TPV

SoSe 2018 **Blatt 5** **17-18/05/2018**

Exercise 10 *Cutoff Functions*

a) Calculate the vacuum energy

$$E_0(L) = \frac{1}{2}\sum_{\omega} \hbar\omega f(\omega), \qquad \omega = \frac{c\pi n}{L}, \quad n = \{1, 2, \dots \infty\} \tag{1}$$

for a one-dimensional system using a sharp cutoff, which corresponds to $f(\omega) = \theta(\omega_c - \omega)$ and show

$$E(a) = \frac{L\hbar\omega_c^2}{4c\pi} + \frac{\hbar\omega_c}{2} \qquad \text{(sharp cutoff)}. \tag{2}$$

When L is divided into segments a and $L - a$, this is independent of a. Thus there will be no force between inserted partitions. *(2 Points)*

b) Show, on the other hand, that any continuous cutoff function will lead to a nonzero cutoff-independent force. To do this, write

$$E_0(L) = \frac{\pi\hbar c}{2L}\sum_{n=1}^{\infty} n\, f\left(\frac{c\pi n}{\omega_c L}\right) \tag{3}$$

Since the argument of f approaches a continuous variable in the limit $\omega_c \to \infty$, we can approximate the sum by an integral, using the Euler-MacLaurin formula:

$$\begin{aligned} G(n) &= \int_{n^2}^{\infty} \mathrm{d}y\, F\left(\frac{\pi\sqrt{y}}{a}\right), \\ \sum_{n=1}^{\infty} G(n) &= \int_0^{\infty} \mathrm{d}n\, G(n) + \frac{1}{2}G(0) - \frac{1}{2!6}G'(0) + \frac{1}{4!30}G'''(0) + \dots \end{aligned} \tag{4}$$

in order to get

$$E_0(L) = \frac{\pi\hbar c}{2L}\int \mathrm{d}n\, n f\left(\frac{\pi n}{\omega_c L}\right) - \frac{\pi\hbar c}{24L} + \dots = \frac{L\omega_c^2}{2\pi}\int \mathrm{d}y\, y f(y) - \frac{\pi\hbar c}{24L} + \mathcal{O}(\omega_c^{-2}). \tag{5}$$

(3 Points)

Exercise 11 *Two photons states and rotations*

Let R be a rotation of the coordinate system about the z-axis through ϕ, and ζ be the one about the x-axis through π:

$$R : \begin{cases} x = x'\cos\phi + y'\sin\phi \\ y = -x'\sin\phi + y'\cos\phi \\ z = z' \end{cases} \qquad \zeta : \begin{cases} x = x' \\ y = -y' \\ z = -z' \end{cases} \tag{6}$$

a) Show that the creation operators transform as follows

$$\begin{aligned} Ra_+^\dagger(\mathbf{k})R^{-1} &= e^{-i\phi}a_+^\dagger(\mathbf{k}) & \zeta a_+^\dagger(\mathbf{k})\zeta^{-1} &= a_+^\dagger(-\mathbf{k}) \\ Ra_+^\dagger(-\mathbf{k})R^{-1} &= e^{i\phi}a_+^\dagger(-\mathbf{k}) & \zeta a_+^\dagger(-\mathbf{k})\zeta^{-1} &= a_+^\dagger(\mathbf{k}) \\ Ra_-^\dagger(-\mathbf{k})R^{-1} &= e^{i\phi}a_-^\dagger(\mathbf{k}) & \zeta a_-^\dagger(\mathbf{k})\zeta^{-1} &= a_-^\dagger(-\mathbf{k}) \\ Ra_-^\dagger(-\mathbf{k})R^{-1} &= e^{-i\phi}a_-^\dagger(-\mathbf{k}) & \zeta a_-^\dagger(-\mathbf{k})\zeta^{-1} &= a_-^\dagger(\mathbf{k}). \end{aligned} \tag{7}$$

(2 Points)

b) One can obtain interesting information about a state of two photons by examining its behavior under rotations and reflections. Consider two photons with momenta $\mathbf{k}$ and $-\mathbf{k}$. There are 4 independent states of polarization, which can be classified according to circular polarizations:

$$\begin{aligned} |++\rangle &= a_+^\dagger(\mathbf{k})a_+^\dagger(-\mathbf{k})|0\rangle \\ |+-\rangle &= a_+^\dagger(\mathbf{k})a_-^\dagger(-\mathbf{k})|0\rangle \\ |-+\rangle &= a_-^\dagger(\mathbf{k})a_+^\dagger(-\mathbf{k})|0\rangle \\ |--\rangle &= a_-^\dagger(\mathbf{k})a_-^\dagger(-\mathbf{k})|0\rangle \end{aligned} \tag{8}$$

Verify that, in terms of states with linear polarization,

$$\begin{aligned} |++\rangle + |--\rangle &= \left[a_1^\dagger(\mathbf{k})a_1^\dagger(-\mathbf{k}) - a_2^\dagger(\mathbf{k})a_2^\dagger(-\mathbf{k})\right]|0\rangle \\ |++\rangle - |--\rangle &= \left[a_1^\dagger(\mathbf{k})a_2^\dagger(-\mathbf{k}) + a_2^\dagger(\mathbf{k})a_1^\dagger(-\mathbf{k})\right]|0\rangle \\ |+-\rangle &= \left[a_1^\dagger(\mathbf{k})a_1^\dagger(-\mathbf{k}) + a_2^\dagger(\mathbf{k})a_2^\dagger(-\mathbf{k}) + ia_1^\dagger(\mathbf{k})a_2^\dagger(-\mathbf{k}) - ia_2^\dagger(\mathbf{k})a_1^\dagger(-\mathbf{k})\right]|0\rangle \\ |-+\rangle &= \left[a_1^\dagger(\mathbf{k})a_1^\dagger(-\mathbf{k}) + a_2^\dagger(\mathbf{k})a_2^\dagger(-\mathbf{k}) - ia_1^\dagger(\mathbf{k})a_2^\dagger(-\mathbf{k}) + ia_2^\dagger(\mathbf{k})a_1^\dagger(-\mathbf{k})\right]|0\rangle. \end{aligned} \tag{9}$$

From this, note that the polarizations of the two photons are correlated:

- in the state $|++\rangle + |--\rangle$ the planes are parallel;
- in the state $|++\rangle - |--\rangle$ the planes are orthogonal;
- in the state $|+-\rangle$ and $|-+\rangle$ the planes have equal probability of being parallel or orthogonal.

(2 Points)