TPV

SoSe 2018 **Blatt 1** **19-20/04/2018**

Exercise 1

Consider the (free-field) Klein-Gordon equation given by

$$\Box\phi(\mathbf{x},t) - \frac{m^2c^2}{\hbar^2}\phi(\mathbf{x},t) = 0, \qquad \Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} \tag{1}$$

and let $\phi(\mathbf{x},t)$ be a solution to it. Defining $\psi(\mathbf{x},t)$ such that

$$\phi(\mathbf{x},t) = \psi(\mathbf{x},t)e^{-imc^2t/\hbar}, \tag{2}$$

a) determine under which condition $\psi(\mathbf{x},t)$ will satisfy the non-relativistic Schrödinger equation $i\hbar\frac{\partial}{\partial t}\psi(\mathbf{x},t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x},t)$. Give a physical interpretation when ϕ is a plane wave solution. *(1 Point)*

Exercise 2 *Charged harmonic oscillator in a variable electric field*

A one-dimensional harmonic oscillator is composed of a particle of mass m, charge q and potential energy $V(\hat{x}) = \frac{1}{2}m\omega^2\hat{x}^2$. We assume that the particle is placed in an electric field $E(t)$ parallel to the x-axis and time-dependent, so that the potential energy:

$$\hat{W}(t) = -qE(t)\hat{x} \tag{3}$$

has to be added to $V(\hat{x})$. Let $|\psi(0)\rangle$ be the state of the system at $t = 0$.

a) Write the hamiltonian $\hat{H}(t)$ of the particle in terms of the operators $\hat{a}$ and $\hat{a}^\dagger$ (annihilation and creation operators of the simple harmonic oscillator). Evaluate the commutators of $\hat{a}$ and $\hat{a}^\dagger$ with $\hat{H}(t)$. *(1 Point)*

b) Let $\alpha(t)$ be the number defined by:

$$\alpha(t) = \langle\psi(t)|\,\hat{a}\,|\psi(t)\rangle \tag{4}$$

where $|\psi(t)\rangle$ is the normalized state vector at time t of the particle under study. Using the previous results, show that $\alpha(t)$ satisfies the differential equation:

$$\frac{\partial}{\partial t}\alpha(t) = -\mathrm{i}\omega\alpha(t) + \mathrm{i}\lambda(t) \tag{5}$$

where

$$\lambda(t) = \frac{q}{\sqrt{2m\hbar\omega}}E(t). \tag{6}$$

Integrate this differential equation. At time t, what are the mean values of the position and momentum of the particle? *(1 Point)*

c) The ket $|\phi(t)\rangle$ is defined by:

$$|\phi(t)\rangle = (\hat{a} - \alpha(t))|\psi(t)\rangle \tag{7}$$

where $\alpha(t)$ is the value calculated in b. Using the results of questions a and b, show that the evolution of $|\phi(t)\rangle$ is given by:

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|\phi(t)\rangle = (\hat{H}(t) + \hbar\omega)|\phi(t)\rangle \tag{8}$$

How does the form of $|\phi(t)\rangle$ vary with time? *(1 Point)*

d) Assuming that $|\psi(0)\rangle$ is an eigenvector of $\hat{a}$ with eigenvalue $\alpha(0)$, show that $|\psi(t)\rangle$ is also an eigenvector of $\hat{a}$, and evaluate its eigenvalue. Find at time t the mean value of the unperturbed Hamiltonian

$$\hat{H}_0 = \hat{H}(t) - \hat{W}(t) \tag{9}$$

as a function of $\alpha(0)$. Give the root-mean-square deviations

$$\Delta x = \sqrt{\langle \hat{x}^2\rangle_t - \langle \hat{x}\rangle_t^2}, \quad \Delta p = \sqrt{\langle \hat{p}^2\rangle_t - \langle \hat{p}\rangle_t^2}, \quad \Delta H_0 = \sqrt{\langle \hat{H}_0^{\,2}\rangle_t - \langle \hat{H}_0\rangle_t^2}; \tag{10}$$

how do they change in time? *(2 Points)*