TPV

SoSe 2018 **Blatt 3** **03-04/05/2018**

Exercise 6 *Free electromagnetic field as a harmonic oscillator*

The free electromagnetic field in the Coulomb gauge ($\nabla \cdot \mathbf{A}(\mathbf{r}, t) = 0$) is fully described by a vector potential $\mathbf{A}(\mathbf{r}, t)$ which can be decomposed in the following form:

$$\mathbf{A}(\mathbf{r}, t) = \sum_{\lambda} q_{\lambda}(t)\mathbf{f}_{\lambda}(\mathbf{r}). \tag{1}$$

For a box of volume V and length L, the functions $\mathbf{f}_{\lambda}(\mathbf{r})$ are given by

$$\mathbf{f}_{\lambda}(\mathbf{r}) = \sqrt{\frac{8\pi c^2}{L^3}}\hat{\varepsilon}_{\lambda} \begin{cases} \cos\left(\mathbf{k}_{\lambda} \cdot \mathbf{r}\right), \\ \sin\left(\mathbf{k}_{\lambda} \cdot \mathbf{r}\right), \end{cases} \qquad \mathbf{k}_{\lambda} = \frac{2\pi}{L}\left(n_x, n_y, n_z\right) \tag{2}$$

with $n_x, n_y \in \mathbb{Z}$, $n_z \in \mathbb{N}^*$ and possible polarizations $\hat{\varepsilon}_{\lambda} = \hat{\varepsilon}_{\lambda}^{(1)}, \hat{\varepsilon}_{\lambda}^{(2)} \perp \mathbf{k}_{\lambda}$. The $\mathbf{f}_{\lambda}(\mathbf{r})$ form a complete orthonormal basis satisfying the orthogonality relation

$$\int \mathbf{f}_{\lambda'}(\mathbf{r})\mathbf{f}_{\lambda}(\mathbf{r})\,\mathrm{d}\mathbf{x} = \delta_{\lambda'\lambda}\mathcal{N}, \tag{3}$$

$\mathcal{N}$ being some normalization constant. For the given $\mathbf{f}_{\lambda}(\mathbf{r})$ the Coulomb-gauge condition follows directly. If the vector potential satisfies the periodic boundary conditions the functions $\mathbf{f}_{\lambda}(\mathbf{r})$ will be solutions to the following equation:

$$\nabla^2\mathbf{f}_{\lambda}(\mathbf{r}) + \frac{\omega_{\lambda}^2}{c^2}\mathbf{f}_{\lambda}(\mathbf{r}) = 0 \tag{4}$$

a) Write an Hamiltonian density for the free electromagnetic field $\mathcal{H} = \frac{1}{8\pi}\left(|\mathbf{E}|^2 + |\mathbf{B}|^2\right)$ in terms of $q_{\lambda}(t)$ and $\mathbf{f}_{\lambda}(\vec{r})$ functions.

(1 Point)

b) Prove that the Hamiltonian of the electromagnetic field $H = \int_V \mathcal{H}\,\mathrm{d}\mathbf{r}$ can be written as the Hamiltonian of a collection of harmonic oscillators:

$$H = \sum_{\lambda}\left(\frac{1}{2}p_{\lambda}^2 + \frac{\omega_{\lambda}^2}{2}q_{\lambda}^2\right) \tag{5}$$

Hint 1 : To derive the Hamiltonian use the relation:

$$\nabla \cdot (\mathbf{A} \times (\nabla \times \mathbf{B})) = (\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{B}) - \mathbf{A} \cdot (\nabla \times (\nabla \times \mathbf{B})). \tag{6}$$

Hint 2 : Use the Ostrogradsky-Gauss theorem (divergence theorem) and the periodic boundary conditions: $\mathbf{f}_{\lambda}(x, y, z) = \mathbf{f}_{\lambda}(x + L, y, z) = \mathbf{f}_{\lambda}(x, y + L, z) = \mathbf{f}_{\lambda}(x, y, z + L)$.

(3 Points)

Exercise 7 *The energy-momentum tensor*

a) Show that the energy-momentum tensor density defined by:

$$\mathcal{T}_{\mu\nu} = -\frac{\partial \phi}{\partial x_\nu} \frac{\partial \mathcal{L}}{\partial \left(\partial \phi / \partial x_\mu \right)} + \mathcal{L} \delta_{\mu\nu} \tag{7}$$

satisfies the continuity equation:

$$\frac{\partial \mathcal{T}_{\mu\nu}}{\partial x_\mu} = 0. \tag{8}$$

when the Euler-Lagrangian equation for ϕ is assumed. *(1 Point)*

b) Show that each component of the four-vector

$$P_\mu(t) = -i \int \mathcal{T}_{4\mu} \mathrm{d}^3\mathbf{x} \tag{9}$$

is constant in time if ϕ vanishes sufficiently rapidly at infinity. (The integration is over three dimensional space at a given instant t.) *(1 Point)*

c) Derive the Hamiltonian density $\mathcal{H}$ and show that for the real scalar field it holds:

$$\mathcal{H} = -\mathcal{T}_{44}. \tag{10}$$

(1 Point)