$. Furthermore, let $\mathcal{C}$ be a rational polyhedral subcone of $\mathcal{S}_\mathbb{R}$. Then $f_{|\mathcal{C}\cap\mathcal{S}}$ is additive up to truncation if and only if $f^\sharp_{|\mathcal{C}}$ is linear.

If $\mu\colon \mathcal{S} \to \mathrm{Div}(X)$ is an additive map and $\mathbf{m} = \mathbf{Mob}_\mu$ is such that for every $s \in \mathcal{S}$ there is a positive integer ι_s such that $\mathbf{m}_{|\mathbb{N}\iota_s s}$ is an additive map, then we have

$$\mathbf{m}^\sharp(s) = \overline{\mu(s)} - \sum \big(\operatorname{ord}_E \|\mu(s)\|\big)E, \tag{5.2}$$

where the sum runs over all geometric valuations E on X.

Proof. The construction will show that $f^\sharp$ is the unique function with the stated properties. To start with, fix a point $s \in \mathcal{S}_\mathbb{Q}$ and let κ be a positive integer such that $\kappa s \in \mathcal{S}$. Set

$$f^\sharp(s) := \frac{f(\iota_{\kappa s}\kappa s)}{\iota_{\kappa s}\kappa}.$$

This is well-defined: take another κ' such that $\kappa' s \in \mathcal{S}$. Then by the definition of the index we have

$$f(\iota_{\kappa s}\iota_{\kappa' s}\kappa\kappa' s) = \iota_{\kappa s}\kappa f(\iota_{\kappa' s}\kappa' s) = \iota_{\kappa' s}\kappa' f(\iota_{\kappa s}\kappa s),$$

so $f(\iota_{\kappa s}\kappa s)/\iota_{\kappa s}\kappa = f(\iota_{\kappa' s}\kappa' s)/\iota_{\kappa' s}\kappa'$.

Now let $s \in \mathcal{S}_\mathbb{Q}$, let ξ be a positive rational number and let λ be a sufficiently divisible positive integer such that $\lambda\xi s \in \mathcal{S}$. Then

$$f^\sharp(\xi s) = \frac{f\big((\iota_{\lambda\xi s}\lambda)\xi s\big)}{\iota_{\lambda\xi s}\lambda} = \xi\frac{f\big((\iota_{\lambda\xi s}\lambda\xi)s\big)}{\iota_{\lambda\xi s}\lambda\xi} = \xi f^\sharp(s),$$

so $f^\sharp$ is positively homogeneous (with respect to rational scalars). It is also superadditive: let $s_1, s_2 \in \mathcal{S}_\mathbb{Q}$ and let κ be a sufficiently divisible positive integer such that $f(\kappa s_1) = f^\sharp(\kappa s_1)$, $f(\kappa s_2) = f^\sharp(\kappa s_2)$ and $f\big(\kappa(s_1+s_2)\big) = f^\sharp\big(\kappa(s_1+s_2)\big)$. By superadditivity of f we have

$$f(\kappa s_1) + f(\kappa s_2) \leq f\big(\kappa(s_1 + s_2)\big),$$

so dividing the inequality by κ we obtain superadditivity of $f^\sharp$.

Let E be any divisor on X, respectively any geometric valuation E over X, when $G \subset \mathrm{WDiv}(X)$, respectively $G \subset \mathbf{Div}(X)$. Consider the function $f_E^\sharp$ given by $f_E^\sharp(s) = \mathrm{mult}_E\, f^\sharp(s)$. Proposition 4.2 applied to each $f_E^\sharp$ shows that $f^\sharp$ extends to a superlinear function on the whole $\mathcal{S}_\mathbb{R}$.

For the statement on cones, necessity is clear. Assume $f^\sharp|_\mathcal{C}$ is linear, and by Lemma 4.3 let $e_1, \dots, e_n$ be generators of $\mathcal{C} \cap \mathcal{S}$. For $s_0 = e_1 + \cdots + e_n$ we have

$$f^\sharp(s_0) = f^\sharp(e_1) + \cdots + f^\sharp(e_n). \tag{5.3}$$

Let μ be a positive integer such that $f(\mu s_0) = f^\sharp(\mu s_0)$ and $f(\mu e_i) = f^\sharp(\mu e_i)$ for all i. From (5.3) we obtain

$$f(\mu s_0) = f(\mu e_1) + \cdots + f(\mu e_n),$$

and Lemma 4.4 implies that $f^\sharp$ is additive on the truncation $\widehat{\mathcal{S}} = \sum \mathbb{N}\mu e_i$ of $\mathcal{C} \cap \mathcal{S}$.

Equation (5.2) is a restatement of the definition given above. □

Definition 5.9. In the context of Lemma 5.8, the function $f^\sharp$ is called *the straightening of f*.

Remark 5.10. In the context of the assumptions of Lemma 5.8, let $s \in \mathcal{S}$ and let λ be a positive integer such that $f^\sharp(\lambda s) = f(\lambda s)$. Then for every positive integer μ we have

$$f(\mu\lambda s) \geq \mu f(\lambda s) = \mu f^\sharp(\lambda s) = f^\sharp(\mu\lambda s) \geq f(\mu\lambda s),$$

so $f(\mu\lambda s) = \mu f(\lambda s)$. Therefore the index ι_s is the smallest integer λ such that $f^\sharp(\lambda s) = f(\lambda s)$.

To conclude this section, I prove a result that will be crucial in the constructions in Chapter 6.

Proposition 5.11. *Let X be a variety, $\mathcal{S} \subset \mathbb{N}^r$ a finitely generated saturated monoid and $\mu\colon \mathcal{S} \to \mathrm{WDiv}(X)^{\kappa\geq 0}$ an additive map. Let $\mathcal{L}$ be a finitely generated submonoid of $\mathcal{S}$ and assume $R(X,\mu(\mathcal{S}))$ is finitely generated. Then $R(X,\mu(\mathcal{L}))$ is finitely generated. Moreover, the map $\mathbf{m} = \mathbf{Mob}_{\mu|\mathcal{L}}$ is piecewise additive up to truncation. In particular, there is a positive integer p such that $\mathbf{Mob}_\mu(ips) = i\,\mathbf{Mob}_\mu(ps)$ for every $i \in \mathbb{N}$ and every $s \in \mathcal{L}$.*

Proof. Denote $\mathcal{M} = \mathcal{L}_\mathbb{R} \cap \mathbb{N}^r$. By Lemma 5.5(2), $R(X,\mu(\mathcal{M}))$ is finitely generated, and by the proof of [ELM+06, Theorem 4.1], there is a finite rational polyhedral subdivision $\mathcal{M}_\mathbb{R} = \bigcup \Delta_i$ such that for every geometric valuation E on X, the map $\mathrm{ord}_E \|\cdot\|$ is $\mathbb{Q}$-additive on $\Delta_i \cap \mathcal{M}_\mathbb{Q}$ for every i. Since for every saturated rank 1 submonoid $\mathcal{R} \subset \mathcal{M}$ the algebra $R(X,\mu(\mathcal{R}))$ is finitely generated by Lemma 5.5(2), the map $\mathbf{m}_{\mathcal{R}\cap\mathcal{L}}$ is additive up to truncation by Lemma 5.6 and thus there is the well-defined straightening $\mathbf{m}^\sharp\colon \mathcal{L}_\mathbb{Q} \to \mathbf{Mob}(X)_\mathbb{Q}$ since $\mathcal{M}_\mathbb{Q} = \mathcal{L}_\mathbb{Q}$. Then (5.2) implies that the map $\mathbf{m}^\sharp_{|\Delta_i\cap\mathcal{L}_\mathbb{Q}}$ is $\mathbb{Q}$-additive for every i, hence by Lemma 5.8 the map $\mathbf{m}$ is piecewise additive up to truncation, and therefore $R(X,\mu(\mathcal{L}))$ is finitely generated by Lemma 5.5(3). □

5.2 Shokurov Algebras on Curves

In this section I define higher rank analogues of algebras defined in [Sho03, Cor07], and I prove a possibly surprising finite generation result on curves.

Definition 5.12. Let X be a variety, let $\mathcal{S}$ be a monoid and let $\mathbf{m}\colon \mathcal{S} \to \mathbf{Mob}(X)$ be a superadditive map. Let $\mathbf{F}$ be a b-divisor on X with $\lceil \mathbf{F} \rceil \geq 0$.

We say the system $\mathbf{m}(\mathcal{S})$ is $\mathbf{F}$-*saturated* (or that it satisfies the saturation condition with respect to $\mathbf{F}$) if for all $s, s_1, \dots, s_n \in \mathcal{S}$ such that $s = \xi_1 s_1 + \cdots + \xi_n s_n$ for some non-negative *rational* numbers ξ_i, there is a model $Y_{s,s_1,\dots,s_n} \to X$ such that for all models $Y \to Y_{s,s_1,\dots,s_n}$ we have

$$\mathrm{Mob}\lceil \xi_1 \mathbf{m}(s_1)_Y + \cdots + \xi_n \mathbf{m}(s_n)_Y + \mathbf{F}_Y \rceil \leq \mathbf{m}(s)_Y.$$

If the models $Y_{s,s_1,\dots,s_n}$ do not depend on $s, s_1, \dots, s_n$, we say the system $\mathbf{m}(\mathcal{S})$ is *uniformly* $\mathbf{F}$*-saturated*.

Remark 5.13. It is important to understand that the numbers ξ_i in the previous definition are rational, and that s is not merely an integral combination of s_i. This fact is crucial in proofs.

Lemma 5.14. *Let X be a variety, let $\mathcal{S}$ be a monoid and let $\mathbf{m}\colon \mathcal{S} \to \mathbf{Mob}(X)$ be a superadditive map. Let $\mathbf{F}$ be a b-divisor on X with $\lceil \mathbf{F} \rceil \geq 0$. The system $\mathbf{m}(\mathcal{S})$ is $\mathbf{F}$-saturated if and only if for all $s \in \mathcal{S}$ and all positive integers λ and μ, there is a model $Y_{s,\lambda,\mu} \to X$ such that for all models $Y \to Y_{s,\lambda,\mu}$ we have*

$$\operatorname{Mob} \left\lceil \tfrac{\lambda}{\mu} \mathbf{m}(\mu s)_Y + \mathbf{F}_Y \right\rceil \leq \mathbf{m}(\lambda s)_Y.$$

Proof. Necessity is clear. For sufficiency, fix $s, s_1, \dots, s_n \in \mathcal{S}$ and fix non-negative rational numbers ξ_i such that $s = \xi_1 s_1 + \cdots + \xi_n s_n$. Let λ be a positive integer such that $\lambda \xi_i \in \mathbb{N}$ for all i. Then on all models Y higher than $Y_{s,1,\lambda}$ we have

$$\begin{aligned} \operatorname{Mob} \left\lceil \sum \xi_i \mathbf{m}(s_i)_Y + \mathbf{F}_Y \right\rceil &= \operatorname{Mob} \left\lceil \tfrac{1}{\lambda} \sum \lambda \xi_i \mathbf{m}(s_i)_Y + \mathbf{F}_Y \right\rceil \\ &\leq \operatorname{Mob} \left\lceil \tfrac{1}{\lambda} \mathbf{m}(\lambda s)_Y + \mathbf{F}_Y \right\rceil \leq \mathbf{m}(s)_Y. \end{aligned}$$

Therefore we can take $Y_{s,s_1,\dots,s_n} := Y_{s,1,\lambda}$. ☐

Definition 5.15. Let (X, Δ) be a relative weak Fano klt pair projective over an affine variety Z where $K_X + \Delta$ is $\mathbb{Q}$-Cartier, and let $\mathcal{S} \subset \mathbb{N}^r$ be a finitely generated monoid. A *Shokurov algebra* on X is the b-divisorial algebra $R(X, \mathbf{m}(\mathcal{S}))$, where $\mathbf{m}\colon \mathcal{S} \to \mathbf{Mob}(X)$ is a superadditive map such that the system $\mathbf{m}(\mathcal{S})$ is bounded and $\mathbf{A}(X, \Delta)$-saturated.

The next result says that saturation is preserved under restriction.

Lemma 5.16. *Let (X, Δ) be a relative weak Fano pair projective over an affine variety Z and let S be a prime component in Δ. Let $\mathcal{S}$ be a finitely generated monoid and assume the system of mobile b-divisors $\{\mathbf{M}_s\}_{s \in \mathcal{S}}$ on X is $(\mathbf{A}(X, \Delta) + \widehat{S})$-saturated. Assume $S \not\subset \operatorname{Supp} \mathbf{M}_{sX}$ for any $s \in \mathcal{S}$. Then the system $\{\mathbf{M}_{s|S}\}_{s \in \mathcal{S}}$ on S is $\mathbf{A}(S, \operatorname{Diff}(\Delta - S))$-saturated.*

Proof. This is analogous to [Cor07, Lemma 2.3.43, Lemma 2.4.3]. Denote $\mathbf{A} = \mathbf{A}(X, \Delta)$ and $\mathbf{A}^0 = \mathbf{A}(S, \operatorname{Diff}(\Delta - S))$. The claim follows as soon as we have the surjectivity of the restriction map

$$H^0\big(Y,\lceil \sum \xi_i \mathbf{M}_{s_iY} + (\mathbf{A}+\widehat{S})_Y\rceil\big) \to H^0\big(\widehat{S}_Y,\lceil \sum \xi_i \mathbf{M}_{s_iY|\widehat{S}_Y} + \mathbf{A}^0_{\widehat{S}_Y}\rceil\big)$$

for all $\xi_i \in \mathbb{Q}_+$ and all $s_i \in \mathcal{S}$, on log resolutions $f\colon Y = Y_{s_1,\dots,s_n} \to X$ where $\mathbf{M}_{s_iY}$ is free for every i. The obstruction to surjectivity is the group

$$H^1\big(Y,\lceil \sum \xi_i \mathbf{M}_{s_iY} + \mathbf{A}_Y\rceil\big) = H^1\big(Y, K_Y + \lceil -f^*(K_X+\Delta) + \sum \xi_i \mathbf{M}_{s_iY}\rceil\big).$$

But this group vanishes by Kawamata-Viehweg vanishing since $-(K_X+\Delta)$ is nef and big and all $\mathbf{M}_{s_iY}$ are nef. □

Definition 5.17. Let $\pi\colon X \to Z$ be a projective morphism of varieties, let $\mathcal{S} \subset \mathbb{N}^r$ be a finitely generated monoid and let $\delta\colon \mathcal{S} \to \mathbb{N}$ be an additive map. Assume $\{\mathbf{B}_s\}_{s\in\mathcal{S}}$ is a system of effective $\mathbb{Q}$-b-divisors on X such that

(1) the system $\{\delta(s)\mathbf{B}_s\}_{s\in\mathcal{S}}$ is superadditive and bounded,

(2) for each $s \in \mathcal{S}$ there is a divisor Δ_s on X such that $K_X + \Delta_s$ is klt and $\lim\limits_{\kappa\to\infty} \frac{1}{\kappa}\mathbf{B}_{\kappa s X} \leq \Delta_s$,

(3) for each $s \in \mathcal{S}$ there is a model Y_s over X and a mobile b-divisor $\mathbf{M}_s$ such that

$$\mathbf{M}_{sY} = \operatorname{Mob}\big(\delta(s)(K_Y + \mathbf{B}_{sY})\big)$$

for every model Y over Y_s.

