TPV

SoSe 2018 **Blatt 11** **05-06/07/2018**

Exercise 20 *Energy-momentum tensor for the scalar field*

It is given the complex scalar field

$$\hat{\psi}(\mathbf{x}, t) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left[\hat{a}_{\mathbf{p}} e^{i(\mathbf{p}\cdot\mathbf{x} - E_{\mathbf{p}} t)/\hbar} + \hat{b}^{\dagger}_{\mathbf{p}} e^{-i(\mathbf{p}\cdot\mathbf{x} - E_{\mathbf{p}} t)/\hbar} \right], \tag{1}$$

where $E_{\mathbf{p}} = +\sqrt{\mathbf{p}^2 c^2 + m^2 c^4}$ is the energy related to momentum $\mathbf{p}$, m the mass, V the volume and the Hamiltonian in second quantization reads

$$\hat{H} = \sum_{\mathbf{p}} E_{\mathbf{p}} \left(\hat{a}^{\dagger}_{\mathbf{p}} \hat{a}_{\mathbf{p}} + \hat{b}^{\dagger}_{\mathbf{p}} \hat{b}_{\mathbf{p}} \right), \tag{2}$$

with $\hat{a}_{\mathbf{p}}$, $\hat{b}_{\mathbf{p}}$ and $\hat{a}^{\dagger}_{\mathbf{p}}$, $\hat{b}^{\dagger}_{\mathbf{p}}$ creation and annihilation operators satisfying the following commutation relations:

$$\left[\hat{a}_{\mathbf{p}},\, \hat{a}^{\dagger}_{\mathbf{p}'} \right] = \delta_{\mathbf{p},\mathbf{p}'} = \left[\hat{b}_{\mathbf{p}},\, \hat{b}^{\dagger}_{\mathbf{p}'} \right], \tag{3}$$

$$\left[\hat{a}_{\mathbf{p}},\, \hat{a}_{\mathbf{p}'} \right] = 0 = \left[\hat{b}_{\mathbf{p}},\, \hat{b}_{\mathbf{p}'} \right]. \tag{4}$$

a) Show that $cP_j = -i \int_V \mathrm{d}\mathbf{x}\, \mathcal{T}_{4j}$ can be written in second quantization as

$$P_j = \sum_{\mathbf{p}} p_j \left(\hat{a}^{\dagger}_{\mathbf{p}} \hat{a}_{\mathbf{p}} + \hat{b}^{\dagger}_{\mathbf{p}} \hat{b}_{\mathbf{p}} \right), \tag{5}$$

where

$$\mathcal{T}_{\mu\nu} = -\frac{\partial \psi}{\partial x_\nu} \frac{\partial \mathcal{L}}{\partial \left(\partial \psi / \partial x_\mu \right)} - \frac{\partial \psi^*}{\partial x_\nu} \frac{\partial \mathcal{L}}{\partial \left(\partial \psi^* / \partial x_\mu \right)} + \mathcal{L} \delta_{\mu\nu} \tag{6}$$

is the energy-momentum tensor and

$$\mathcal{L} = -\hbar^2 c^2 \left[\partial_\mu \psi^* \partial_\mu \psi + \left(\frac{mc}{\hbar} \right)^2 |\psi|^2 \right] \tag{7}$$

is the Lagrangian density. *(1 Point)*

b) Show that $\left[\hat{P}_\mu, \hat{a}_{\mathbf{p}} \right] = -p_\mu \hat{a}_{\mathbf{p}}$. *(1 Point)*

c) Using the previous point establish that $\hat{P}_\mu$ is the generator of space-time translations by showing that $\left[\hat{P}_\mu, \hat{\psi}(\mathbf{x}, t) \right] = i\hbar \partial_\mu \hat{\psi}(\mathbf{x}, t)$. *(1 Point)*

d) Let $|K\rangle$ be an eigenstate of $\hat{P}_\mu$ such that $\hat{P}_\mu|K\rangle = K_\mu|K\rangle$, show then that the quantity $\langle K|\hat{\psi}(\mathbf{x},t)\hat{\psi}(\mathbf{x}',t')|K\rangle$ is translationally invariant

$$\langle K|\hat{\psi}(\mathbf{x},t)\hat{\psi}(\mathbf{x}',t')|K\rangle = \langle K|\hat{\psi}(\mathbf{x}-\mathbf{x}',t-t')\hat{\psi}(0,0)|K\rangle \tag{8}$$

(2 Points)

Hint: Use the fact that

$$\hat{\psi}(\mathbf{x},t) = e^{-i\hat{P}\cdot x/\hbar}\hat{\psi}(0,0)e^{i\hat{P}\cdot x/\hbar} \tag{9}$$

where we have used the notation

$$a \cdot b = \mathbf{a}\cdot\mathbf{b} - a_0 b_0, \qquad \textit{with } a = (\mathbf{a}, ia_0), \quad b = (\mathbf{b}, ib_0), \tag{10}$$

already introduced in the lectures.

e) The designation of particle and antiparticle is a matter of convention and we can freely reverse the labels. In order to see this, given the scalar field in Eq. (1), construct an operator that takes $\hat{a}_\mathbf{p}$ and $\hat{a}^\dagger_\mathbf{p}$ to $\hat{b}_\mathbf{p}$ and $\hat{b}^\dagger_\mathbf{p}$ and vice versa and commutes with the Hamiltonian of Eq. (2)

$$\begin{aligned} \hat{C}\hat{\psi}(\mathbf{x},t)\hat{C}^{-1} &= e^{i\gamma}\hat{\psi}^\dagger(\mathbf{x},t), \\ \left[\hat{C},\hat{H}\right] &= 0, \\ \hat{C}^\dagger\hat{C} &= 1, \\ \hat{C}^2 &= 1, \end{aligned} \tag{11}$$

where γ is an arbitrary phase. *(2 Points)*