TPV

SoSe 2018 **Blatt 10** **28-29/06/2018**

Exercise 20 *Spin-0 particle*

Let $\hat{\psi}$ be a scalar field with spin equal to zero. The quantised field operator is hermitian and given by

$$\hat{\psi}(\mathbf{x},t) = \frac{1}{\sqrt{V}}\sum_{\mathbf{p}} \frac{1}{\sqrt{2E_{\mathbf{p}}}}\left[\hat{c}_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x}-E_{\mathbf{p}}t)/\hbar} + \hat{c}^{\dagger}_{\mathbf{p}} e^{-i(\mathbf{p}\cdot\mathbf{x}-E_{\mathbf{p}}t)/\hbar}\right], \tag{1}$$

where $E_{\mathbf{p}} = +\sqrt{\mathbf{p}^2c^2 + m^2c^4}$ is the energy related to momentum $\mathbf{p}$, m the mass, V the volume and $\hat{c}^{\dagger}_{\mathbf{p}}$, $\hat{c}_{\mathbf{p}}$ some creation and annihilation operators.

The field describes a particle of only one species, that is equal to its own antiparticle. Show that the Hamiltonian

$$\hat{H} = \frac{\hbar^2c^2}{2}\int_V \mathrm{d}^3x \left[\frac{1}{c^2}\left(\frac{\partial\hat{\psi}}{\partial t}\right)^2 + \left(\nabla\hat{\psi}\right)^2 + \frac{m^2c^2}{\hbar^2}\hat{\psi}^2\right] \tag{2}$$

can be written as

$$\hat{H} = \frac{1}{2}\sum_{\mathbf{p}} E_{\mathbf{p}}\left(\hat{c}^{\dagger}_{\mathbf{p}}\hat{c}_{\mathbf{p}} + \hat{c}_{\mathbf{p}}\hat{c}^{\dagger}_{\mathbf{p}}\right). \tag{3}$$

Calculate the eigenvalues of $\hat{H}$ for $\hat{c}^{\dagger}_{\mathbf{p}}$, $\hat{c}_{\mathbf{p}}$ being either bosonic or fermionic creation and annihilation operators, respectively. Argue whether the excitations of the real scalar field are bosonic or fermionic fields.

Evaluate the four-current

$$\hat{j}_{\mu} = -i\hbar\left[\hat{\psi}^{\dagger}\partial_{\mu}\hat{\psi} - \left(\partial_{\mu}\hat{\psi}^{\dagger}\right)\hat{\psi}\right], \tag{4}$$

and discuss which charge the particles may have. *(2 Points)*

Exercise 21 *Dipole interaction*

We want to understand the non relativistic interaction between an atom and the electromagnetic field in case of spontaneous emission. Hereby we consider the atom to be initially in the excited state $|e\rangle$ and the electromagnetic field to be in the vacuum state $|vac\rangle$. After the emission the atom decays to the ground state $|g\rangle$ and the electromagnetic field is in the state $|1_\lambda\rangle$ *i. e.* one excitation of the field in mode $\lambda = \left(\mathbf{k}_\lambda, \boldsymbol{\epsilon}_\lambda\right)$. The frequency of the mode is $\omega_\lambda = c|\mathbf{k}_\lambda|$ and it is resonant to the atomic transition frequency $\omega_0 = \omega_e - \omega_g$, where $\hat{H}_0|g\rangle = \hbar\omega_g|g\rangle$ and $\hat{H}_0|e\rangle = \hbar\omega_e|e\rangle$. The matrix element in the lowest order perturbation theory is given by

$$\langle e, vac|\hat{H}_1^{(1)}|g, 1_{\lambda'}\rangle = -\frac{e}{mc}\langle e, vac|\hat{\mathbf{p}}\cdot\hat{\mathbf{A}}^{\perp}(\hat{\mathbf{R}},t)|g, 1_{\lambda'}\rangle, \tag{5}$$

where $\hat{\mathbf{p}}$ is the relative momentum and the transverse vector potential evaluated at the center of mass $\hat{\mathbf{R}}$ is

$$\hat{\mathbf{A}}^{\perp}(\hat{\mathbf{R}},t) = \sum_{\lambda}\sqrt{\frac{2\pi\hbar c^2}{V\omega_\lambda}}\boldsymbol{\epsilon}_\lambda\left(\hat{a}_\lambda e^{i\mathbf{k}_\lambda\cdot\hat{\mathbf{R}}-i\omega_\lambda t} + \hat{a}^{\dagger}_\lambda e^{-i\mathbf{k}_\lambda\cdot\hat{\mathbf{R}}+i\omega_\lambda t}\right), \tag{6}$$

with $\boldsymbol{\epsilon}_\lambda \perp \mathbf{k}_\lambda$, V is the quantisation volume and $\hat{a}^\dagger_\lambda$, $\hat{a}_\lambda$ are the creation and annihilation operators of the electromagnetic field.

Show that the matrix element in Eq. (5) in the lowest order perturbation theory can be written as

$$-\langle e, vac|\hat{\mathbf{d}} \cdot \hat{\mathbf{E}}^\perp(\hat{\mathbf{R}}, t)|g, 1_{\lambda'}\rangle, \tag{7}$$

where $\hat{\mathbf{d}} = e\hat{\mathbf{r}}$ is the atomic dipole moment ($\hat{\mathbf{p}}$ is the conjugate momentum to $\hat{\mathbf{r}}$) and $\hat{\mathbf{E}}^\perp(\hat{\mathbf{R}}, t)$ is the transverse electric field given by:

$$\hat{\mathbf{E}}^\perp(\hat{\mathbf{R}}, t) = i \sum_\lambda \sqrt{\frac{2\pi\hbar\omega_\lambda}{V}} \boldsymbol{\epsilon}_\lambda \Big(\hat{a}_\lambda e^{i\mathbf{k}_\lambda \cdot \mathbf{R} - i\omega_\lambda t} - \hat{a}^\dagger_\lambda e^{-i\mathbf{k}_\lambda \cdot \mathbf{R} + i\omega_\lambda t}\Big). \tag{8}$$

Hint: Use that, at the lowest order in an expansion in $\hat{\mathbf{A}}^\perp(\hat{\mathbf{R}}, t)$, $\hat{\mathbf{p}} \simeq m\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{r}}$ *and that* $\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{r}} \simeq -\frac{1}{i\hbar}\Big[\hat{H}_0, \hat{\mathbf{r}}\Big]$. *(2 Points)*