Let $\mathbf{m}\colon \mathcal{S} \to \mathbf{Mob}(X)$ be the superadditive map given by $\mathbf{m}(s) = \mathbf{M}_s$ for all $s \in \mathcal{S}$. If the system $\mathbf{m}(\mathcal{S})$ is $\mathbf{F}$-saturated for a b-divisor $\mathbf{F}$ with $\lceil \mathbf{F}\rceil \geq 0$, we say the system $\mathbf{m}(\mathcal{S})$ is *adjoint* and that the algebra $R(X,\mathbf{m}(\mathcal{S}))$ is an *adjoint algebra* on X.

I pose the following two natural conjectures.

Conjecture A. *Let (X,Δ) be a relative weak Fano klt pair projective over a normal affine variety Z, where $K_X+\Delta$ is $\mathbb{Q}$-Cartier. Let $\mathcal{S} \subset \mathbb{N}^r$ be a finitely generated monoid and let $\mathbf{m}\colon \mathcal{S} \to \mathbf{Mob}(X)$ be a superadditive map such that the system $\mathbf{m}(\mathcal{S})$ is bounded and $\mathbf{A}(X,\Delta)$-saturated. Let $\mathcal{C}$ be a rational polyhedral cone in $\operatorname{int}\mathcal{S}_\mathbb{R}$. Then the Shokurov algebra $R(X,\mathbf{m}(\mathcal{C}\cap\mathcal{S}))$ is a finitely generated $\mathcal{O}_Z$-algebra.*

Conjecture B. *Let $\pi\colon X \to Z$ be a projective morphism between normal varieties, let $\mathcal{S} \subset \mathbb{N}^r$ be a finitely generated monoid and let $\mathbf{m}\colon \mathcal{S} \to \mathbf{Mob}(X)$ be a superadditive map such that the system $\mathbf{m}(\mathcal{S})$ is adjoint. Let $\mathcal{C}$ be a rational polyhedral cone in $\operatorname{int}\mathcal{S}_{\mathbb{R}}$. Then the adjoint algebra $R(X, \mathbf{m}(\mathcal{C}\cap\mathcal{S}))$ is a finitely generated $\mathcal{O}_Z$-algebra.*

Ideally we would like the conjectures to extend to the whole cone $\mathcal{S}_{\mathbb{R}}$, however this is in general not possible, see Remark 5.18.

Remark 5.18. The formulations of Conjectures A and B are in general the best possible, that is we cannot extend the results to the boundary of the cone $\mathcal{S}_{\mathbb{R}}$. For let X be a variety, let $\mathcal{S} = \mathbb{N}^2$ and assume $\mathbf{m}\colon \mathcal{S} \to \mathbf{Mob}(X)$ is a superadditive map such that the system $\mathbf{m}(\mathcal{S})$ is bounded and $\mathbf{F}$-saturated. Let $\mathbf{n}\colon \mathcal{S} \to \mathbf{Mob}(X)$ be the superadditive map given by

$$\mathbf{n}(s) = \begin{cases} \mathbf{m}(s), & s \in \mathcal{S}_{\mathbb{R}} \setminus \operatorname{int}\mathcal{S}_{\mathbb{R}}, \\ \mathbf{m}(2s), & s \in \operatorname{int}\mathcal{S}_{\mathbb{R}}. \end{cases}$$

Since saturation is the property of rays by Lemma 5.14, the system $\mathbf{n}(\mathcal{S})$ is again $\mathbf{F}$-saturated. However the algebra $R(X, \mathbf{n}(\mathcal{S}))$ is not finitely generated since the map $\mathbf{n}^\sharp$ is not continuous on the whole $\mathcal{S}_{\mathbb{R}}$.

I will confirm Conjectures A and B on an affine curve.

Theorem 5.19. *Let X be an affine curve, let $\mathcal{S}$ be a finitely generated submonoid of $\mathbb{N}^r$ and let $\mathbf{m}\colon \mathcal{S} \to \mathbf{Mob}(X)$ be a superadditive map such that the system $\mathbf{m}(\mathcal{S})$ is bounded and $\mathbf{F}$-saturated. Let $\mathcal{C}$ be a rational polyhedral cone in $\operatorname{int}\mathcal{S}_{\mathbb{R}}$.*

Then the algebra $R(X, \mathbf{m}(\mathcal{C}\cap\mathcal{S}))$ is finitely generated.

Remark 5.20. Observe that on a curve b-divisors are just the usual divisors. Also all divisors move in the corresponding linear systems, so the saturation condition reads

$$\lceil \tfrac{\mu}{\nu}\mathbf{m}(\nu s) + \mathbf{F} \rceil \le \mathbf{m}(\mu s)$$

for every $s \in \mathcal{S}$ and all positive integers μ and ν. By boundedness, for every $s \in \mathcal{S}$ the limit $\lim\limits_{\mu\to\infty} \frac{1}{\mu}\mathbf{m}(\mu s)$ exists, and therefore the map $\mathbf{m}_{|\mathbb{N}s}$ is additive up to truncation by Lemma 5.21 below. Thus there exists the well-defined straightening

$\mathbf{m}^\sharp$. Furthermore the map $\mathbf{m}^\sharp|_\mathcal{C}$ is $\mathbb{Q}$-PL if and only if for every prime divisor E in the support of $\mathbf{m}(\mathcal{S})$ the function $\mathbf{m}^\sharp_E|_\mathcal{C}$ is $\mathbb{Q}$-PL, see the proof of Lemma 5.8. Also the saturation condition on a curve is a component-wise condition, so from now on I assume the system $\mathbf{m}(\mathcal{S})$ is supported at a point.

Lemma 5.21. *Let X be an affine curve and let $\mathbf{m}\colon \mathbb{N} \to \operatorname{Mob}(X)$ be a bounded superadditive map such that the system $\mathbf{m}(\mathbb{N})$ is supported at a point P and $\mathbf{F}$-saturated. Then $\mathbf{m}$ is additive up to truncation.*

Proof. Let $\mathbf{F} = -fP$ with $0 \leq f < 1$, and let $\mathbf{m}(\nu) = m_\nu P \geq 0$ for every $\nu \in \mathbb{N}$. Denote $d_\nu = m_\nu/\nu$ and $d = \lim\limits_{\nu\to\infty} d_\nu$. The saturation condition when $\nu \to \infty$ becomes

$$\lceil \mu d - f \rceil \leq \mu d_\mu$$

for all $\mu > 0$. If $d \notin \mathbb{Q}$, then there exists $\mu \in \mathbb{N}$ such that $\{\mu d\} > f$ and therefore

$$\mu d_\mu \leq \mu d < \lceil \mu d - f \rceil \leq \mu d_\mu,$$

a contradiction. Thus for every $\kappa \in \mathbb{N}$ such that $\kappa d \in \mathbb{Z}$ we have

$$\kappa d_\kappa \leq \kappa d = \lceil \kappa d - f \rceil \leq \kappa d_\kappa,$$

and so $d = d_\kappa$ and $\mathbf{m}_{\kappa\mathbb{N}}$ is additive for any such κ. □

Lemma 5.22. *Let X be an affine curve, let $\mathcal{S}$ be a finitely generated monoid and let $\mathbf{m}\colon \mathcal{S} \to \operatorname{Mob}(X)$ be a superadditive map such that the system $\mathbf{m}(\mathcal{S})$ is bounded, supported at a point P and $\mathbf{F}$-saturated. Let $\mathbf{m}^\sharp$ be the straightening of $\mathbf{m}$, see Remark 5.20.*

Then there exists a constant $0 < b \leq 1/2$ with the following property: for each $s \in \mathcal{S}$ either $\mathbf{m}^\sharp(s) = \mathbf{m}(s)$ or $\mathbf{m}^\sharp(s) = \mathbf{m}(s) + e_sP$ for some e_s with $b \leq e_s \leq 1 - b$.

Proof. Let $\mathbf{F} = -fP$ with $f < 1$. Fix $s \in \mathcal{S}$ and assume $\mathbf{m}^\sharp(s) \neq \mathbf{m}(s)$. Then there is the smallest positive integer λ such that $\mathbf{m}\big((\lambda+1)s\big) \neq (\lambda+1)\mathbf{m}(s)$; in particular

$$\mathbf{m}(\lambda s) = \lambda\mathbf{m}(s) \quad \text{and} \quad \mathbf{m}\big((\lambda+1)s\big) = (\lambda+1)\mathbf{m}(s) + e_{\lambda s}P$$

for some $e_{\lambda s} \geq 1$. From the saturation condition we have

$$\left\lceil \tfrac{\lambda}{\lambda+1}\mathbf{m}\big((\lambda+1)s\big) - fP \right\rceil \leq \mathbf{m}(\lambda s),$$

that is

$$\left\lceil \mathbf{m}(\lambda s) + \tfrac{\lambda}{\lambda+1}e_{\lambda s}P - fP \right\rceil \leq \mathbf{m}(\lambda s).$$

This implies $\frac{\lambda}{\lambda+1} \leq f$, and so $\frac{1}{\lambda+1} \geq 1 - f$. Therefore

$$\mathbf{m}^\sharp(s) \geq \tfrac{1}{\lambda+1}\mathbf{m}\big((\lambda+1)s\big) = \mathbf{m}(s) + \tfrac{1}{\lambda+1}e_{\lambda s}P \geq \mathbf{m}(s) + (1-f)P.$$

On the other hand, let κ be a positive integer such that $\mathbf{m}^\sharp(\kappa s) = \mathbf{m}(\kappa s)$. Then saturation gives

$$\left\lceil \tfrac{1}{\kappa}\mathbf{m}(\kappa s) - fP \right\rceil \leq \mathbf{m}(s),$$

i.e. $\lceil \mathbf{m}^\sharp(s) - fP \rceil \leq \mathbf{m}(s)$. Hence

$$\mathbf{m}^\sharp(s) - \mathbf{m}(s) \leq fP.$$

In particular if $f \leq 1/2$ then $\mathbf{m}^\sharp(s) = \mathbf{m}(s)$ for every $s \in \mathcal{S}$. Set $b := \min\{1 - f, 1/2\}$. □

Lemma 5.23. *Let X be an affine curve, let $\mathcal{S}$ be a finitely generated monoid and let $\mathbf{m}\colon \mathcal{S} \to \mathrm{Mob}(X)$ be a superadditive map such that the system $\mathbf{m}(\mathcal{S})$ is bounded, supported at a point P and $\mathbf{F}$-saturated. Let b be the constant from Lemma 5.22. Then for each $s \in \mathcal{S}$ we have $\iota_s \leq 1/b$.*

Proof. By Lemma 5.21 and Remark 5.20, there exists a well-defined straightening $\mathbf{m}^\sharp$ of $\mathbf{m}$. Observe that Lemma 5.22 implies that $\mathbf{m}(s) = \lfloor \mathbf{m}^\sharp(s) \rfloor$ for each $s \in \mathcal{S}$, and this in turn implies that the index ι_s is the smallest integer λ such that $\mathbf{m}^\sharp(\lambda s)$ is an integral divisor (cf. Remark 5.10).

Now fix $s \in \mathcal{S}$, assume $\iota_s > 1$ and let $\mathbf{m}^\sharp(\iota_s s) = \mathbf{m}(\iota_s s) = \mu_s P$. Notice that ι_s and μ_s must be coprime: otherwise assume p is a prime dividing both ι_s and μ_s. Then

$$\mathbf{m}^\sharp\big(\tfrac{\iota_s}{p}s\big) = \tfrac{\mu_s}{p}P$$

is an integral divisor and so ι_s is not the index of s, a contradiction. Therefore there is an integer $1 \leq \kappa \leq \iota_s - 1$ such that $\kappa\mu_s \equiv 1 \pmod{\iota_s}$, and therefore

$$\mathbf{m}^\sharp(\kappa s) = \tfrac{\kappa\mu_s}{\iota_s}P \quad \text{and} \quad \mathbf{m}(\kappa s) = \tfrac{\kappa\mu_s - 1}{\iota_s}P.$$

Combining this with Lemma 5.22 we obtain

$$bP \leq \mathbf{m}^\sharp(\kappa s) - \mathbf{m}(\kappa s) = \tfrac{1}{\iota_s}P,$$

and finally $\iota_s \leq 1/b$. □

Finally we have

Proof of Theorem 5.19. By Lemma 4.3 the monoid $\mathcal{S}' = \mathcal{C} \cap \mathcal{S}$ is finitely generated and let $e_1, \ldots, e_n$ be its generators. We have $\mathcal{S}'_\mathbb{R} = \mathcal{C}$ and $\mathbf{m}^\sharp$ is continuous on $\mathcal{S}'_\mathbb{R}$. Setting $\kappa := \lfloor 1/b \rfloor!$ for b as in Lemma 5.22, and taking the truncation $\widehat{\mathcal{S}} = \sum_{i=1}^n \mathbb{N}\kappa e_i$ of $\mathcal{S}'$, we have that $\mathbf{m}^\sharp(s) = \mathbf{m}(s)$ for every $s \in \widehat{\mathcal{S}}$ by Lemma 5.23 and $\mathcal{S}'_\mathbb{R} = \widehat{\mathcal{S}}_\mathbb{R}$. By Remark 5.20, I assume the system $\mathbf{m}(\mathcal{S})$ is supported at a point.

