TPV

SoSe 2018 **Blatt 6** **24-25/05/2018**

Exercise 12 *Photons with circular polarization*

Consider the two modes at wave vector $\mathbf{k}_\lambda$ and linear polarizations $\boldsymbol{\epsilon}_1$ and $\boldsymbol{\epsilon}_2$, such that $\mathbf{k}\cdot\boldsymbol{\epsilon}_1 = \mathbf{k}\cdot\boldsymbol{\epsilon}_2 = \boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_2 = 0$ and $\boldsymbol{\epsilon}_1\cdot\boldsymbol{\epsilon}_1 = \boldsymbol{\epsilon}_2\cdot\boldsymbol{\epsilon}_2 = 1$.
The energy is $\hat{H} = \hbar\omega\left(\hat{a}_1^\dagger a_1 + \hat{a}_2^\dagger a_2\right)$ where $\omega = c|\mathbf{k}|$ and a_j is the annihilation operator of a photon with wave-vector $\mathbf{k}$ and polarization $\boldsymbol{\epsilon}_j$, satisfying the commutation relation

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij}. \tag{1}$$

a) Take the modes with circular polarization

$$\begin{aligned}\hat{a}_+ &= -\frac{\hat{a}_1 + i\hat{a}_2}{\sqrt{2}}\\ \hat{a}_- &= \frac{\hat{a}_1 - i\hat{a}_2}{\sqrt{2}}\end{aligned} \tag{2}$$

show that

$$\begin{aligned}\left[\hat{a}_\pm, \hat{a}_\pm^\dagger\right] &= 1,\\ \left[\hat{a}_\mp, \hat{a}_\pm^\dagger\right] &= 0,\end{aligned} \tag{3}$$

and the Hamiltonian can be rewritten as $\hat{H} = \hbar\omega\left(\hat{a}_+^\dagger a_+ + \hat{a}_-^\dagger a_-\right)$. *(1 Point)*

b) Perform an infinitesimal rotation about $\mathbf{k}$ of $\delta\phi \ll 1$. Write $\boldsymbol{\epsilon}_1'$, $\boldsymbol{\epsilon}_2'$ and show that $\delta\boldsymbol{\epsilon}_\pm' = \mp i\delta\phi\epsilon_\pm$, where $\delta\boldsymbol{\epsilon}_\pm' = \boldsymbol{\epsilon}_\pm' - \boldsymbol{\epsilon}_\pm$ therefore the photon with $\boldsymbol{\epsilon}_\pm$ has spin component $\pm\hbar$.
(1 Point)

c) Discuss why the chemical potential of the photons is zero.
(1 Point)

Exercise 13 *Properties of the gamma matrices*

The Dirac-Hamiltonian operator is given by

$$\hat{H} = c\,\boldsymbol{\alpha}\cdot\mathbf{p} + \alpha_t mc^2 \tag{4}$$

with $\alpha_\mu = (\alpha_x, \alpha_y, \alpha_z, \alpha_t)$. It holds that $\alpha_\mu^\dagger = \alpha_\mu$, $\mathrm{Tr}\{\alpha_\mu\} = 0$ as well as $\{\alpha_\mu, \alpha_\nu\} = 2\delta_{\mu\nu}$, with the anticommutator defined as $\{A, B\} = AB + BA$. Beside, the Dirac Equation in the van-der-Waerden-Form is given by

$$\left(\gamma_\mu \frac{\partial}{\partial x_\mu} + \frac{mc}{\hbar}\right)\psi(\vec{x}, t) = 0\ , \tag{5}$$

where the γ-matrices are defined as $\gamma_j = -i\alpha_t\alpha_j$ for $j = 1, 2, 3$ and $\gamma_4 = \alpha_t$. Show the following properties, without explicitly writing the gamma matrices:

1. the anticommutators $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ hold ,
2. the γ_μ are hermitian,
3. the trace of the γ_μ vanishes.

(1 Point)