TPV

SoSe 2018 **Blatt 4** **10-11.05.2018**

Exercise 8 *Commutation relations for the electromagnetic field*

Every vector potential $\mathbf{A}(\mathbf{r})$ can be written as the sum of a longitudinal and a transverse component:

$$\mathbf{A}(\mathbf{r}) = \mathbf{A}_{\parallel}(\mathbf{r}) + \mathbf{A}_{\perp}(\mathbf{r})\,, \tag{1}$$

with $\nabla \times \mathbf{A}_{\parallel} = 0$ and $\nabla \cdot \mathbf{A}_{\perp} = 0$. The longitudinal and transverse δ-functions are defined through the relations

$$\mathbf{A}_{\parallel}(\mathbf{r}) = \sum_m \int d^3x\; \delta^{\parallel}_{lm}(\mathbf{r}-\mathbf{r}')\, A_m(\mathbf{r}')\,, \tag{2}$$

$$\mathbf{A}_{\perp}(\mathbf{r}) = \sum_m \int d^3x\; \delta^{\perp}_{lm}(\mathbf{r}-\mathbf{r}')\, A_m(\mathbf{r}')\,, \tag{3}$$

for $l, m, n \in \{x, y, z\}$ and they satisfy the following identities:

$$\delta^{*}_{lm}(\mathbf{r}) = \delta^{*}_{ml}(\mathbf{r})\,, \qquad \delta^{*}_{lm}(\mathbf{r}) = \delta^{*}_{lm}(-\mathbf{r})\,, \qquad \frac{\partial}{\partial x_l}\delta^{||}_{mn}(\mathbf{r}-\mathbf{r}') = \frac{\partial}{\partial x_m}\delta^{||}_{ln}(\mathbf{r}-\mathbf{r}')\,, \tag{4}$$

$$\sum_l \frac{\partial}{\partial x_l}\delta^{||}_{lm}(\mathbf{r}-\mathbf{r}') = \frac{\partial}{\partial x_m}\delta(\mathbf{r}-\mathbf{r}')\,, \qquad \sum_l \frac{\partial}{\partial x_l}\delta^{\perp}_{lm}(\mathbf{r}-\mathbf{r}') = 0\,, \tag{5}$$

where $*$ is an index which runs over the components $* = \{\|, \perp\}$. Beside, their Fourier transforms have the properties

$$\delta^{||}_{lm}(\mathbf{k}) = \frac{k_l k_m}{|\mathbf{k}|^2}\,, \qquad \delta^{\perp}_{lm}(\mathbf{k}) = \delta_{lm} - \frac{k_l k_m}{|\mathbf{k}|^2}\,. \tag{6}$$

a) Show, first, that the following relations hold:

$$\begin{aligned}
\sum_{\vec{\epsilon}_\lambda} \epsilon_{\lambda,l}\epsilon_{\lambda,m} &= \delta_{lm} - \frac{k_{\lambda,l}k_{\lambda,m}}{|\mathbf{k}_\lambda|^2}\,, \\
\sum_{\vec{\epsilon}_\lambda} \epsilon_{\lambda,l}\frac{(\mathbf{k}_\lambda \times \boldsymbol{\epsilon}_\lambda)_m}{|\mathbf{k}_\lambda|} &= \sum_n \varepsilon_{lmn}\frac{k_{\lambda,n}}{|\mathbf{k}_\lambda|}\,, \\
\sum_{\boldsymbol{\epsilon}_\lambda} \frac{(\mathbf{k}_\lambda \times \boldsymbol{\epsilon}_\lambda)_l}{|\mathbf{k}_\lambda|}\frac{(\mathbf{k}_\lambda \times \vec{\epsilon}_\lambda)_m}{|\mathbf{k}_\lambda|} &= \delta_{lm} - \frac{k_{\lambda,l}k_{\lambda,m}}{|\mathbf{k}_\lambda|^2}\,,
\end{aligned} \tag{7}$$

$l, m, n \in \{x, y, z\}$ and ε_{lmn} being the Levi-Civita-Tensor. Using Eqs. (7), show that the cartesian components of the fields in the Coulomb gauge satisfy the commutator:

$$\left[\hat{E}_x(\mathbf{r},t), \hat{B}_y(\mathbf{r}',t)\right] = 4\pi i c\hbar \frac{\partial}{\partial z}\delta(\mathbf{r}-\mathbf{r}\,')\,; \tag{8}$$

the fields being given by:

$$\hat{\mathbf{E}}(\mathbf{r}) = \int d^3k \sum_{\epsilon_\lambda} i\mathcal{E}_k \boldsymbol{\epsilon}_\lambda \left[\hat{a}_{\epsilon_\lambda} e^{i\mathbf{k}\cdot\mathbf{r}} - \hat{a}^\dagger_{\epsilon_\lambda} e^{-i\mathbf{k}\cdot\mathbf{r}}\right] , \tag{9}$$

$$\hat{\mathbf{B}}(\mathbf{r}) = \int d^3k \sum_{\epsilon_\lambda} i\mathcal{B}_k \left(\frac{\mathbf{k}}{|\mathbf{k}|} \times \boldsymbol{\epsilon}_\lambda\right) \left[\hat{a}_{\epsilon_\lambda} e^{i\mathbf{k}\cdot\mathbf{r}} - \hat{a}^\dagger_{\epsilon_\lambda} e^{-i\mathbf{k}\cdot\mathbf{r}}\right] , \tag{10}$$

with $\mathcal{B}_k = \mathcal{E}_k = \sqrt{\hbar\omega/(4\pi^2)}$. *(2 Punkte)*

Hint: The inverse Fourier-transform has been here defined with the factor $(2\pi)^{-3}$.

b) In the Heisenberg picture the operators of the free fields are given by: gegeben durch

$$\hat{\mathbf{E}}(\mathbf{r}, t) = \int d^3k \sum_{\epsilon_\lambda} i\mathcal{E}_k \boldsymbol{\epsilon}_\lambda \left[\hat{a}_\lambda e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} - \hat{a}^\dagger_\lambda e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] , \tag{11}$$

$$\hat{\mathbf{B}}(\mathbf{r}, t) = \int d^3k \sum_{\epsilon_\lambda} i\mathcal{B}_k \left(\frac{\mathbf{k}}{|\mathbf{k}|} \times \boldsymbol{\epsilon}_\lambda\right) \left[\hat{a}_\lambda e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} - \hat{a}^\dagger_\lambda e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}\right] . \tag{12}$$

Evaluate the commutator $[\hat{E}_x(\mathbf{r}_1,\, t_1), \hat{B}_y(\mathbf{r}_2,\, t_2)]$, and discuss the possibility of a simultaneous measurement of two field components for different space-time points ($|\boldsymbol{\rho}|^2 = c^2\tau^2$, $|\boldsymbol{\rho}|^2 > c^2\tau^2$ und $|\boldsymbol{\rho}|^2 < c^2\tau^2$), wobei $\mathbf{r}_1 - \mathbf{r}_2 = \boldsymbol{\rho}$, $t_1 - t_2 = \tau$. *(2 Punkte)*

Hint: The frequency is a function of $|\mathbf{k}|$*:* $\omega = c|\mathbf{k}|$.

Exercise 9 *Gauge invariance*

The Lagrangian density for the electromagnetic field in the presence of an external current density j_μ is:

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu} F_{\mu\nu} - j_\mu A_\mu, \tag{13}$$

a) what is the condition on j_μ in order ensure the gauge invariance? *(1 Punkt)*