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Quantum Optics and Cold Atoms

SoSe 2014 **Sheet 2** **6 May 2014**

(Due on 13. May 2014)

Question 1

a) In a two-level system the transition between the ground state $|g\rangle$ and the excited state $|e\rangle$ has the transition frequency ω_0.

Exactly solve the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle_t = H|\psi\rangle_t \tag{1}$$

with $|\psi\rangle_0 = \alpha(0)|g\rangle + \beta(0)|e\rangle$, $|\alpha(t)|^2 + |\beta(t)|^2 = 1$ and

$$H = -\frac{\hbar\gamma}{2}\sigma_z + \hbar\Omega(\sigma^+ + \sigma^-) \tag{2}$$

where $\gamma = \omega - \omega_0$. *(1 Point)*

b) Solve the Schrödinger equation using perturbation theory to first order when $|\psi\rangle_0 = |g\rangle$. *(1 Point)*

Question 2

Assume that a third level $|i\rangle$ at frequency $\omega_1 > \omega_0$ can be coupled to state $|g\rangle$ via radiation. Starting from

$$H = \hbar\omega_0|e\rangle\langle e| + \hbar\omega_1|i\rangle\langle i| \tag{3}$$

$$+ \hbar\Omega(|e\rangle\langle g|e^{-i\omega t} + |g\rangle\langle e|e^{i\omega t}) \tag{4}$$

$$+ \hbar\Omega'(|i\rangle\langle g|e^{-i\omega t} + |g\rangle\langle i|e^{i\omega t}) \tag{5}$$

a) Find the representation in which the Hamiltonian is time independent. *(1 Point)*

b) Determine the condition under which the coupling to level $|i\rangle$ can be neglected and the system can be reduced to two levels. *(1 Point)*

Question 3

Consider the dynamics of the density matrix for Question 1

$$\frac{\partial \rho}{\partial t} = \frac{1}{i\hbar}[H, \rho] + \Gamma(\sigma \rho \sigma^+ - \frac{1}{2}\sigma^+ \sigma \rho - \frac{1}{2}\sigma^+ \sigma) \tag{6}$$

where $\Gamma > 0$ and

$$H = \frac{\hbar \omega_0}{2}\sigma_z + \hbar\Omega(\sigma^+ e^{-i\omega t} + \sigma e^{i\omega t}) \; . \tag{7}$$

a) Determine the form of the master equation when H is moved to the reference frame which is time independent.

(1 Point)

b) Write the optical Bloch equation

(0.5 Point)

c) Solve the optical Bloch equations for

$$\rho_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \; ; \; \Omega = 0 \; .$$

Determine $\mathrm{Tr}\{\rho^2\}$ as a function of time.

(0.5 Point)

d) Solve the optical Bloch equations for $\Gamma = 0$, $\Omega > 0$, $\Delta > 0$ and

$$\rho_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \; .$$

Determine $\mathrm{Tr}\{\rho^2\}$ as a function of time. *(0.5 Point)*

Question 4

Show that for a two-level system $\rho = \frac{1}{2}(\mathbb{1}_2 + \underline{U} \cdot \underline{\sigma})$ with

$$\underline{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \tag{8}$$

and

$$\underline{U} = (U_x, U_y, U_z) \tag{9}$$

is a real vector $\underline{U} \in \mathbb{R}^3$.
Show that there is positive semi-definiteness when $|\underline{U}| \leq 1$. (*hint: evaluate the eigenvalues of the density matrix*) *(1 Point)*