By Corollary 4.7 applied to the monoid $\widehat{\mathcal{S}}$ the map $\mathbf{m}^\sharp|_{\widehat{\mathcal{S}}_\mathbb{R}}$ is $\mathbb{Q}$-PL and thus the algebra $R(X, \mathbf{m}(\mathcal{S}'))$ is finitely generated by Lemmas 5.8 and 5.5(3). □

Chapter 6

Finite Generation of the Canonical Ring

In this Chapter I establish the first step in a project to prove finite generation of the canonical ring without the Minimal Model Program. I prove:

Theorem 6.1. *Let (X, Δ) be a projective klt pair and assume Property $\mathcal{L}_A^G$ in dimensions $\leq \dim X$. Then the canonical ring $R(X, K_X + \Delta)$ is finitely generated.*

As explained in Chapter 1, there are several issues when trying to prove the finite generation by induction on the dimension. The main conceptual problem is the finite generation of the kernel of the restriction map. Note that the "kernel issue" did not exist in the case of pl flips, since the relative Picard number $= 1$ ensured that the kernel was a principal ideal, at least after shrinking the base and passing to a truncation. However, the proof of Lemma 3.18 models the general lines of the proof in Section 6.2.

It is natural to try and restrict to a component of Δ, the issue of course being that (X, Δ) does not have log canonical centres. Therefore I allow restrictions to components of some effective divisor $D \sim_{\mathbb{Q}} K_X+\Delta$, and a tie-breaking-like technique allows to create log canonical centres. Algebras encountered this way are, in effect, plt algebras, and proving their restriction is finitely generated is technically the most involved part of the proof, see Section 6.1.

Since the algebras I consider are of higher rank, not all divisors will have the same log canonical centres. I therefore restrict to available centres, and lift generators from algebras that live on different divisors. Since the restrictions will also be algebras of higher rank, the induction process must start from them.

Thus, the main technical result of this chapter is the following.

Theorem 6.2. *Let X be a smooth projective variety, and for $i = 1, \dots, \ell$ let $D_i = k_i(K_X + \Delta_i + A)$, where A is an ample $\mathbb{Q}$-divisor and $(X, \Delta_i + A)$ is a log smooth log canonical pair with $|D_i| \neq \emptyset$. Assume Property $\mathcal{L}_A^G$ in dimensions $\leq \dim X$. Then the Cox ring $R(X; D_1, \dots, D_\ell)$ is finitely generated.*

Property $\mathcal{L}_A^G$ in the statement of Theorems 6.1 and 6.2 describes the convex geometry of the set of log canonical pairs with big boundaries in terms of divisorial components of the stable base loci. More precisely:

Property $\mathcal{L}_A^G$. *Let X be a smooth variety projective over an affine variety Z, B a simple normal crossings divisor on X and A a general ample $\mathbb{Q}$-divisor. Let $V \subset \operatorname{Div}(X)_{\mathbb{R}}$ be the vector space spanned by the components of B and let $\mathcal{L}_V = \{\Theta \in V : (X, \Theta)$ is log canonical$\}$; this is a rational polytope in V. Then for any component G of B, the set*

$$\mathcal{L}_A^G = \{\Phi \in \mathcal{L}_V : G \not\subset \mathbf{B}(K_X + \Phi + A)\}$$

is a rational polytope.

As a demonstration, I show how results of [BCHM06] imply Property $\mathcal{L}_A^G$. Of course, a hope is that this will be proved without Mori theory.

Proposition 6.3. *Property $\mathcal{L}_A^G$ follows from the MMP.*

Proof. Let K_X be a divisor with $\mathcal{O}_X(K_X) \simeq \omega_X$ and $\operatorname{Supp} A \not\subset \operatorname{Supp} K_X$, and let Λ be the monoid in $\operatorname{Div}(X)$ generated by the components of K_X, B and A. Let $\iota \colon \Lambda \to \Lambda$ be the identity map, and denote $\mathcal{S} = \mathbb{R}_+(K_X + A + \mathcal{L}_V) \cap \Lambda$. Since $\mathcal{L}_V$ is a rational polytope, $\mathcal{S}$ is a finitely generated monoid and let D_i be generators of $\mathcal{S}$. By [BCHM06, Corollary 1.1.9], the Cox ring $R(X; D_1, \dots, D_k)$ is finitely generated, thus so is the algebra $R(X, \mathcal{S})$ by projection. The set $\mathcal{M} = \{D \in \mathcal{S} : |D|_{\mathbb{Q}} \neq \emptyset\}$ is a convex cone, and therefore finitely generated since $R(X, \mathcal{S})$ is finitely generated, so I can assume $\mathcal{M} = \mathcal{S}$. By Proposition 5.11, the map $\mathbf{Mob}_\iota$ is piecewise additive up to truncation, which proves that the closure $\mathcal{C}$ of the set $(\mathcal{L}_A^G)_\mathbb{Q}$ is a rational polytope, and I claim it equals $\mathcal{L}_A^G$. Otherwise there exists $\Phi \in \mathcal{L}_A^G \backslash \mathcal{C}$, and therefore the convex hull of the set $\mathcal{C} \cup \{\Phi\}$, which is by convexity a subset of $\mathcal{L}_A^G$, contains a rational point $\Phi' \in \mathcal{L}_A^G \backslash \mathcal{C}$, a contradiction. $\square$

6.1 Restricting Plt Algebras

In this section I establish one of the technically most difficult steps in the proof of Theorem 6.2. Crucial results and techniques will be those used to prove Nonvanishing theorem in [Hac08] using methods developed in [HM08], and the techniques of Chapter 4.

The key result is Theorem 3.19, which also immediately implies:

Corollary 6.4. *Let $\pi\colon X \to Z$ be a projective morphism to a normal affine variety Z, where $(X, \Delta = S + A + B)$ is a purely log terminal pair, $S = \lfloor \Delta \rfloor$ is irreducible, (X, S) is log smooth, A is a general ample $\mathbb{Q}$-divisor and $(S, \Omega + A_{|S})$ is canonical, where $\Omega = (\Delta - S)_{|S}$. Assume $S \not\subset \mathbf{B}(K_X + \Delta)$, and let $\Phi_m = \Omega \wedge \frac{1}{m} \operatorname{Fix} |m(K_X + \Delta)|_S$ for every m such that $m\Delta$ is Cartier. Then*

$$|m(K_S + \Omega - \Phi_m)| + m\Phi_m = |m(K_X + \Delta)|_S.$$

The following lemma shows that finite generation implies certain boundedness on the convex geometry of boundaries, and it will be used in the proof of Theorem 6.6 below.

Lemma 6.5. *Let $(X, \Delta = B + A)$ be a log smooth klt pair, where A is a general ample $\mathbb{Q}$-divisor, B is an effective $\mathbb{R}$-divisor, and assume that no component of B is in $\mathbf{B}(K_X + \Delta)$. Assume Property $\mathcal{L}_A^G$ and Theorem 6.2 in dimension $\dim X$. Let $V \subset \operatorname{Div}(X)_\mathbb{R}$ be the vector space spanned by the components of B and $W \subset V$ the smallest rational affine subspace containing B. Then there is a constant $\eta > 0$ and a positive integer r such that if $\Phi \in W$ and k is a positive integer such that $\|\Phi - B\| < \eta$ and $k(K_X + \Phi + A)/r$ is Cartier, then no component of B is in $\operatorname{Fix} |k(K_X + \Phi + A)|$.*

Proof. Let K_X be a divisor such that $\mathcal{O}_X(K_X) \simeq \omega_X$ and $\operatorname{Supp} A \not\subset \operatorname{Supp} K_X$, and let $\Lambda \subset \operatorname{Div}(X)$ be the monoid spanned by components of K_X, B and A. Let G be a components of B. By Property $\mathcal{L}_A^G$ there is a rational polytope $\mathcal{P} \subset W$ such that $\Delta \in \operatorname{relint} \mathcal{P}$ and $G \not\subset \mathbf{B}(K_X+\Phi+A)$ for every $\Phi \in \mathcal{P}$. Let $D_1, \ldots, D_\ell$ be generators of $\mathbb{R}_+(K_X + A + \mathcal{P}) \cap \Lambda$. By Theorem 6.2 the Cox ring $R(X; D_1, \ldots, D_\ell)$ is finitely generated, and thus so is the algebra $R(X, \Lambda)$ by projection. By Proposition 5.11 there is a rational polyhedral cone $\mathcal{C} \subset \Lambda_\mathbb{R}$ such that $\Delta \in \mathcal{C}$ and the map $\mathbf{Mob}_{\iota|\mathcal{C}\cap\Lambda^{(r)}}$ is additive for some positive integer r, where $\iota\colon \Lambda \to \Lambda$ is the identity map. In particular, if $\Phi \in \mathcal{C}\cap\mathcal{P}$ and $k(K_X+\Phi+A)/r$ is Cartier, then $G \not\subset \operatorname{Fix}|k(K_X+\Phi+A)|$. Pick η such that $\Phi \in \mathcal{C} \cap \mathcal{P}$ whenever $\Phi \in W$ and $\|\Phi - \Delta\| < \eta$. We can take η and r to work for all components of B, and we are done. □

The rest of this section is devoted to proving the following main technical result.

Theorem 6.6. *Let X be a smooth variety, S a smooth prime divisor and A a very general ample $\mathbb{Q}$-divisor on X. For $i = 1, \ldots, \ell$ let $D_i = k_i(K_X + \Delta_i)$, where $(X, \Delta_i = S + B_i + A)$ is a log smooth plt pair with $\lfloor \Delta_i \rfloor = S$ and $|D_i| \neq \emptyset$. Assume Property $\mathcal{L}_A^G$ in dimensions $\leq \dim X$ and Theorem 6.2 in dimension $\dim X - 1$. Then the algebra $\operatorname{res}_S R(X; D_1, \ldots, D_\ell)$ is finitely generated.*

Proof. Step 1. I first show that we can assume $S \notin \operatorname{Fix}|D_i|$ for all i.

To prove this, let K_X be a divisor with $\mathcal{O}_X(K_X) \simeq \omega_X$ and $\operatorname{Supp} A \not\subset \operatorname{Supp} K_X$, and let Λ be the monoid in $\operatorname{Div}(X)$ generated by the components of K_X and all Δ_i. Denote $\mathcal{C}_S = \{P \in \Lambda_\mathbb{R} : S \notin \mathbf{B}(P)\}$. By Property $\mathcal{L}_A^G$, the set $\mathcal{A} = \sum_i \mathbb{R}_+ D_i \cap \mathcal{C}_S$ is a rational polyhedral cone.

The monoid $\sum_{i=1}^\ell \mathbb{R}_+ D_i \cap \Lambda$ is finitely generated and let $P_1, \ldots, P_q$ be its generators with $P_i = D_i$ for $i = 1, \ldots, \ell$. Let $\mu\colon \bigoplus_{i=1}^q \mathbb{N}e_i \to \operatorname{Div}(X)$ be an additive map from a simplicial monoid such that $\mu(e_i) = P_i$. Therefore $\mathcal{S} = \mu^{-1}(\mathcal{A}\cap\Lambda)\cap\bigoplus_{i=1}^\ell \mathbb{N}e_i$ is a finitely generated monoid and let $h_1, \ldots, h_m$ be generators of $\mathcal{S}$, and observe that $\mu(h_i)$ is a multiple of an adjoint bundle for every i.

Since $\operatorname{res}_S H^0(X, \mu(s)) = 0$ for every $s \in \big(\bigoplus_{i=1}^\ell \mathbb{N}e_i\big)\backslash\mathcal{S}$, we have that the algebra $\operatorname{res}_S R(X, \mu(\bigoplus_{i=1}^\ell \mathbb{N}e_i)) = \operatorname{res}_S R(X; D_1, \ldots, D_\ell)$ is finitely generated if and only if

$\operatorname{res}_S R(X, \mu(\mathcal{S}))$ is. Since we have the diagram

$$\begin{array}{ccc} R(X;\mu(h_1),\dots,\mu(h_m)) & \longrightarrow & R(X,\mu(\mathcal{S})) \\ \downarrow & & \downarrow \\ \operatorname{res}_S R(X;\mu(h_1),\dots,\mu(h_m)) & \longrightarrow & \operatorname{res}_S R(X,\mu(\mathcal{S})) \end{array}$$

where the horizontal maps are natural projections and the vertical maps are restrictions to S, it is enough to prove that the algebra $\operatorname{res}_S R(X;\mu(h_1),\dots,\mu(h_m))$ is finitely generated. By passing to a truncation, I can assume further that $S \notin \operatorname{Fix}|\mu(h_i)|$ for $i = 1,\dots,m$.

Step 2. Therefore I can assume $\mathcal{S} = \bigoplus_{i=1}^{\ell} \mathbb{N}e_i$ and $\mu(e_i) = D_i$ for every i. For $s = \sum_{i=1}^{\ell} t_i e_i \in \mathcal{S}_{\mathbb{Q}}$ and $t_s = \sum_{i=1}^{\ell} t_i k_i$, denote $\Delta_s = \sum_{i=1}^{\ell} t_i k_i \Delta_i / t_s$ and $\Omega_s = (\Delta_s - S)_{|S}$. Observe that

$$R(X; D_1,\dots,D_\ell) = \bigoplus_{s\in\mathcal{S}} H^0(X, t_s(K_X + \Delta_s)).$$

In this step I show that we can assume that $(S, \Omega_s + A_{|S})$ is terminal for every $s \in \mathcal{S}_{\mathbb{Q}}$.

Let $\sum F_k = \bigcup_i \operatorname{Supp} B_i$, and denote $\mathbf{B}_i = \mathbf{B}(X, \Delta_i)$ and $\mathbf{B} = \mathbf{B}(X, S + \nu \sum_k F_k + A)$, where $\nu = \max_{i,k}\{\operatorname{mult}_{F_k} B_i\}$. By Lemma 2.7 there is a log resolution $f\colon Y \to X$ such that the components of $\{\mathbf{B}_Y\}$ do not intersect, and denote $D_i' = k_i(K_Y + \mathbf{B}_{iY})$. Observe that

$$R(X; D_1,\dots,D_\ell) \simeq R(Y; D_1',\dots,D_\ell'). \tag{6.1}$$

Since $B_i \leq \nu \sum_k F_k$, by comparing discrepancies we see that the components of $\{\mathbf{B}_{iY}\}$ do not intersect for every i, and notice that $f^*A = f_*^{-1}A \leq \mathbf{B}_{iY}$ for every i since A is very general. For $s = \sum_{i=1}^{\ell} t_i e_i \in \mathcal{S}_{\mathbb{Q}}$ and $t_s = \sum_{i=1}^{\ell} t_i k_i$, denote $\Delta_s' = \sum_{i=1}^{\ell} t_i k_i \mathbf{B}_{iY}/t_s$. Let H be a small effective f-exceptional $\mathbb{Q}$-divisor such that $A' \sim_{\mathbb{Q}} f^*A - H$ is a general ample $\mathbb{Q}$-divisor, and let $T = f_*^{-1}S$. Then, setting $\Psi_s = \Delta_s' - f^*A - T + H \geq 0$ and $\Omega_s' = \Psi_{s|T} + A'_{|T}$, the pair $(T, \Omega_s' + A'_{|T})$ is terminal and $K_Y + T + \Psi_s + A' \sim_{\mathbb{Q}} K_Y + \Delta_s'$. Now replace X by Y, S by T, Δ_s by $T + \Psi_s + A'$ and Ω_s by Ω_s'.

