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Quantum Optics and Cold Atoms

SoSe 2014 **Sheet 4 (Part 2)** **26 June 2014**

(Due on 1st and 14th of July 2014)

Question 3 *Born-Markov master equation*

The Born-Markov master equation for an optical dipole with the transition frequency ω_0, which is coupled with the modes of the free electromagnetic field is given by:

$$\frac{\partial}{\partial t}\rho = \frac{1}{i\hbar}[H_0,\rho] \quad + \quad \frac{\gamma}{2}\left(\langle n(\omega_0)\rangle + 1\right)\left(2\sigma\rho\sigma^\dagger - \sigma^\dagger\sigma\rho - \rho\sigma^\dagger\sigma\right) \\ \quad + \quad \frac{\gamma}{2}\langle n(\omega_0)\rangle\left(2\sigma^\dagger\rho\sigma - \sigma\sigma^\dagger\rho - \rho\sigma\sigma^\dagger\right), \tag{1}$$

where ρ is the density matrix describing the state of the dipole at the time t, H_0 governs the dipole coherent dynamics and includes the frequency shift due to the coupling with the reservoir, $\sigma = |g\rangle\langle e|$ and $\sigma^\dagger = |e\rangle\langle g|$.

a) Determine the mean number of photons at frequency ω_0

$$\langle n(\omega_0)\rangle = \frac{1}{e^{\beta\hbar\omega_0} - 1} \tag{2}$$

for $\omega_0 = 2\pi \cdot 10^{14}$ Hz (optical domain) and $\omega_0 = 2\pi \cdot 10^{9}$ Hz (microwave domain) at $T = 300$K. Argue (i) that for optical frequencies $\langle n(\omega_0)\rangle \ll 1$ and (ii) when this implies that its contribution in equation (1) can be typically neglected. *(0.5 Point)*

Tip: *Here are the values of the fundamental constants (CGS units) in equation (1) where* $\beta = 1/(k_B T)$*: the Planck's constant is* $\hbar = 1.06 \cdot 10^{-27}$ *erg·s, and the Boltzmann constant, which in fact was also introduced by Planck, is* $k_B = 1.38 \cdot 10^{-16}$ *erg·K*$^{-1}$*.*

b) The damping rate in equation (1) is:

$$\gamma = 2\mathbf{Re}\int_0^\infty d\tau \sum_\lambda |g_\lambda|^2 e^{i(\omega_0 - \omega_\lambda)\tau} \tag{3}$$

with $\sum_\lambda := \sum_{\underline{k}_\lambda}\sum_{\underline{\epsilon}_\lambda \perp \underline{k}_\lambda}$ the sum over the modes of the electromagnetic field. Here $g_\lambda = -\frac{\underline{d}\cdot\underline{E}_\lambda}{\hbar}$, where $\underline{d}$ is the dipole moment and $\underline{E}_\lambda = \sqrt{\frac{2\pi\hbar\omega_\lambda}{V}}\underline{\epsilon}_\lambda$ (in Gauss-units), with $\omega_\lambda = c|\underline{k}_\lambda|$ and V the quantization volume. Cast the sum into an integral, valid for large volumes

$$\sum_{\underline{k}_\lambda} \rightarrow \frac{V}{8\pi^3}\int d\underline{k} \tag{4}$$

and use spherical coordinates to show that γ can be cast into the form:

$$\gamma = 2\mathbf{Re} \int_0^\infty d\tau \int_0^\infty d\omega \;\; \omega^3 e^{i(\omega_0 - \omega)\tau} \left[\alpha \int d\Omega \sum_{\underline{\epsilon} \perp \underline{k}} |\underline{d} \cdot \underline{\epsilon}|^2 \right], \tag{5}$$

with $\alpha = 1/(4\pi^2 \hbar c^3)$ and Ω the solid angle, which determines the direction $\hat{n}$ of the photon wave vector: $d\Omega = (d\cos\theta)(d\phi)$ with $-1 \leq \cos\theta \leq 1$, $0 \leq \phi \leq 2\pi$, and $\hat{n} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$. *(2 Point)*

c) Use that $\mathbf{Re} \int_0^\infty d\tau e^{i(\omega_0-\omega)\tau} = \pi\delta(\omega_0 - \omega)$ and show that

$$\gamma = \frac{4|\underline{d}|^2 \omega_0^3}{3\hbar c^3} \int d\Omega P(\Omega), \tag{6}$$

where

$$P(\Omega) = \frac{3}{8\pi} \int \sum_{\underline{\epsilon} \perp \hat{n}} \frac{|\underline{d} \cdot \underline{\epsilon}|^2}{|\underline{d}|^2}. \tag{7}$$

(1 Point)

d) Show that $\int d\Omega P(\Omega) = 1$ using that $\sum_{\underline{\epsilon} \perp \hat{n}} |\underline{d} \cdot \underline{\epsilon}|^2 = |\underline{d}|^2 - |\underline{d} \cdot \hat{n}|^2$ and setting $\hat{n}$ along the $\hat{z}$ axis, so that $|\underline{d} \cdot \hat{n}| = |\underline{d}| \cos\theta$. *(1 Point)*

Note that:

$$\gamma = \frac{4}{3} \frac{|\underline{d}|^2 \omega_0^3}{\hbar c^3} \tag{8}$$

is the Einstein's A coefficient for spontaneous emission in the CGS (Gauss) units. In the SI units

$$\gamma_{\text{SI}} = \frac{\gamma_{\text{cgs}}}{4\pi\epsilon_0} = \frac{|\underline{d}|^2 \omega_0^3}{3\pi\epsilon_0 \hbar c^3}. \tag{9}$$

Question 4 *Unraveling the master equation*

Consider the master equation for a damped Harmonic oscillator:

$$\frac{\partial}{\partial t}\rho = \frac{1}{i\hbar} [H_0, \rho] + \kappa \left(2a\rho a^\dagger - a^\dagger a \rho - \rho a^\dagger a \right), \tag{10}$$

where a and $a^\dagger$ are the photon annihilation and creation operator, H_0 governs the field coherent dynamics and κ is the photon damping rate.

Write the equation in terms of an effective, non hermitian Hamiltonian and a jump operator. Write $\rho = \sum_k \rho^{(k)}$ and determine the exact form of $\rho^{(k)}$ as the k-th order term in the power expansion in terms of the jump operator. Discuss the result in terms of quantum trajectories. *(3 Point)*