TPV

SoSe 2018 **Blatt 7** **01/06/2018**

Exercise 14 *Spin of the relativistic wave-function*

In the non-relativistic theory of spins the spin is described by the Pauli matrices σ_j. The extension to 4-components Dirac spinors is then given by the following operator:

$$\hat{\Sigma}_j = \frac{1}{2i}\left(\hat{\gamma}_k\hat{\gamma}_l - \hat{\gamma}_l\hat{\gamma}_k\right) \tag{1}$$

with (j,k,l) being a cyclic permutation of $(1,2,3)$.
The solutions for the Dirac equation for a free particle moving with momentum $\mathbf{p}$ are given by:

$$\psi^{(j)}(\mathbf{x},t) = \mathcal{N} u^{(j)}(\mathbf{p}) e^{i(\mathbf{p}\cdot\mathbf{x}-Et)/\hbar} \quad \text{with} \quad \mathcal{N} = \sqrt{\left(|E|+mc^2\right)/(2|E|V)} \tag{2}$$

$$u^{(1)}(\mathbf{p}) = \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}, \qquad u^{(2)}(\mathbf{p}) = \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ -\frac{cp_z}{E+mc^2} \end{pmatrix},$$

$$u^{(3)}(\mathbf{p}) = \begin{pmatrix} -\frac{cp_z}{|E|+mc^2} \\ -\frac{c(p_x+ip_y)}{|E|+mc^2} \\ 1 \\ 0 \end{pmatrix}, \qquad u^{(4)}(\mathbf{p}) = \begin{pmatrix} -\frac{c(p_x-ip_y)}{|E|+mc^2} \\ \frac{cp_z}{|E|+mc^2} \\ 0 \\ 1 \end{pmatrix},$$

where p_x, p_y, p_z are the components of the momentum $\mathbf{p}$, m is the mass, E is the energy, V is the volume.

a) Assume that at time $t=0$ the wave function of an electron is given by

$$\psi(\mathbf{x},0) = \frac{1}{\sqrt{V}} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} e^{ip_z z/\hbar}, \tag{3}$$

where a, b, c, d are independent from the space-time coordinates and they have to satisfy $|a|^2+|b|^2+|c|^2+|d|^2 = 1$. Find the probabilities to measure the electron at time $t>0$ with positive energy and spin component along $\mathbf{e}_z$ in: i) spin up state, ii) spin down state, as well as the probabilities to measure the electron at time $t>0$, with negative energy and spin component along $\mathbf{e}_z$ in: iii) spin up state, iv) spin down state. *(2 Points)*

b) Show that the helicity operator, defined as the projection of $\hat{\Sigma}_j$ along the direction of $\mathbf{p}$, commutes with the Hamiltonian operator of the free electron $\hat{H} = c\hat{\boldsymbol{\alpha}}\cdot\hat{\mathbf{p}} + \alpha_t mc^2$:

$$\left[\hat{\boldsymbol{\Sigma}}\cdot\mathbf{p}/|\mathbf{p}|, \hat{H}\right] = 0 \tag{4}$$

What follows out of this? Does this also hold for the projections $\hat{\boldsymbol{\Sigma}}\cdot\mathbf{e}_n$ along arbitrary directions $\mathbf{e}_n$? *(2 Points)*

c) Determine the common eigenstates of $\hat{H}$ and $\hat{\Sigma}_x$ in the case of $\mathbf{p} = p\,\mathbf{e}_x$. Determine the corresponding eigenvalues.
Remark: the $\psi^{(j)}(\mathbf{x}, 0)$ *of Eq.* (2) *are common eigenstates for* $\hat{\Sigma}_z$ *and* $\hat{H}$.

(1 Point)

Exercise 15 *Transformation of the spinor*

Transform the solutions for the free electron problem given in Eq. (2), in the case of $\mathbf{p} = p\,\mathbf{e}_x$, to a moving reference frame with $v = \beta c$, where $\beta = cp/E$. The Lorentz transformation for a boost in the x-direction is given by

$$S_{Lor} = \cosh\frac{\chi}{2} - \hat{\alpha}_x \sinh\frac{\chi}{2}, \tag{5}$$

for $\tanh\chi = \beta$. Compare these results with the solutions of the electron at rest. Check whether the states are normalized. *(2 Points)*