Step 3. For every $s \in \mathcal{S}$, denote $F_s = \frac{1}{t_s}\operatorname{Fix}|t_s(K_X + \Delta_s)|_S$ and $F_s^\sharp = \liminf\limits_{m\to\infty} F_{ms}$.

Define the maps $\Theta\colon \mathcal{S}\to \mathrm{Div}(S)_\mathbb{Q}$ and $\Theta^\sharp\colon \mathcal{S}\to \mathrm{Div}(S)_\mathbb{Q}$ by

$$\Theta(s)=\Omega_s-\Omega_s\wedge F_s,\qquad \Theta^\sharp(s)=\Omega_s-\Omega_s\wedge F_s^\sharp.$$

Then, denoting $\Theta_s=\Theta(s)$ and $\Theta_s^\sharp=\Theta^\sharp(s)$, we have

$$\mathrm{res}_S\, R(X;D_1,\dots,D_\ell)\simeq\bigoplus_{s\in\mathcal{S}}H^0(S,t_s(K_S+\Theta_s))\tag{6.2}$$

by Corollary 6.4. Furthermore, for $s\in\mathcal{S}$ let $\varepsilon>0$ be a rational number such that $\varepsilon(K_X+\Delta_s)+A$ is ample. Then by Theorem 3.19 we have

$$|k_s(K_S+\Omega_s-\Phi_s)|+k_s\Phi_s\subset|k_s(K_X+\Delta_s)|_S$$

for any Φ_s and k_s such that $k_s\Delta_s,k_s\Phi_s\in\mathrm{Div}(X)$ and $\Omega_s\wedge(1-\frac{\varepsilon}{k_s})F_s\leq\Phi_s\leq\Omega_s$. Then similarly as in the proof of Theorem 3.20, by Lemma 6.5 we have that $\Omega_s\wedge F_s^\sharp$ is rational and

$$\mathrm{res}_S\, R(X,K_X+\Delta_s)^{(k_s^\sharp)}\simeq R(S,K_S+\Theta_s^\sharp)^{(k_s^\sharp)},\tag{6.3}$$

where $k_s^\sharp\Theta_s^\sharp$ and $k_s^\sharp\Delta_s$ are both Cartier. Note also, by the same proof, that $G\not\subset\mathbf{B}(K_S+\Theta_s^\sharp)$ for every component G of $\Theta_s^\sharp$. In particular, $\Theta_{k_s^\sharp ps}=\Theta_{k_s^\sharp s}=\Theta_s^\sharp$ for every $p\in\mathbb{N}$.

Define maps $\lambda\colon\mathcal{S}\to\mathrm{Div}(S)_\mathbb{Q}$ and $\lambda^\sharp\colon\mathcal{S}\to\mathrm{Div}(S)_\mathbb{Q}$ by

$$\lambda(s)=t_s(K_S+\Theta_s),\qquad \lambda^\sharp(s)=t_s(K_S+\Theta_s^\sharp).$$

Then $\lambda^\sharp$ extends to a function on $\mathcal{S}_\mathbb{R}$, and by Theorem 6.9 below, there is a finite rational polyhedral subdivision $\mathcal{S}_\mathbb{R}=\bigcup\mathcal{C}_i$ such that the map $\lambda^\sharp$ is linear on each $\mathcal{C}_i$. In particular, there is a sufficiently divisible positive integer κ such that $\kappa\lambda^\sharp(s)$ is Cartier for every $s\in\mathcal{S}$, and thus $\kappa\lambda^\sharp(s)=\lambda(\kappa s)$ for every $s\in\mathcal{S}$. Therefore the restriction of λ to $\mathcal{S}_i^{(\kappa)}$ is additive, where $\mathcal{S}_i=\mathcal{S}\cap\mathcal{C}_i$. If $s_1^i,\dots,s_z^i$ are generators of $\mathcal{S}_i^{(\kappa)}$, then the Cox ring $R(S;\lambda(s_1^i),\dots,\lambda(s_z^i))$ is finitely generated by Theorem 6.2, and so is the algebra $R(S,\lambda(\mathcal{S}_i^{(\kappa)}))$ by projection. Hence the algebra $\bigoplus_{s\in\mathcal{S}}H^0(S,\lambda(s))$ is finitely generated, and this together with (6.2) finishes the proof. □

It remains to prove that the map $\lambda^\sharp$ is rationally piecewise linear. Firstly we have the following result, which can be viewed as a global version of Lemma 6.5. Recall that $\mathcal{S} = \bigoplus_{i=1}^{\ell} \mathbb{N}e_i$.

Lemma 6.7. *There is a positive integer r such that the following stands. If $\Psi \in \operatorname{Div}(S)_{\mathbb{Q}}$ is such that $\operatorname{Supp}\Psi \subset \bigcup_{i=1}^{\ell} \operatorname{Supp}(\Omega_{e_i} - A_{|S})$ and no component of Ψ is in $\mathbf{B}(K_S + \Psi + A_{|S})$, then no component of Ψ is in $\operatorname{Fix}|k(K_S + \Psi + A_{|S})|$ for every k with $k(\Psi + A_{|S})/r$ Cartier.*

Proof. Let $\sum_{j=1}^{q} G_j = \bigcup_{i=1}^{\ell} \operatorname{Supp}(\Omega_{e_i} - A_{|S})$, and for each j let $\mathcal{P}_{G_j} = \{\Xi \in \sum_j [0,1]G_j : G_j \not\subset \mathbf{B}(K_S + \Xi + A_{|S})\}$. Each $\mathcal{P}_{G_j}$ is a rational polytope by Property $\mathcal{L}_A^G$. Let K_S be a divisor such that $\mathcal{O}_S(K_S) \simeq \omega_S$ and $\operatorname{Supp} A \not\subset \operatorname{Supp} K_X$, let $\mathcal{P}$ be the convex hull of all rational polytopes $K_S + A_{|S} + \mathcal{P}_{G_j}$, and set $\mathcal{C} = \mathbb{R}_+\mathcal{P}$. Observe that $K_S + \Psi + A_{|S} \in \mathcal{C}$. Let $G_{q+1}, \dots, G_w$ be the components of $K_S + A_{|S}$ not equal to G_j for $j = 1, \dots, q$, and let $\Lambda = \bigoplus_{j=1}^{w} \mathbb{N}G_j$. Then by Theorem 6.2 in dimension $\dim S$ the algebra $R(S, \mathcal{C} \cap \Lambda)$ is finitely generated and the map $\mathbf{Mob}_{\iota|\mathcal{C}\cap\Lambda^{(r)}}$ is piecewise additive for some r by Proposition 5.11, where $\iota\colon \Lambda \to \Lambda$ is the identity map. In particular, if $G_j \not\subset \mathbf{B}(K_S + \Psi + A_{|S})$ and $k(\Psi + A_{|S})/r$ is Cartier, then $G_j \not\subset \operatorname{Fix}|k(K_S + \Psi + A_{|S})|$. □

Theorem 6.8. *For any $s, t \in \mathcal{S}_{\mathbb{R}}$ we have*

$$\lim_{\varepsilon \downarrow 0} \Theta^{\sharp}_{s+\varepsilon(t-s)} = \Theta^{\sharp}_{s}.$$

Proof. Step 1. First we will prove that $\Theta_s^{\sigma} = \Theta_s^{\sharp}$, where

$$\Theta_s^{\sigma} = \Omega_s - \Omega_s \wedge N_{\sigma}\|K_X + \Delta_s\|_S,$$

cf. Remark 2.24. I am closely following the proof of [Hac08, Theorem 7.16]. Let r be a positive integer as in Lemma 6.7, let $\phi < 1$ be the smallest positive coefficient of $\Omega_s - \Theta_s^{\sigma}$ if it exists, and set $\phi = 1$ otherwise. Let $V \subset \operatorname{Div}(X)_{\mathbb{R}}$ and $W \subset \operatorname{Div}(S)_{\mathbb{R}}$ be the smallest rational affine spaces containing Δ_s and Θ_s^{σ} respectively. Let $0 < \eta \ll 1$ be a rational number such that $\eta(K_X + \Delta_s) + \frac{1}{2}A$ is ample, and if $\Delta' \in V$ with $\|\Delta' - \Delta_s\| < \eta$, then $\Delta' - \Delta_s + \frac{1}{2}A$ is ample. Then by Lemma 2.27 there are rational points $(\Delta_i, \Theta_i) \in V \times W$ and integers $k_i \gg 0$ such that:

(1) we may write $\Delta_s = \sum r_i \Delta_i$ and $\Theta_s^{\sigma} = \sum r_i \Theta_i$, where $r_i > 0$ and $\sum r_i = 1$,

(2) $k_i\Delta_i/r$ are integral and $\|\Delta_s - \Delta_i\| < \phi\eta/2k_i$,

(3) $k_i\Theta_i/k_s$ are integral, $\|\Theta_s^\sigma - \Theta_i\| < \phi\eta/2k_i$ and observe that $\Theta_i \leq \Omega_i$ since $k_i \gg 0$ and $(\Delta_i, \Theta_i) \in V \times W$.

Step 2. Set $A_i = A/k_i$ and $\Omega_i = (\Delta_i - S)_{|S}$. In this step I prove that for any component $P \in \operatorname{Supp}\Omega_s$, and for any $l > 0$ sufficiently divisible, we have

$$\operatorname{mult}_P(\Omega_i \wedge \tfrac{1}{l}\operatorname{Fix}|l(K_X + \Delta_i + A_i)|_S) \leq \operatorname{mult}_P(\Omega_i - \Theta_i). \tag{6.4}$$

If $\phi = 1$, (6.4) follows immediately from Lemma 2.25. Now assume $0 < \phi < 1$. Since $\|\Omega_s - \Omega_i\| < \phi\eta/2k_i$ and $\|\Theta_s^\sigma - \Theta_i\| < \phi\eta/2k_i$, it suffices to show that

$$\operatorname{mult}_P(\Omega_i \wedge \tfrac{1}{l}\operatorname{Fix}|l(K_X + \Delta_i + A_i)|_S) \leq (1 - \tfrac{\eta}{k_i})\operatorname{mult}_P(\Omega_s - \Theta_s^\sigma).$$

Let $\delta > \eta/k_i$ be a rational number such that $\delta(K_X + \Delta_i) + \frac{1}{2}A_i$ is ample. Since

$$K_X + \Delta_i + A_i = (1-\delta)(K_X + \Delta_i + \tfrac{1}{2}A_i) + \big(\delta(K_X + \Delta_i) + \tfrac{1+\delta}{2}A_i\big),$$

we have

$$\operatorname{ord}_P\|K_X + \Delta_i + A_i\|_S \leq (1-\delta)\operatorname{ord}_P\|K_X + \Delta_i + \tfrac{1}{2}A_i\|_S,$$

and thus

$$\operatorname{mult}_P \tfrac{1}{l}\operatorname{Fix}|l(K_X + \Delta_i + A_i)|_S \leq (1 - \tfrac{\eta}{k_i})\sigma_P\|K_X + \Delta_i\|_S$$

for l sufficiently divisible, cf. Lemma 2.25.

Step 3. In this step we prove that there exists an effective divisor H' on X not containing S such that for all sufficiently divisible positive integers m we have

$$|m(K_S + \Theta_i)| + m(\Omega_i - \Theta_i) + (mA_i + H')_{|S} \subset |m(K_X + \Delta_i) + mA_i + H'|_S. \tag{6.5}$$

First observe that since $S \not\subset \mathbf{B}(K_X + \Delta_i)$, we have $S \not\subset \operatorname{Bs}|m(K_X + \Delta_i + A_i)|$ for m sufficiently divisible. Assume further that m is divisible by l, for l as in Step 2. Let $f\colon Y \to X$ be a log resolution of $(X, \Delta_i + A_i)$ and of $|m(K_X + \Delta_i + A_i)|$. Let

$\Gamma = \mathbf{B}(X, \Delta_i + A_i)_Y$ and $E = K_Y + \Gamma - f^*(K_X + \Delta_i + A_i)$, and define

$$\Xi = \Gamma - \Gamma \wedge \tfrac{1}{m} \operatorname{Fix} |m(K_Y + \Gamma)|.$$

We have that $m(K_Y + \Xi)$ is Cartier, $\operatorname{Fix} |m(K_Y + \Xi)| \wedge \Xi = 0$ and $\operatorname{Mob}(m(K_Y + \Xi))$ is free. Since $\operatorname{Fix} |m(K_Y + \Xi)| + \Xi$ has simple normal crossings support, it follows that $\mathbf{B}(K_Y + \Xi)$ contains no log canonical centres of $(Y, \lceil \Xi \rceil)$. Let $T = f_*^{-1} S, \Gamma_T = (\Gamma - T)_{|T}$ and $\Xi_T = (\Xi - T)_{|T}$, and consider a section

$$\sigma \in H^0(T, \mathcal{O}_T(m(K_T + \Xi_T))) = H^0(T, \mathcal{J}_{\|m(K_T+\Xi_T)\|}(m(K_T + \Xi_T))).$$

(cf. Lemma 2.15). By Theorem 2.18, there is an ample divisor H on Y such that if $\tau \in H^0(T, \mathcal{O}_T(H))$, then $\sigma \cdot \tau$ is in the image of the homomorphism

$$H^0(Y, \mathcal{O}_Y(m(K_Y + \Xi) + H)) \to H^0(T, \mathcal{O}_T(m(K_Y + \Xi) + H)).$$

Therefore

$$|m(K_T + \Xi_T)| + m(\Gamma_T - \Xi_T) + H_{|T} \subset |m(K_Y + \Gamma) + H|_T. \tag{6.6}$$

We claim that

$$\Omega_i + A_{i|S} \geq (f_{|T})_* \Xi_T \geq \Theta_i + A_{i|S} \tag{6.7}$$

and so, as $(S, \Omega_i + A_{i|S})$ is canonical, we have

$$|m(K_S + \Theta_i)| + m((f_{|T})_* \Xi_T - \Theta_i) \subset |m(K_S + (f_{|T})_* \Xi_T)| = (f_{|T})_* |m(K_T + \Xi_T)|.$$

Pushing forward the inclusion (6.6), we obtain (6.5) for $H' = f_* H$.

