# TPV

**SoSe 2018** **Blatt 2** **26-27/04/2018**

## Exercise 3 *Quantization of a classical field*

Consider the Lagrangian for a system of $N+2$ masses coupled through an harmonic interaction given by

$$L(\eta, \dot{\eta}) = \sum_{j=0}^{N+1} \left[ \frac{m}{2} \dot{\eta}_j^2 - \frac{\kappa}{2} (\eta_j - \eta_{j+1})^2 \right], \tag{1}$$

where $m$ is the mass, $\kappa$ is a force costant and the $\{\eta_j(t)\}_0^{N+1}$ are the lateral displacements of the masses along a 1D-chain. Beside, assume fixed-endpoint boundary conditions:

$$\eta_0(t) = \eta_{N+1}(t) = 0. \tag{2}$$

a) Derive the equations of motions of the given system. *(1 Point)*

b) Show that the normal modes are

$$\eta_j(t) = \cos(\omega t) \sin(jp) \tag{3}$$

and determine the allowed (discrete) values for $p$ imposing the boundary conditions of Eq. (2). *(2 Points)*

c) Show then that the $N$ independent frequencies of the system are given by:

$$\omega_n^2 = \omega_0^2 \sin^2 \left[ \frac{\pi n}{2(N+1)} \right], \qquad \omega_0 = 2\sqrt{\frac{\kappa}{m}}. \tag{4}$$

Plot $\omega_n/\omega_0$ as a function of $n$ and determine the highest eigen-frequency. *(2 Points)*

d) Solve the previous point assuming periodic boundary conditions (PBC). How does the spectrum change? *(2 Points)*

e) Let's now take the continuum limit:

$$l = Na \quad \text{fixed, while} \qquad a \to 0, \quad N \to \infty, \tag{5}$$

considering the continuous variable $\eta(x,t)$ and performing the replacements:

$$(\eta_{n+1} - \eta_n)^2 \to a^2 \left[ \frac{\partial \eta(x,t)}{\partial x} \right]^2$$
$$\sum_j \left[ \cdot \right] \to \frac{1}{a} \int_0^l \left[ \cdot \right] \mathrm{d}x \tag{6}$$

one gets the Lagrangian:

$$L_{cont} = \frac{1}{2}\int_0^l \left[\rho\left(\frac{\partial\eta(x,t)}{\partial t}\right)^2 - Y\left(\frac{\partial\eta(x,t)}{\partial x}\right)^2\right] \tag{7}$$

assuming finite values for the mass density $\rho = m/a$ and the string tension $Y = \kappa a$.

Derive from Eq. (7) the equations of motions using the Euler-Lagrange equations. Then find the normal modes for the continuous ring (PBC i.e. $\eta(0,t) = \eta(l,t)$) and fixed-endpoint boundary conditions ($\eta(0,t) = 0 = \eta(l,t)$). *(2 Points)*

f) Determine the Hamiltonian $H(p_1,\dots,p_N;\eta_1,\dots,\eta_N)$ for the classical discrete chain, where $p_j$ is the canonical momentum conjugate to the variable $\eta_j$. *(1 Point)*

g) Now, in order to quantize the system, take $\{\hat{p}_j\}_1^N$ and $\{\hat{\eta}_j\}_1^N$ to be hermitian operators satisfying the commutation relations:

$$[\hat{p}_j,\, \hat{\eta}_k] = -i\hbar\delta_{jk} \tag{8}$$

and assume PBC.

Verify that the transformation

$$\begin{aligned} \hat{\eta}_j &= \frac{1}{\sqrt{N}}\sum_{n=-N/2}^{N/2} \hat{Q}_n\, e^{i\,2\pi nj/N}, \\ \hat{p}_j &= \frac{1}{\sqrt{N}}\sum_{n=-N/2}^{N/2} \hat{P}_n\, e^{i\,2\pi nj/N}, \end{aligned} \tag{9}$$

defines a new set of operators $\{\hat{P}_j\}_1^N$ and $\{\hat{Q}_j\}_1^N$ which satisfy the following identities:

$$\begin{aligned} \left[\hat{P}_n^\dagger,\, \hat{Q}_m\right] &= -i\hbar\delta_{nm}, \\ \hat{P}_n^\dagger &= \hat{P}_{-n}, \\ \hat{Q}_n^\dagger &= \hat{Q}_{-n}. \end{aligned} \tag{10}$$

*(1 Point)*

h) Find the transformed Hamiltonian $\hat{\tilde{H}}\left(\hat{P}_{-N/2},\dots,\hat{P}_{N/2};\hat{Q}_{-N/2},\dots,\hat{Q}_{N/2}\right)$ and determine its eigenvalues. *(1 Point)*

## **Exercise 4** *Maxwell equations*

Show that, given the Maxwell tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & B_3 & -B_2 & -iE_1 \\ -B_3 & 0 & B_1 & -iE_2 \\ B_2 & -B_1 & 0 & -iE_3 \\ iE_1 & iE_2 & iE_3 & 0 \end{pmatrix} \tag{11}$$

and the 4−current

$$j_\mu = (\mathbf{j}, ic\rho), \tag{12}$$

a)

$$\frac{\partial F_{\mu\nu}}{\partial x_\nu} = \frac{j_\mu}{4\pi c} \tag{13}$$

gives the inhomogeneous Maxwell equations; *(2 Points)*

b) while the homogeneous ones are given by

$$\frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} = 0. \tag{14}$$

*(2 Points)*

## Exercise 5 *Boson/Fermion creation and annihilation operators*

The basic commutation relation for the boson annihilation and creation operator is

$$\left[\hat{b}, \hat{b}^\dagger\right] = \mathbf{1}, \qquad \left[\hat{b}, \hat{b}\right] = 0 = \left[\hat{b}^\dagger, \hat{b}^\dagger\right]; \tag{15}$$

while the basic anticommutation relation for the fermion annihilation and creation operator is

$$\left\{\hat{f}, \hat{f}^\dagger\right\} = \mathbf{1}, \qquad \left\{\hat{f}, \hat{f}\right\} = 0 = \left\{\hat{f}^\dagger, \hat{f}^\dagger\right\}. \tag{16}$$

Starting from these definitions show that:

a) the eigenstate $|n_B\rangle$ of $\hat{b}^\dagger\hat{b}$ have the properties:

$$\begin{aligned} \hat{b}^\dagger\hat{b}|n_B\rangle &= n_B\,|n_B\rangle, \qquad n_B \in \mathbb{N}, \\ \hat{b}|n_B\rangle &= \sqrt{n_B}\,|n_B - 1\rangle, \\ \hat{b}^\dagger|n_B\rangle &= \sqrt{n_B + 1}\,|n_B + 1\rangle; \end{aligned} \tag{17}$$

*(1 Point)*

b) the eigenstate $|n_F\rangle$ of $\hat{f}^\dagger\hat{f}$ have the properties:

$$\begin{aligned} \hat{f}^\dagger\hat{f}|n_F\rangle &= n_F\,|n_F\rangle, \qquad n_F \in \left\{0, 1\right\}, \\ \hat{f}|n_F\rangle &= \sqrt{n_F}\,|n_F - 1\rangle, \\ \hat{f}^\dagger|n_F\rangle &= \sqrt{1 - n_F},|n_F + 1\rangle. \end{aligned} \tag{18}$$

*(1 Point)*