We will now prove the inequality (6.7) claimed above. We have $\Xi_T \leq \Gamma_T$ and $(f_{|T})_* \Gamma_T = \Omega_i + A_{i|S}$ and so the first inequality follows.

In order to prove the second inequality, let P be any prime divisor on S and let $P' = (f_{|T})_*^{-1} P$. Assume that $P \subset \operatorname{Supp} \Omega_i$, and thus $P' \subset \operatorname{Supp} \Gamma_T$. Then there is a component Q of the support of Γ such that

$$\operatorname{mult}_{P'} \operatorname{Fix} |m(K_Y + \Gamma)|_T = \operatorname{mult}_Q \operatorname{Fix} |m(K_Y + \Gamma)|$$

and $\operatorname{mult}_{P'}\Gamma_T = \operatorname{mult}_Q \Gamma$. Therefore

$$\operatorname{mult}_{P'}\Xi_T = \operatorname{mult}_{P'}\Gamma_T - \min\{\operatorname{mult}_{P'}\Gamma_T, \operatorname{mult}_{P'}\tfrac{1}{m}\operatorname{Fix}|m(K_Y+\Gamma)|_T\}.$$

Notice that $\operatorname{mult}_{P'}\Gamma_T = \operatorname{mult}_P(\Omega_i + A_{i|S})$ and since $E_{|T}$ is exceptional, we have that

$$\operatorname{mult}_{P'}\operatorname{Fix}|m(K_Y+\Gamma)|_T = \operatorname{mult}_P \operatorname{Fix}|m(K_X+\Delta_i+A_i)|_S.$$

Therefore $(f_{|T})_*\Xi_T = \Omega_i + A_{i|S} - \Omega_i \wedge \frac{1}{m}\operatorname{Fix}|m(K_X+\Delta_i+A_i)|_S$. The inequality now follows from Step 2.

Step 4. In this step we prove

$$|k_i(K_S+\Theta_i)| + k_i(\Omega_i - \Theta_i) \subset |k_i(K_X+\Delta_i)|_S. \tag{6.8}$$

For any $\Sigma \in |k_i(K_S+\Theta_i)|$ and any $m > 0$ sufficiently divisible, we may choose a divisor $G \in |m(K_X+\Delta_i)+mA_i+H|$ such that $G_{|S} = \frac{m}{k_i}\Sigma + m(\Omega_i - \Theta_i) + (mA_i+H)_{|S}$. If we define $\Lambda = \frac{k_i-1}{m}G + \Delta_i - S - A$, then

$$k_i(K_X+\Delta_i) \sim_{\mathbb{Q}} K_X + S + \Lambda + A_i - \tfrac{k_i-1}{m}H,$$

where $A_i - \frac{k_i-1}{m}H$ is ample as $m \gg 0$. By Lemma 2.12, we have a surjective homomorphism

$$H^0(X, \mathcal{J}_{S,\Lambda}(k_i(K_X+\Delta_i))) \to H^0(S, \mathcal{J}_{\Lambda_{|S}}(k_i(K_X+\Delta_i))).$$

Since (S,Ω_i) is canonical, $(S, \Omega_i + \frac{k_i-1}{m}H_{|S})$ is klt as $m \gg 0$, and so $\mathcal{J}_{\Omega_t + \frac{k_i-1}{m}H_{|S}} = \mathcal{O}_S$. Since

$$\Lambda_{|S} - (\Sigma + k_i(\Omega_i - \Theta_i)) = \tfrac{k_i-1}{m}G_{|S} + \Omega_i - A_{|S} - (\Sigma + k_i(\Omega_i - \Theta_i)) \le \Omega_i + \tfrac{k_i-1}{m}H_{|S},$$

then by Lemma 2.10(3) we have $\mathcal{I}_{\Sigma + k_i(\Omega_i - \Theta_i)} \subset \mathcal{J}_{\Lambda_{|S}}$, and so

$$\Sigma + k_i(\Omega_i - \Theta_i) \in |k_i(K_X+\Delta_i)|_S,$$

which proves (6.8).

Step 5. There are ample divisors A_n with $\operatorname{Supp} A_n \subset \operatorname{Supp}(\Delta_s - S)$ such that $\|A_n\| \to 0$ and $\Delta_s + A_n$ are $\mathbb{Q}$-divisors. Observe that $\Theta_s^\sigma = \lim_{n\to\infty} \Theta_n^\sigma$ with

$$\Theta_n^\sigma = \Omega_n - \Omega_n \wedge N_\sigma\|K_X + \Delta_n\|_S,$$

where $\Delta_n = \Delta_s + A_n$ and $\Omega_n = (\Delta_n - S)_{|S}$. Note that

$$N_\sigma\|K_X + \Delta_n\|_S = \sum \operatorname{ord}_P \|K_X + \Delta_n\|_S \cdot P$$

for all prime divisors P on S for all n, cf. Remark 2.24. But then as in Step 3 of the proof of Theorem 6.6, no component of Θ_n^σ is in $\mathbf{B}(K_S + \Theta_n^\sigma)$, and thus, by Property $\mathcal{L}_A^G$ and since $\Theta_n^\sigma \geq \Theta_s^\sigma$ for every n, no component of Θ_s^σ is in $\mathbf{B}(K_S + \Theta_s^\sigma)$. Since k_i is divisible by r and $\Theta_i \in W$, by (6.8) we have

$$\Omega_i - \Theta_i \geq \Omega_i \wedge \tfrac{1}{k_i} \operatorname{Fix} |k_i(K_X + \Delta_i)|_S \geq \Omega_i - \Theta_i^\sharp,$$

and so $\Theta_i^\sharp \geq \Theta_i$, where

$$\Theta_i^\sharp = \Omega_i - \Omega_i \wedge \liminf_{m\to\infty} \tfrac{1}{m} \operatorname{Fix} |m(K_X + \Delta_i)|_S.$$

Let P be a prime divisor on S. If $\operatorname{mult}_P \Theta_s^\sigma = 0$, then $\operatorname{mult}_P \Theta_s^\sharp = 0$ since $\Theta_s^\sigma \geq \Theta_s^\sharp$ by Lemma 2.25. Otherwise $\operatorname{mult}_P \Theta_i > 0$ for all i and thus $\operatorname{mult}_P \Theta_i^\sharp > 0$. Therefore by concavity we have

$$\operatorname{mult}_P \Theta_s^\sharp \geq \sum r_i \operatorname{mult}_P \Theta_i^\sharp \geq \sum r_i \operatorname{mult}_P \Theta_i = \operatorname{mult}_P \Theta_s^\sigma,$$

proving the claim from Step 1.

Step 6. Now let C be an ample $\mathbb{Q}$-divisor such that $\Delta_t - \Delta_s + C$ is ample. Then by the claim from Step 1 and by Lemma 2.25,

$$\begin{aligned}\Omega_s - \Theta_s^\sharp &= \Omega_s \wedge \lim_{\varepsilon\downarrow 0} \big(\sum \operatorname{ord}_P \|K_X + \Delta_s + \varepsilon(\Delta_t - \Delta_s + C)\|_S \cdot P\big)\\ &\leq \Omega_s \wedge \lim_{\varepsilon\downarrow 0} \big(\sum \operatorname{ord}_P \|K_X + \Delta_s + \varepsilon(\Delta_t - \Delta_s)\|_S \cdot P\big) \leq \Omega_s - \Theta_s^\sharp,\end{aligned}$$

where the last inequality follows from convexity. Therefore that inequality is an equality, and this completes the proof. $\square$

Now, let Z be a prime divisor on S and let $\mathcal{L}_Z$ be the closure in $\mathcal{S}_\mathbb{R}$ of the set $\{s \in \mathcal{S}_\mathbb{R} : \operatorname{mult}_Z \Theta^\sharp_s > 0\}$. Then $\mathcal{L}_Z$ is a closed cone. Let $\lambda^\sharp_Z \colon \mathcal{S}_\mathbb{R} \to \mathbb{R}$ be the function given by $\lambda^\sharp_Z(s) = \operatorname{mult}_Z \lambda^\sharp(s)$, and similarly for $\Theta^\sharp_Z$.

Theorem 6.9. *For every prime divisor Z on S, the map $\lambda^\sharp_Z$ is rationally piecewise linear. Therefore, $\lambda^\sharp$ is rationally piecewise linear.*

Proof. Let $G_1, \dots, G_w$ be prime divisors on X not equal to S and $\operatorname{Supp} A$ such that $\operatorname{Supp}(\Delta_s - S - A) \subset \sum G_i$ for every $s \in \mathcal{S}$. Let $\nu = \max\{\operatorname{mult}_{G_i} \Delta_s : s \in \mathcal{S}, i = 1, \dots, w\} < 1$, and let $0 < \eta \ll 1 - \nu$ be a rational number such that $A - \eta \sum G_i$ is ample. Let $A' \sim_\mathbb{Q} A - \eta \sum G_i$ be a general ample $\mathbb{Q}$-divisor. Define $\Delta'_s = \Delta_s - A + \eta \sum G_i + A' \geq 0$, and observe that $\Delta'_s \sim_\mathbb{Q} \Delta_s$, $\lfloor \Delta'_s \rfloor = S$ and $(S, (\Delta'_s - S)_{|S})$ is terminal.

Define the map $\chi \colon \mathcal{S} \to \operatorname{Div}(X)$ by $\chi(s) = \kappa t_s(K_X + \Delta'_s)$, for κ sufficiently divisible. Then as before, we can construct maps $\tilde{\Theta}^\sharp \colon \mathcal{S}_\mathbb{R} \to \operatorname{Div}(S)_\mathbb{R}$, $\tilde{\lambda}^\sharp \colon \mathcal{S}_\mathbb{R} \to \operatorname{Div}(S)_\mathbb{R}$ and $\tilde{\lambda}^\sharp_Z \colon \mathcal{S}_\mathbb{R} \to \mathbb{R}$ associated to χ. By construction, $\operatorname{ord}_E \|\tilde{\lambda}^\sharp_s / \kappa t_s\|_S = \operatorname{ord}_E \|\lambda^\sharp_s / t_s\|_S$, and thus $\operatorname{mult}_Z \tilde{\Theta}^\sharp_s = \operatorname{mult}_Z \Theta^\sharp_s + \eta$ for every $s \in \mathcal{L}_Z$. Let $\tilde{\mathcal{L}}_Z$ be the closure in $\mathcal{S}_\mathbb{R}$ of the set $\{s \in \mathcal{S}_\mathbb{R} : \operatorname{mult}_Z \tilde{\Theta}^\sharp_s > 0\}$, and thus $\mathcal{L}_Z$ is the closure in $\mathcal{S}_\mathbb{R}$ of the set $\{s \in \mathcal{S}_\mathbb{R} : \operatorname{mult}_Z \tilde{\Theta}^\sharp_s > \eta\}$. Note that $\operatorname{mult}_Z \tilde{\Theta}^\sharp_s \geq \eta$ for every $s \in \mathcal{L}_Z$ by Theorem 6.8. Now for every face $\mathcal{F}$ of $\mathcal{S}_\mathbb{R}$, either $\mathcal{F} \cap \mathcal{L}_Z \subset \operatorname{relint}(\mathcal{F} \cap \tilde{\mathcal{L}}_Z)$ or $\partial(\mathcal{F} \cap \mathcal{L}_Z) \cap \partial(\mathcal{F} \cap \tilde{\mathcal{L}}_Z) \subset \partial \mathcal{F}$. Therefore by compactness there is a rational polyhedral cone $\mathcal{M}_Z$ such that $\mathcal{L}_Z \subset \mathcal{M}_Z \subset \tilde{\mathcal{L}}_Z$, and so the map $\tilde{\lambda}^\sharp_Z|_{\mathcal{M}_Z}$ is *superlinear.*

By Theorem 6.11 below, for any 2-plane $H \subset \mathbb{R}^\ell$ the map $\tilde{\lambda}^\sharp_Z|_{\mathcal{M}_Z \cap H}$ is piecewise linear, and thus $\tilde{\lambda}^\sharp_Z|_{\mathcal{M}_Z}$ is piecewise linear by Lemma 4.13.

To prove that $\tilde{\lambda}^\sharp_Z|_{\mathcal{M}_Z}$ is rationally piecewise linear, let $k = \dim \mathcal{M}_Z$ and let $\mathcal{M}_Z = \bigcup \mathcal{C}_m$ be a finite polyhedral decomposition such that $\tilde{\lambda}^\sharp_Z|_{\mathcal{C}_m}$ is linear for every m. Let $\mathcal{H}$ be a hyperplane which contains a common $(k-1)$-dimensional face of cones $\mathcal{C}_i$ and $\mathcal{C}_j$ and assume $\mathcal{H}$ is not rational. By Step 1 of the proof of Lemma 4.10 there is a point $s \in \mathcal{C}_i \cap \mathcal{C}_j$ such that the minimal affine rational space containing s has dimension $k-1$. Then as in Step 1 of the proof of Theorem 6.11 there is an k-dimensional cone $\widetilde{\mathcal{C}}$ such that $s \in \operatorname{int} \widetilde{\mathcal{C}}$ and the map $\tilde{\lambda}^\sharp_Z|_{\widetilde{\mathcal{C}}}$ is linear. But then the cones $\widetilde{\mathcal{C}} \cap \mathcal{C}_i$ and $\widetilde{\mathcal{C}} \cap \mathcal{C}_i$ are k-dimensional and linear extensions of $\tilde{\lambda}^\sharp_Z|_{\mathcal{C}_i}$ and $\tilde{\lambda}^\sharp_Z|_{\mathcal{C}_j}$ coincide since they are equal to the linear extension of $\tilde{\lambda}^\sharp_Z|_{\widetilde{\mathcal{C}}}$, a contradiction.

Therefore all $(k-1)$-dimensional faces of the cones $\mathcal{C}_i$ belong to rational hyperplanes and thus $\mathcal{C}_i$ are rational cones.

Therefore the map $\tilde{\lambda}^\sharp_Z|_{\mathcal{M}_Z}$ is rationally piecewise linear, and since $\mathcal{L}_Z$ is the closure of the set $\{s \in \mathcal{S}_\mathbb{R} : \operatorname{mult}_Z \tilde{\Theta}^\sharp_s > \eta\}$, we have that $\mathcal{L}_Z$ is a rational polyhedral cone, the map $\tilde{\lambda}^\sharp_Z|_{\mathcal{L}_Z}$ is rationally piecewise linear, and therefore so is $\lambda^\sharp_Z$. Now it is trivial that $\lambda^\sharp$ is a rationally piecewise linear map. □

Thus it remains to prove that $\lambda^\sharp_Z|_{\mathcal{M}_Z \cap H}$ is piecewise linear for every 2-plane $H \subset \mathbb{R}^\ell$. As in Step 1 of the proof of Theorem 6.6, by replacing $\mathcal{S}_\mathbb{R}$ by $\mathcal{M}_Z$ and $\lambda^\sharp_Z$ by $\tilde{\lambda}^\sharp_Z$, it is enough to assume, and I will until the end of the section, that $\lambda^\sharp_Z$ is a superlinear function on $\mathcal{S}_\mathbb{R}$ for a fixed prime divisor Z on S.

Let C_s be a local Lipschitz constant of $\Theta^\sharp$ around $s \in \mathcal{S}_\mathbb{R}$ in the smallest rational affine space containing s. For every $s \in \mathcal{S}$, let ϕ_s be the smallest coefficient of $\Omega_s - \Theta^\sharp_s$.

Theorem 6.10. *Fix $s \in \mathcal{S}_\mathbb{R}$ and let $U \subset \mathbb{R}^\ell$ be the smallest rational affine subspace containing s. If $\phi_s > 0$, let $0 < \delta \ll 1$ be a rational number such that $\phi_u > 0$ for $u \in U$ with $\|u - s\| \leq \delta$, set $\phi = \min\{\phi_u : u \in U, \|u - s\| \leq \delta\}$ and let $0 < \varepsilon \ll \delta$ be a rational number such that $(C_s/\phi + 1)\varepsilon(K_X + \Delta_s) + A$ is ample. If $\phi_s = 0$ and $\operatorname{Supp} \Delta_s = \sum F_i$, let $0 < \varepsilon \ll 1$ be a rational number such that $\sum f_i F_i + A$ is ample for any $f_i \in (-\varepsilon, \varepsilon)$, and set $\phi = 1$. Let $t \in U \cap \mathcal{S}_\mathbb{Q}$ and $k_t \gg 0$ be an integer such that $\|t - s\| < \varepsilon/k_t$, $k_t\Delta_t/r$ is Cartier for r as in Lemma 6.7 and $S \not\subset \mathbf{B}(K_X + \Delta_t)$. Then for any divisor Θ on S such that $\Theta \leq \Omega_t$, $\|\Theta - \Theta^\sharp_s\| < \phi\varepsilon/k_t$ and $k_t\Theta/r$ is Cartier we have*

$$|k_t(K_S + \Theta)| + k_t(\Omega_t - \Theta) \subset |k_t(K_X + \Delta_t)|_S.$$

Proof. Set $A_t = A/k_t$. I first prove that for any component $P \in \operatorname{Supp} \Omega_s$, and for any $l > 0$ sufficiently divisible, we have

$$\operatorname{mult}_P(\Omega_t \wedge \tfrac{1}{l} \operatorname{Fix} |l(K_X + \Delta_t + A_t)|_S) \leq \operatorname{mult}_P(\Omega_t - \Theta). \tag{6.9}$$

Assume first that $\phi_s = 0$. Then in particular $\operatorname{ord}_P \|K_X + \Delta_s\|_S = 0$ and $\Delta_t - \Delta_s + A_t$

is ample since $\|\Delta_t - \Delta_s\| < \varepsilon/k_t$, so

$$\begin{aligned}\operatorname{ord}_P \|K_X + \Delta_t + A_t\|_S &= \operatorname{ord}_P \|K_X + \Delta_s + (\Delta_t - \Delta_s + A_t)\|_S \\ &\leq \operatorname{ord}_P \|K_X + \Delta_s\|_S = 0.\end{aligned}$$

Since for l sufficiently divisible we have

$$\operatorname{mult}_P \tfrac{1}{l} \operatorname{Fix} |l(K_X + \Delta_t + A_t)|_S = \operatorname{ord}_P \|K_X + \Delta_t + A_t\|_S \tag{6.10}$$

as in Step 3 of the proof of Theorem 6.6, we obtain (6.9).

Now assume that $\phi_s \neq 0$ and set $C = C_s/\phi$. By Lipschitz continuity we have $\|\Theta_t^\sharp - \Theta_s^\sharp\| < C\phi\varepsilon/k_t$, so $\|\Theta_t^\sharp - \Theta\| < (C+1)\phi\varepsilon/k_t$. Therefore it suffices to show that

$$\operatorname{mult}_P(\Omega_t \wedge \tfrac{1}{l} \operatorname{Fix} |l(K_X + \Delta_t + A_t)|_S) \leq (1 - \tfrac{C+1}{k_t}\varepsilon) \operatorname{mult}_P(\Omega_t - \Theta_t^\sharp).$$

Since $k_t \gg 0$, we can choose a rational number $\eta > (C+1)\varepsilon/k_t$ such that $\eta(K_X + \Delta_t) + A_t$ is ample. From

$$K_X + \Delta_t + A_t = (1-\eta)(K_X + \Delta_t) + (\eta(K_X + \Delta_t) + A_t),$$

we have

$$\operatorname{ord}_P \|K_X + \Delta_t + A_t\|_S \leq (1-\eta) \operatorname{ord}_P \|K_X + \Delta_t\|_S,$$

and thus by (6.10),

$$\operatorname{mult}_P \tfrac{1}{l} \operatorname{Fix} |l(K_X + \Delta_t + A_t)|_S \leq (1 - \tfrac{C+1}{k_t}\varepsilon) \operatorname{ord}_P \|K_X + \Delta_t\|_S$$

for l sufficiently divisible.

Now the theorem follows as in Steps 3 and 4 of the proof of Theorem 6.8. □

Finally, we have

Theorem 6.11. *Fix $s \in \mathcal{S}_\mathbb{R}$ and let R be a ray in $\mathcal{S}_\mathbb{R}$ not containing s. Then there exists a ray $R' \subset \mathbb{R}_+ s + R$ not containing s such that the map $\lambda_Z^\sharp|_{\mathbb{R}_+ s + R'}$ is linear. In particular, for every 2-plane $H \subset \mathbb{R}^\ell$, the map $\lambda_Z^\sharp|_{\mathcal{S}_\mathbb{R} \cap H}$ is piecewise linear.*

Proof. Step 1. Let $U \subset \mathbb{R}^\ell$ be the smallest rational affine space containing s. In this step I prove that the map $\Theta^\sharp$ is linear in a neighbourhood of s contained in U.

Let ε and ϕ be as in Theorem 6.10. Let $W \subset \mathbb{R}^\ell$ and $V \subset \mathrm{Div}(S)_\mathbb{R}$ be the smallest rational affine spaces containing s and $\Theta_s^\sharp$ respectively, and let r be as in Lemma 6.7. By Lemma 2.27, there exist rational points $(t_i, \Theta'_{t_i}) \in W \times V$ and integers $k_{t_i} \gg 0$ such that:

(1) we may write $s = \sum r_{t_i} t_i$, $\Delta_s = \sum r_{t_i}\Delta_{t_i}$ and $\Theta_s^\sharp = \sum r_{t_i}\Theta'_{t_i}$, where $r_{t_i} > 0$ and $\sum r_{t_i} = 1$,

(2) $k_{t_i}\Delta_{t_i}/r$ are integral and $\|s - t_i\| < \varepsilon/k_{t_i}$,

(3) $k_{t_i}\Theta'_{t_i}/r$ are integral, $\|\Theta_s^\sharp - \Theta'_{t_i}\| < \phi\varepsilon/k_{t_i}$ and note that $\Theta'_{t_i} \leq \Omega_{t_i}$ since $k_{t_i} \gg 0$ and $(t_i, \Theta'_{t_i}) \in W \times V$.

Observe that $S \not\subset \mathbf{B}(K_X + \Delta_{t_i})$ since $t_i \in W$ for every i and $\varepsilon \ll 1$ by Property $\mathcal{L}_A^G$. By Theorem 6.10 we have that

$$|k_{t_i}(K_S + \Theta'_{t_i})| + k_{t_i}(\Omega_{t_i} - \Theta'_{t_i}) \subset |k_{t_i}(K_X + \Delta_{t_i})|_S.$$

Since $\Theta'_{t_i} \in V$ and $k_{t_i}\Theta'_{t_i}/r$ is Cartier, no component of Θ'_{t_i} is in $\mathrm{Fix}\,|k_{t_i}(K_S + \Theta'_{t_i})|$ for every i by Lemma 6.7. In particular,

$$\Omega_{t_i} - \Theta'_{t_i} \geq \Omega_{t_i} \wedge \tfrac{1}{k_{t_i}}\,\mathrm{Fix}\,|k_{t_i}(K_X + \Delta_{t_i})|_S \geq \Omega_{t_i} - \Theta^\sharp_{t_i},$$

and so

$$\Theta^\sharp_{t_i} \geq \Theta'_{t_i}.$$

But by assumption (1) and since the map $\Theta^\sharp_Z$ is concave, we have

$$\Theta^\sharp_Z(s) \geq \sum r_{t_i}\Theta^\sharp_Z(t_i) \geq \sum r_{t_i}\,\mathrm{mult}_Z\,\Theta'_{t_i} = \Theta^\sharp_Z(s),$$

which proves the statement by Lemma 4.5.

Step 2. Now assume $s \in \mathcal{S}_\mathbb{Q}$, $\phi_s = 0$ and fix $u \in R$ such that s and u belong to a rational affine subspace $\mathcal{P}$ of $\mathbb{R}^\ell$. Let $\Delta\colon \bigoplus_{i=1}^\ell \mathbb{R}e_i \to \mathrm{Div}(X)_\mathbb{R}$ be a linear map given by $\Delta(p_i) = \Delta_{p_i}$ for linearly independent points $p_1, \ldots, p_\ell \in \mathcal{P} \cap \mathcal{S}_\mathbb{Q}$, and then extended linearly. Observe that $\Delta(p) = \Delta_p$ for every $p \in \mathcal{P} \cap \mathcal{S}_\mathbb{R}$.

Let W be the smallest rational affine subspace containing s and u. If there is a sequence $s_n \in (s,u]$ such that $\lim_{n\to\infty} s_n = s$ and $\phi_{s_n} = 0$, then $\lambda^\sharp$ is linear on the cone $\mathbb{R}_+ s + \mathbb{R}_+ s_1$ by Lemma 4.5.

Therefore we can assume that there are rational numbers $0 < \varepsilon, \eta \ll 1$ such that for all $v \in [s,u]$ with $0 < \|v - s\| < 2\varepsilon$ we have $\phi_v > 0$, that for every prime divisor P on S, we have either $\operatorname{mult}_P \Omega_v > \operatorname{mult}_P \Theta_v^\sharp$ or $\operatorname{mult}_P \Omega_v = \operatorname{mult}_P \Theta_v^\sharp$ and either $\operatorname{mult}_P \Theta_v^\sharp = 0$ or $\operatorname{mult}_P \Theta_v^\sharp > 0$ for all such v, and that $\Delta_v - \Delta_s + \Xi + A$ is ample for all such v and for any divisor Ξ such that $\operatorname{Supp} \Xi \subset \operatorname{Supp} \Delta_s \cup \operatorname{Supp} \Delta_u$ and $\|\Xi\| < \eta$.

Pick $t \in (s,u]$ such that $\|s - t\| < \varepsilon/k_s$, $k_s s$ is integral and the smallest rational affine subspace containing t is precisely W. Let $0 < \delta \ll 1$ be a rational number such that $\phi_v > 0$ for $v \in W$ with $\|v - t\| \leq \delta$, set $\phi = \min\{\phi_v : v \in W, \|v - t\| \leq \delta\}$ and let $0 < \xi \ll \min\{\delta, \varepsilon\}$ be a rational number such that $(C_t/\phi + 1)\xi(K_X + \Delta_t) + A$ is ample. Denote by $V \subset \operatorname{Div}(S)_\mathbb{R}$ the smallest rational affine space containing $\Theta_s^\sharp = \Omega_s$ and $\Theta_t^\sharp$, and let r be as in Lemma 6.7. Then by Lemma 2.28 there exist rational points $(t_i, \Theta'_{t_i}) \in W \times V$ and integers $k_{t_i} \gg 0$ such that:

(1) we may write $t = \sum r_{t_i} t_i$, $\Delta_t = \sum r_{t_i} \Delta_{t_i}$ and $\Theta_t^\sharp = \sum r_{t_i} \Theta'_{t_i}$, where $r_{t_i} > 0$ and $\sum r_{t_i} = 1$,

(2) $t_1 = s$, $\Theta'_{t_1} = \Theta_{t_1}^\sharp = \Omega_{t_1}$, $k_{t_1} = k_s$,

(3) $k_{t_i}\Delta_{t_i}/r$ are integral and $\|t - t_i\| < \xi/k_{t_i}$ for $i = 2, \dots, n-1$,

(4) $\Theta'_{t_i} \leq \Omega_{t_i}$, $k_{t_i}\Theta'_{t_i}/r$ are integral, $\|\Theta_t^\sharp - \Theta'_{t_i}\| < \phi\xi/k_{t_i}$ and (t_i, Θ'_{t_i}) belong to the smallest rational affine space containing $(t, \Theta_t^\sharp)$ for $i = 2, \dots, n-1$,

(5) $\Delta_t = \frac{k_{t_1}}{k_{t_1}+k_{t_n}}\Delta_{t_1} + \frac{k_{t_n}}{k_{t_1}+k_{t_n}}\Delta_{t_n} + \Psi$, where $k_{t_n}\Delta_{t_n}/r$ is integral, $\|t - t_n\| < \varepsilon/k_{t_n}$ and $\|\Psi\| < \eta/(k_{t_1} + k_{t_n})$,

(6) $\Theta_t^\sharp = \frac{k_{t_1}}{k_{t_1}+k_{t_n}}\Theta'_{t_1} + \frac{k_{t_n}}{k_{t_1}+k_{t_n}}\Theta'_{t_n} + \Phi$, where $\Theta'_{t_n} \leq \Omega_{t_n}$, $k_{t_n}\Theta'_{t_n}/r$ is integral, $\|\Theta_t^\sharp - \Theta'_{t_n}\| < \varepsilon/k_{t_n}$ and $\|\Phi\| < \eta/(k_{t_1} + k_{t_n})$.

Observe also that $\operatorname{Supp}\Psi \subset \operatorname{Supp}\Delta_t$ and $\operatorname{Supp}\Phi \subset \operatorname{Supp}\Theta_t^\sharp$ by Remarks 2.29 and 2.30 applied to the linear map Δ defined at the beginning of Step 2. Then by

Theorem 6.10,

$$|k_{t_i}(K_S+\Theta'_{t_i})|+k_{t_i}(\Omega_{t_i}-\Theta'_{t_i})\subset|k_{t_i}(K_X+\Delta_{t_i})|_S$$

for $i=2,\dots,n-1$. Let P be a component in $\operatorname{Supp}\Omega_t$ and denote $A_{t_n}=A/k_{t_n}$. I claim that

$$\operatorname{mult}_P(\Omega_{t_n}\wedge\tfrac{1}{l}\operatorname{Fix}|l(K_X+\Delta_{t_n}+A_{t_n})|_S)\leq\operatorname{mult}_P(\Omega_{t_n}-\Theta'_{t_n}) \tag{6.11}$$

for $l\gg 0$ sufficiently divisible. Assume first that $\operatorname{mult}_P\Theta^\sharp_t=0$. Then $\operatorname{mult}_P\Theta^\sharp_s=0$ by the choice of ε, and thus $\operatorname{mult}_P\Theta'_{t_n}=0$ since $\Theta'_{t_n}\in V$. Therefore

$$\operatorname{mult}_P(\Omega_{t_n}\wedge\tfrac{1}{l}\operatorname{Fix}|l(K_X+\Delta_{t_n}+A_{t_n})|_S)\leq\operatorname{mult}_P\Omega_{t_n}=\operatorname{mult}_P(\Omega_{t_n}-\Theta'_{t_n}).$$

Now assume that $\operatorname{mult}_P\Theta^\sharp_t>0$. Then for l sufficiently divisible we have

$$\operatorname{mult}_P\tfrac{1}{l}\operatorname{Fix}|l(K_X+\Delta_{t_n}+A_{t_n})|_S=\operatorname{ord}_P\|K_X+\Delta_{t_n}+A_{t_n}\|_S$$

as in Step 3 of the proof of Theorem 6.6, and since $\Delta_t-\Delta_{t_1}-\frac{k_{t_1}+k_{t_n}}{k_{t_1}}\Psi+A$ is ample by the choice of η,

$$\begin{aligned}\operatorname{mult}_P(\Omega_{t_n}\wedge\tfrac{1}{l}\operatorname{Fix}|l(K_X+\Delta_{t_n}+A_{t_n})|_S)&\leq\operatorname{ord}_P\|K_X+\Delta_{t_n}+A_{t_n}\|_S\\&=\operatorname{ord}_P\big\|K_X+\Delta_t+\tfrac{k_{t_1}}{k_{t_n}}\big(\Delta_t-\Delta_{t_1}-\tfrac{k_{t_1}+k_{t_n}}{k_{t_1}}\Psi+A\big)\big\|_S\\&\leq\operatorname{ord}_P\|K_X+\Delta_t\|_S=\operatorname{mult}_P(\Omega_t-\Theta^\sharp_t).\end{aligned}$$

Combining assumptions (5) and (6) above we have

$$\Omega_t-\Theta^\sharp_t\leq\Omega_t-\Theta^\sharp_t+\tfrac{k_{t_1}}{k_{t_n}}\big(\Omega_t-\Theta^\sharp_t-\tfrac{k_{t_1}+k_{t_n}}{k_{t_1}}(\Psi_{|S}-\Phi)\big)=\Omega_{t_n}-\Theta'_{t_n},$$

and (6.11) is proved. Furthermore, we can choose $\varepsilon\ll 1$ and $k_{t_n}\gg 0$ such that $S\not\subset\mathbf{B}(K_X+\Delta_{t_n})$. Otherwise, if we denote $\mathcal{Q}=\{p\in\mathcal{S}_\mathbb{R}:S\not\subset\mathbf{B}(K_X+\Delta_p)\}$, $\mathcal{Q}$ is a rational polyhedral cone by Property $\mathcal{L}^G_A$, and $t\in\partial\mathcal{Q}$ for every $t\in[s,u]$ with $0<\|t-s\|\ll 1$, and thus $s\in\partial\mathcal{Q}$. But then for $0<\|t-s\|\ll 1$, s and t belong to the same face of $\mathcal{Q}$, and so does t_n, a contradiction. Therefore as in the proof of

Theorem 6.8 we have

$$|k_{t_n}(K_S+\Theta'_{t_n})|+k_{t_n}(\Omega_{t_n}-\Theta'_{t_n})\subset|k_{t_n}(K_X+\Delta_{t_n})|_S.$$

Denote $\sum G_j=\operatorname{Supp}(\Omega_s-A_{|S})\cup\operatorname{Supp}(\Omega_u-A_{|S})$, and let $\mathcal{Q}'=\{\Xi\in\sum_j[0,1]G_j : Z\not\subset\mathbf{B}(K_S+\Xi+A_{|S})\}$. Then by Property $\mathcal{L}^G_A$, $\mathcal{Q}'$ is a rational polytope and $\Theta^\sharp_p\in\mathcal{Q}'$ for every $p\in\mathcal{S}_\mathbb{R}$. Therefore as above and by Theorem 6.8, if $\varepsilon\ll 1$ then $Z\not\subset\mathbf{B}(K_S+\Theta'_{t_n})$, and as in Step 1 we have that $\lambda^\sharp_Z$ is linear on the cone $\sum_{i=1}^n\mathbb{R}_+t_i$, and in particular on the cone $\mathbb{R}_+s+\mathbb{R}_+t$.

Step 3. Assume now that $s\in\mathcal{S}_\mathbb{Q}$, $\phi_s>0$ and fix $u\in R$. Let again W be the smallest rational affine space containing s and u. Let $0<\xi\ll 1$ be a rational number such that $\phi_v>0$ for $v\in[s,u]$ with $\|v-s\|\leq 2\xi$, that for every prime divisor P on S we have either $\operatorname{mult}_P\Omega_v>\operatorname{mult}_P\Theta^\sharp_v$ or $\operatorname{mult}_P\Omega_v=\operatorname{mult}_P\Theta^\sharp_v$ for all such v, and let $\phi=\min\{\phi_v : v\in[s,u],\|v-s\|\leq 2\xi\}$.

Let k_s be a positive integer such that $k_s\Delta_s/r$ and $k_s\Theta^\sharp_s/r$ are integral, where r is as in Lemma 6.7. Let us first show that there is a real number $0<\varepsilon\ll\xi$ such that $(C_t/\phi+1)\varepsilon(K_X+\Delta_v)+A$ is ample for all $v\in\mathcal{S}_\mathbb{R}$ such that $\|v-s\|<2\xi$, where $\|t-s\|=\varepsilon/k_s$. If $\Theta^\sharp$ is locally Lipschitz around s this is straightforward. Otherwise, assume $\Theta^\sharp$ is not locally Lipschitz around s and assume we cannot find such ε. But then there is a sequence $s_n\in(s,u]$ such that $\lim\limits_{n\to\infty}s_n=s$ and $C_{s_n}\|s_n-s\|\geq M$, where M is a constant and $C_{s_n}\to\infty$. Since a local Lipschitz constant is the maximum of local slopes of the concave function $\Theta^\sharp|_{[s,u]}$, we have that

$$\frac{\Theta^\sharp_{s_n}-\Theta^\sharp_s}{\|s_n-s\|}>C_{s_n}.$$

Therefore

$$\Theta^\sharp_{s_n}-\Theta^\sharp_s>C_{s_n}\|s_n-s\|\geq M$$

for all $n\in\mathbb{N}$, which contradicts Theorem 6.8.

Increase ε a bit, and pick $t\in(s,u]$ such that $\|s-t\|<\varepsilon/k_s$, the smallest rational subspace containing t is precisely W and $(C_t/\phi+1)\varepsilon(K_X+\Delta_v)+A$ is ample for all $v\in\mathcal{S}_\mathbb{R}$ such that $\|v-s\|<2\varepsilon$. In particular, $\Theta^\sharp$ is locally Lipschitz in a neighbourhood of t contained in W. Furthermore, by changing ϕ slightly I can assume that $\phi \le \min\{\phi_v : v \in W, \|v-t\| \ll 1\}$. Denote by V the smallest rational affine space containing $\Theta_s^\sharp$ and $\Theta_t^\sharp$, and let r be as in Lemma 6.7. Then by Lemma 2.28 there exist rational points $(t_i, \Theta'_{t_i}) \in W \times V$ and integers $k_{t_i} \gg 0$ such that:

(1) we may write $t = \sum r_{t_i} t_i$, $\Delta_t = \sum r_{t_i} \Delta_{t_i}$ and $\Theta_t^\sharp = \sum r_{t_i} \Theta'_{t_i}$, where $r_{t_i} > 0$ and $\sum r_{t_i} = 1$,

(2) $t_1 = s$, $\Theta'_{t_1} = \Theta^\sharp_{t_1}$, $k_{t_1} = k_s$,

(3) $k_{t_i}\Delta_{t_i}/r$ are integral and $\|t - t_i\| < \varepsilon/k_{t_i}$ for all i,

(4) $\Theta'_{t_i} \le \Omega_{t_i}$, $k_{t_i}\Theta'_{t_i}/r$ are integral and $\|\Theta_t^\sharp - \Theta'_{t_i}\| < \phi\varepsilon/k_{t_i}$.

Observe that similarly as in Step 2 we have $S \not\subset \mathbf{B}(K_X + \Delta_{t_i})$ for all i, and therefore by Theorem 6.10,

$$|k_{t_i}(K_S + \Theta'_{t_i})| + k_{t_i}(\Omega_{t_i} - \Theta'_{t_i}) \subset |k_{t_i}(K_X + \Delta_{t_i})|_S$$

for all i. Then we finish as in Step 2.

Step 4. Assume in this step that $s \in \mathcal{S}_\mathbb{R}$ is a non-rational point and fix $u \in R$. By Step 1 there is a rational cone $\mathcal{C} = \sum_{i=1}^k \mathbb{R}_+ g_i$ with $g_i \in \mathcal{S}_\mathbb{Q}$ and $k > 1$ such that $\lambda_Z^\sharp$ is linear on $\mathcal{C}$ and $s = \sum \alpha_i g_i$ with all $\alpha_i > 0$. Consider the rational point $g = \sum_{i=1}^k g_i$. Then by Step 2 there is a point $s' = \alpha g + \beta u$ with $\alpha, \beta > 0$ such that the map $\lambda_Z^\sharp$ is linear on the cone $R_+ g + \mathbb{R}_+ s'$. Now we have

$$\lambda_Z^\sharp\Big(\sum g_i + s'\Big) = \lambda_Z^\sharp(g + s') = \lambda_Z^\sharp(g) + \lambda_Z^\sharp(s') = \sum \lambda_Z^\sharp(g_i) + \lambda_Z^\sharp(s'),$$

so the map $\lambda_Z^\sharp|_{\mathcal{C}+\mathbb{R}_+ s'}$ is linear by Lemma 4.5. Taking $\mu = \max_i\{\frac{\alpha}{\alpha_i\beta}\}$ and taking a point $\hat{u} = \mu s + u$ in the relative interior of $\mathbb{R}_+ s + R$, it is easy to check that

$$\hat{u} = \sum(\mu\alpha_i - \tfrac{\alpha}{\beta})t_i + \tfrac{1}{\beta}s' \in \mathcal{C} + \mathbb{R}_+ s',$$

so the map $\lambda_Z^\sharp|_{\mathbb{R}_+ s + \mathbb{R}_+ \hat{u}}$ is linear.

Step 5. Finally, let H be any 2-plane in $\mathbb{R}^\ell$. Then by the previous steps, for every ray $R \subset \mathcal{S}_\mathbb{R} \cap H$ there is a polyhedral cone $\mathcal{C}_R$ with $R \subset \mathcal{C}_R \subset \mathcal{S}_\mathbb{R} \cap H$ such that there is a polyhedral decomposition $\mathcal{C}_R = \mathcal{C}_{R,1} \cup \mathcal{C}_{R,2}$ with $\lambda^\sharp_Z|_{\mathcal{C}_{R,1}}$ and $\lambda^\sharp_Z|_{\mathcal{C}_{R,2}}$ being linear maps, and if $R \subset \operatorname{relint}(\mathcal{S}_\mathbb{R} \cap H)$, then $R \subset \operatorname{relint} \mathcal{C}_R$.

Let $S^{\ell-1}$ be the unit sphere. Restricting to the compact set $S^{\ell-1} \cap \mathcal{S}_\mathbb{R} \cap H$ we see that $\lambda^\sharp_Z|_{\mathcal{S}_\mathbb{R} \cap H}$ is piecewise linear. □

6.2 Proof of the Main Result

Proof of Theorem 6.2.

Step 1. I first show that it is enough to prove the theorem in the case when A is a general ample $\mathbb{Q}$-divisor and $(X, \Delta_i + A)$ is a log smooth klt pair for every i.

Let p and k be sufficiently divisible positive integers such that all divisors $k(\Delta_i + pA)$ are very ample and $(p+1)kA$ is very ample. Let $(p+1)kA_i$ be a general section of $|k(\Delta_i + pA)|$ and let $(p+1)kA'$ be a general section of $|(p+1)kA|$. Set $\Delta_i' = \frac{p}{p+1}\Delta_i + A_i$. Then the pairs $(X, \Delta_i' + A')$ are klt and

$$(p+1)k(K_X + \Delta_i + A) \sim (p+1)k(K_X + \Delta_i' + A') =: D_i'$$

for all i. Then a truncation of $R(X; D_1, \dots, D_\ell)$ is isomorphic to $R(X; D_1', \dots, D_\ell')$, so it is enough to prove the latter algebra is finitely generated.

Step 2. Therefore I can assume that $\Delta_i = \sum_{j=1}^N \delta_{ij} F_j$ with $\delta_{ij} \in [0,1)$. Write $K_X + \Delta_i + A \sim_\mathbb{Q} \sum_{j=1}^N f'_{ij} F_j \geq 0$, where $F_j \neq A$ since A is general. By blowing up, and by possibly replacing the pair (X, Δ_i) by (Y, Δ_i') for some model $Y \to X$ as in Step 2 of the proof of Theorem 6.6, I can assume that the divisor $\sum_{j=1}^N F_j$ has simple normal crossings. Thus for every i,

$$K_X \sim_\mathbb{Q} -A + \sum\nolimits_{j=1}^N f_{ij} F_j,$$

where $f_{ij} = f'_{ij} - \delta_{ij} > -1$.

Let $\Lambda = \bigoplus_{j=1}^N \mathbb{N} F_j \subset \operatorname{Div}(X)$ be a simplicial monoid and set $\mathcal{T} = \{(t_1, \dots, t_\ell) : t_i \geq 0, \sum t_i = 1\} \subset \mathbb{R}^\ell$. For each $\tau = (t_1, \dots, t_\ell) \in \mathcal{T}$, denote $\delta_{\tau j} = \sum_i t_i \delta_{ij}$ and $f_{\tau j} = \sum_i t_i f_{ij}$, and observe that $K_X \sim_\mathbb{R} -A + \sum_j f_{\tau j} F_j$. Denote $\mathcal{B}_\tau = \sum_{j=1}^N [\delta_{\tau j} + f_{\tau j}, 1 + f_{\tau j}] F_j \subset \Lambda_\mathbb{R}$ and let $\mathcal{B} = \bigcup_{\tau \in \mathcal{T}} \mathcal{B}_\tau$. It is easy to see that $\mathcal{B}$ is a rational polytope: every point in $\mathcal{B}$ is a barycentric combination of the vertices of $\mathcal{B}_{\tau_1}, \dots, \mathcal{B}_{\tau_\ell}$, where τ_i are the standard basis vectors of $\mathbb{R}^\ell$. Thus $\mathcal{C} = \mathbb{R}_+\mathcal{B}$ is a rational polyhedral cone.

For each $j = 1, \dots, N$ fix a section $\sigma_j \in H^0(X, F_j)$ such that $\operatorname{div} \sigma_j = F_j$. Consider the Λ-graded algebra $\mathfrak{R} = \bigoplus_{s\in\Lambda} \mathfrak{R}_s \subset R(X; F_1, \dots, F_N)$ generated by the elements of $R(X, \mathcal{C} \cap \Lambda)$ and all σ_j; observe that $\mathfrak{R}_s = H^0(X, s)$ for every $s \in \mathcal{C} \cap \Lambda$. I claim that it is enough to show that $\mathfrak{R}$ is finitely generated.

To see this, assume $\mathfrak{R}$ is finitely generated and denote

$$\omega_i = rk_i \sum\nolimits_j (\delta_{ij} + f_{ij})F_j \in \Lambda$$

for r sufficiently divisible and $i = 1, \dots, \ell$. Set $\mathcal{G} = \sum_i \mathbb{R}_+\omega_i \cap \Lambda$ and observe that $\omega_i \sim rD_i$. Then by Lemma 5.5(2) the algebra $R(X, \mathcal{C} \cap \Lambda)$ is finitely generated, and therefore by Proposition 5.11 there is a finite rational polyhedral subdivision $\mathcal{G}_\mathbb{R} = \bigcup_k \mathcal{G}_k$ such that the map $\mathbf{Mob}_{\iota|\mathcal{G}_k\cap\Lambda}$ is additive up to truncation for every k, where $\iota\colon \Lambda \to \Lambda$ is the identity map.

Let $\omega'_1, \dots, \omega'_q$ be generators of $\mathcal{G}$ such that $\omega'_i = \omega_i$ for $i = 1, \dots, \ell$, and let $\pi\colon \bigoplus_{i=1}^q \mathbb{N}\omega'_i \to \mathcal{G}$ be the natural projection. Then the map $\mathbf{Mob}_{\pi|\pi^{-1}(\mathcal{G}_k\cap\Lambda)}$ is additive up to truncation for every k, and thus $R(X, \pi(\bigoplus_{i=1}^q \mathbb{N}\omega'_i))$ is finitely generated by Lemma 5.5(3). Therefore $R(X, \pi(\bigoplus_{i=1}^\ell \mathbb{N}\omega_i)) \simeq R(X; rD_1, \dots, rD_\ell)$ is finitely generated by Lemma 5.5(2), thus $R(X; D_1, \dots, D_\ell)$ is finitely generated by Lemma 5.5(1).

Step 3. Therefore it suffices to prove that $\mathfrak{R}$ is finitely generated. Take a point $\sum_j (f_{\tau j} + b_{\tau j})F_j \in \mathcal{B}\backslash\{0\}$; in particular $b_{\tau j} \in [\delta_{\tau j}, 1]$. Setting

$$r_\tau = \max_{j=1}^{N} \left\{ \frac{f_{\tau j} + b_{\tau j}}{f_{\tau j} + 1} \right\} \quad \text{and} \quad b'_{\tau j} = -f_{\tau j} + \frac{f_{\tau j} + b_{\tau j}}{r_\tau},$$

we have

$$\sum\nolimits_j (f_{\tau j} + b_{\tau j})F_j = r_\tau \sum\nolimits_j (f_{\tau j} + b'_{\tau j})F_j. \tag{6.12}$$

Observe that $r_\tau \in (0, 1]$, $b'_{\tau j} \in [b_{\tau j}, 1]$ and there exists j_0 such that $b'_{\tau j_0} = 1$. For every $j = 1, \dots, N$, let

$$\mathcal{F}_{\tau j} = (1 + f_{\tau j})F_j + \sum\nolimits_{k\neq j} [\delta_{\tau k} + f_{\tau k}, 1 + f_{\tau k}]F_k,$$

and set $\mathcal{F}_j = \bigcup_{\tau\in\mathcal{T}} \mathcal{F}_{\tau j}$, which is a rational polytope. Then $\mathcal{C}_j = \mathbb{R}_+\mathcal{F}_j$ is a rational polyhedral cone, and (6.12) shows that $\mathcal{C} = \bigcup_j \mathcal{C}_j$. Furthermore, since $\sum_j(f_{\tau j} + b_{\tau j})F_j \sim_{\mathbb{R}} K_X + \sum_j b_{\tau j}F_j + A$ for $\tau \in \mathcal{T}$, for every j and for every $s \in \mathcal{C}_j \cap \Lambda$ there is $r_s \in \mathbb{Q}_+$ such that $s \sim_{\mathbb{Q}} r_s(K_X + F_j + \Delta_s + A)$ where $\operatorname{Supp}\Delta_s \subset \sum_{k\neq j} F_k$ and the pair $(X, F_j + \Delta_s + A)$ is log canonical.

Step 4. Assume that the restricted algebra $\operatorname{res}_{F_j} R(X, \mathcal{C}_j \cap \Lambda)$ is finitely generated for every j. I will show that then $\mathfrak{R}$ is finitely generated.

Let $V = \sum_{j=1}^N \mathbb{R}F_j \simeq \mathbb{R}^N$, and let $\|\cdot\|$ be the Euclidean norm on V. By compactness there is a constant C such that every $\mathcal{F}_j \subset V$ is contained in the closed ball centred at the origin with radius C. Let deg denote the total degree function on Λ, i.e. $\deg(\sum_{j=1}^N \alpha_j F_j) = \sum_{j=1}^N \alpha_j$; it induces the degree function on elements of $\mathfrak{R}$. Let M be a positive integer such that, for each j, $\operatorname{res}_{F_j} R(X, \mathcal{C}_j \cap \Lambda)$ is generated by $\{\sigma_{|F_j} : \sigma \in R(X, \mathcal{C}_j \cap \Lambda), \deg\sigma \leq M\}$, and such that $M \geq CN^{1/2} \max\limits_{i,j}\{\frac{1}{1-\delta_{ij}}\}$. By Hölder's inequality we have $\|s\| \geq N^{-1/2}\deg s$ for all $s \in \mathcal{C} \cap \Lambda$, and thus

$$\|s\|/C \geq \max_{i,j}\left\{\frac{1}{1-\delta_{ij}}\right\}$$

for all $s \in \mathcal{C} \cap \Lambda$ with $\deg s \geq M$. Let $\mathcal{H}$ be a finite set of generators of the finite dimensional vector space

$$\bigoplus_{s\in\mathcal{C}\cap\Lambda, \deg s\leq M} H^0(X, s)$$

such that for every j, the set $\{\sigma_{|F_j} : \sigma \in \mathcal{H}\}$ generates $\operatorname{res}_{F_j} R(X, \mathcal{C}_j \cap \Lambda)$. I claim that $\mathfrak{R}$ is generated by $\{\sigma_1, \ldots, \sigma_N\} \cup \mathcal{H}$, with σ_j as in Step 2.

To that end, take any section $\sigma \in \mathfrak{R}$ with $\deg\sigma > M$. By definition, possibly by considering monomial parts of σ and dividing σ by a suitable product of sections σ_j, I can assume that $\sigma \in R(X, \mathcal{C} \cap \Lambda)$. Furthermore, by Step 3 there exists $w \in \{1, \ldots, N\}$ such that $\sigma \in R(X, \mathcal{C}_w \cap \Lambda)$, thus there is $\tau \in \mathcal{T} \cap \mathbb{Q}^\ell$ such that $\sigma \in H^0(X, r_\sigma \sum_j(f_{\tau j} + b_{\tau j})F_j)$ with $b_{\tau w} = 1$. Observe that $r_\sigma \geq \max\limits_{i,j}\{\frac{1}{1-\delta_{ij}}\}$ since $\|\sum_j(f_{\tau j} + b_{\tau j})F_j\| \leq C$, and in particular $\frac{r_\sigma - 1}{r_\sigma} \geq \delta_{\tau w}$ for every $\tau \in \mathcal{T}$.

Therefore by assumption there are elements $\theta_1, \ldots, \theta_z \in \mathcal{H}$ and a polynomial $\varphi \in \mathbb{C}[X_1, \ldots, X_z]$ such that $\sigma_{|F_w} = \varphi(\theta_{1|F_w}, \ldots, \theta_{z|F_w})$. Therefore by (5.1) in Remark

5.3,

$$(\sigma-\varphi(\theta_1,\dots,\theta_z))/\sigma_w\in H^0\big(X,r_\sigma\sum\nolimits_j(f_{\tau j}+b_{\tau j})F_j-F_w\big).$$

Since

$$r_\sigma\sum_j(f_{\tau j}+b_{\tau j})F_j-F_w=r_\sigma\Big((f_{\tau w}+\tfrac{r_\sigma-1}{r_\sigma})F_w+\sum_{j\neq w}(f_{\tau j}+b_{\tau j})F_j\Big),$$

we have $r_\sigma\sum_j(f_{\tau j}+b_{\tau j})F_j-F_w\in\mathcal{C}\cap\Lambda$. We finish by descending induction on $\deg\sigma$.

Step 5. Therefore it remains to show that for each j, the algebra $\operatorname{res}_{F_j}R(X,\mathcal{C}_j\cap\Lambda)$ is finitely generated.

To that end, choose a rational $0<\varepsilon\ll 1$ such that $\varepsilon\sum_{k\in I}F_k+A$ is ample for every $I\subset\{1,\dots,N\}$, and let $A_I\sim_{\mathbb{Q}}\varepsilon\sum_{k\in I}F_k+A$ be a very general ample $\mathbb{Q}$-divisor. Fix j, and for $I\subset\{1,\dots,N\}\backslash\{j\}$ let

$$\mathcal{F}^I_{\tau j}=(1+f_{\tau j})F_j+\sum_{k\in I}[1-\varepsilon+f_{\tau k},1+f_{\tau k}]F_k+\sum_{k\notin I\cup\{j\}}[\delta_{\tau k}+f_{\tau k},1-\varepsilon+f_{\tau k}]F_k.$$

Set $\mathcal{F}^I_j=\bigcup_{\tau\in\mathcal{T}}\mathcal{F}^I_{\tau j}$; these are rational polytopes such that $\mathcal{F}_j=\bigcup_{I\subset\{1,\dots,N\}\backslash\{j\}}\mathcal{F}^I_j$, and therefore $\mathcal{C}^I_j=\mathbb{R}_+\mathcal{F}^I_j$ are rational polyhedral cones such that $\mathcal{C}_j=\bigcup\mathcal{C}^I_j$. Furthermore, for every $s\in\mathcal{C}^I_j\cap\Lambda$ we have $s\sim_{\mathbb{Q}}r_s(K_X+F_j+\Delta_s+A)\sim_{\mathbb{Q}}r_s(K_X+F_j+\Delta'_s+A_I)$, where $\Delta'_s=\Delta_s-\varepsilon\sum_{k\in I}F_k\geq 0$ and $\lfloor F_j+\Delta'_s+A_I\rfloor=F_j$.

Therefore it is enough to prove that $\operatorname{res}_{F_j}R(X,\mathcal{C}^I_j\cap\Lambda)$ is finitely generated for every I. Fix I and let $h_1,\dots,h_m$ be generators of $\mathcal{C}^I_j\cap\Lambda$. Similarly as in Step 1 of the proof of Theorem 6.6, it is enough to prove that the restricted algebra $\operatorname{res}_{F_j}R(X;h_1,\dots,h_m)$ is finitely generated. For p sufficiently divisible, by the argument above we have $ph_v\sim\rho_v(K_X+F_j+B_v+A_I)=:H_v$, where $\lceil B_v\rceil\subset\sum_{k\neq j}F_k$, $\lfloor B_v\rfloor=0$, $\rho_v\in\mathbb{N}$ and A_I is a very general ample $\mathbb{Q}$-divisor. Therefore it is enough to show that $\operatorname{res}_{F_j}R(X;H_1,\dots,H_m)$ is finitely generated by Lemma 5.5(1). But this follows from Theorem 6.6 and the proof is complete. □

Proof of Theorem 6.1. By Theorem 3.10 and by induction on $\dim X$, we may assume $K_X+\Delta$ is big. Write $K_X+\Delta\sim_{\mathbb{Q}}B+C$ with B effective and C ample. Let $f\colon Y\to X$ be a log resolution of $(X,\Delta+B+C)$ and let H be an effective f-exceptional divisor such that $f^*C - H$ is ample. Then writing $K_Y + \Gamma = f^*(K_X + \Delta) + E$, where $\Gamma = \mathbf{B}(X, \Delta)_Y$, we have that $R(Y, K_Y + \Gamma)$ and $R(X, K_X + \Delta)$ have isomorphic truncations. Since $K_Y + \Gamma \sim_{\mathbb{Q}} (f^*B + H + E) + (f^*C - H)$, we may assume from the start that $\operatorname{Supp}(\Delta + B + C)$ has simple normal crossings. Let ε be a small positive rational number and set $\Delta' = (\Delta + \varepsilon B) + \varepsilon C$. Then $K_X + \Delta' \sim_{\mathbb{Q}} (\varepsilon + 1)(K_X + \Delta)$, and $R(X, K_X + \Delta)$ and $R(X, K_X + \Delta')$ have isomorphic truncations, so the result follows from Theorem 6.2. $\square$

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