TPV

SoSe 2018 **Blatt 8** **14-15/06/2018**

Exercise 16 *Interpretation of the Dirac matrix*

Consider the operator $\hat{\mathbf{x}}$, position operator of an electron with mass m and charge e, such that $\left[\hat{x}_j, \hat{p}_k\right] = i\hbar\delta_{jk}$ with $j,\, k = 1, 2, 3$.

a) Given $\hat{H} = c\hat{\boldsymbol{\alpha}} \cdot \hat{\mathbf{p}} + \hat{\alpha}_t mc^2$, with $\left\{\hat{\alpha}_\mu, \hat{\alpha}_\nu\right\} = 2\delta_{\mu\nu}$, $\mu, \nu = 1, 2, 3, 4$ and $\hat{\alpha}_\mu = \hat{\alpha}_\mu^\dagger$, show that (in the Heisenberg picture)

$$\frac{\mathrm{d}\hat{x}_k}{\mathrm{d}t} = c\hat{\alpha}_k, \tag{1}$$

therefore the Dirac matrices $\hat{\alpha}_k$ are velocity components. *(0.5 Points)*

b) The magnetic moment of the electrons is

$$\hat{\boldsymbol{\mu}} = \frac{e}{2}\hat{\mathbf{r}} \times \hat{\boldsymbol{\alpha}}. \tag{2}$$

Show that for an eigenstate of the energy at eigenvalue E, the expectation value of $\hat{\boldsymbol{\mu}}$ is

$$\langle\hat{\boldsymbol{\mu}}\rangle = \frac{ec}{2E}\big(\mathbf{L} + 2\mathbf{S}\big) \tag{3}$$

where $\mathbf{L} = \langle\hat{\mathbf{r}} \times \hat{\mathbf{p}}\rangle$ is the orbital momentum and $\mathbf{S} = \hbar\frac{\langle\hat{\boldsymbol{\sigma}}\rangle}{2}$ is the spin operator.

In order to do this, consider an operator $\langle\mathcal{O}\rangle$ and calculate $\langle\hat{\mathcal{O}}\rangle = \langle E|\hat{\mathcal{O}}|E\rangle$, with $|E\rangle$ a normalized eigenstate of the energy such that $\langle E|E\rangle = 1$, $\hat{H}|E\rangle = E|E\rangle$. Show that it can be written as:

$$\langle\hat{\mathcal{O}}\rangle = \frac{1}{2E}\langle\{\hat{H}, \hat{\mathcal{O}}\}\rangle, \tag{4}$$

with $E \neq 0$ and $\{,\}$ being the anticommutator. Use Eq. (4) to derive Eq. (3). *(2 Points)*

c) Show that

$$\begin{aligned} \frac{\mathrm{d}\hat{\boldsymbol{\alpha}}}{\mathrm{d}t} &= \frac{2}{i\hbar}\Big(\hat{\boldsymbol{\alpha}}\hat{H} - c\mathbf{p}\Big), \\ \frac{\mathrm{d}\hat{\mathbf{p}}}{\mathrm{d}t} &= 0, \\ \frac{\mathrm{d}\hat{H}}{\mathrm{d}t} &= 0. \end{aligned} \tag{5}$$

(0.5 Points)

Exercise 17 *Zitterbewegung*

We will calculate $\langle\hat{\mathbf{x}}(t)\rangle$ of a free electron and compare this dynamics with the one of the non-relativistic motion.

In order to do this we want to determine

$$\langle \hat{\mathbf{x}}(t) \rangle = \int_V \mathrm{d}\mathbf{x}'\, \mathbf{x}' |\psi\big(\mathbf{x}', t\big)|^2 \tag{6}$$

with $|\psi\big(\mathbf{x}', t\big)|^2 = \psi^\dagger\big(\mathbf{x}', t\big)\psi\big(\mathbf{x}', t\big)$. Here

$$\psi(\mathbf{x}, t) = \sum_{j,\mathbf{p}} c_{j,p}\psi^{(j)}(\mathbf{x}, t) \tag{7}$$

$\psi^{(j)}(\mathbf{x}, t) = \dfrac{1}{\sqrt{V}} u^{(j)}(\mathbf{p}) e^{i(\mathbf{p}\cdot\mathbf{x} - E^{(j)}t)/\hbar}$, $j = 1, 2, 3, 4$. The $u^{(j)}$ are given by:

$$u^{(1)}(\mathbf{p}) = \mathcal{N} \begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_z}{E + mc^2} \\ \dfrac{c(p_x + ip_y)}{E + mc^2} \end{pmatrix}, \qquad u^{(2)}(\mathbf{p}) = \mathcal{N} \begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p_x - ip_y)}{E + mc^2} \\ -\dfrac{cp_z}{E + mc^2} \end{pmatrix},$$

$$u^{(3)}(\mathbf{p}) = \mathcal{N} \begin{pmatrix} -\dfrac{cp_z}{|E| + mc^2} \\ -\dfrac{c(p_x + ip_y)}{|E| + mc^2} \\ 1 \\ 0 \end{pmatrix}, \qquad u^{(4)}(\mathbf{p}) = \mathcal{N} \begin{pmatrix} -\dfrac{c(p_x - ip_y)}{|E| + mc^2} \\ \dfrac{cp_z}{|E| + mc^2} \\ 0 \\ 1 \end{pmatrix},$$

with normalization $\mathcal{N} = \sqrt{\big(|E| + mc^2\big)/(2|E|)}$.
Here p_x, p_y, p_z are the components of the momentum $\mathbf{p}$, m is the mass, V is the volume and $E^{(j)} = \pm\sqrt{p_x^2c^2 + p_y^2c^2 + p_z^2c^2 + m^2c^4}$ is the energy (such that $E^{(j)} > 0$ for $j = 1, 2$ and $E^{(j)} < 0$ for $j = 3, 4$).

a) Use Heisenberg picture and Eqs. (5) to show that

$$\ddot{\hat{\alpha}}_k(t) = \frac{2}{i\hbar}\Big(\dot{\hat{\alpha}}_k \hat{H}\Big). \tag{8}$$

Demonstrate that the solution of this equation of motion reads

$$\hat{\alpha}_k(t) = \hat{B}_0 + \dot{\hat{\alpha}}_k(0) e^{-2i\hat{H}t/\hbar}\left(\frac{i\hbar}{2}\right)\hat{H}^{-1}, \qquad \hat{B}_0 = c\hat{p}_k\hat{H}^{-1}. \tag{9}$$

(2 Points)

b) Determine $\hat{x}_k(t)$ using Eqs. (1) and (5) and show that

$$\hat{x}_k(t) = \hat{x}_k(0) + (c^2\hat{p}_k\hat{H}^{-1})t + \frac{i\hbar c}{2}\Big(\hat{\alpha}_k(0) - c\hat{p}_k\hat{H}^{-1}\Big)\hat{H}^{-1}\Big(e^{-2i\hat{H}t/\hbar} - 1\Big). \tag{10}$$

(1 Point)

c) Using Eq. (7) calculate explicitly

$$
\begin{aligned}
\langle c^2 \hat{p}_k \hat{H}^{-1} \rangle &= \int_V \mathrm{d}\mathbf{x}'\, \psi^\dagger(\mathbf{x}') c^2 \hat{p}_k \hat{H}^{-1} \psi(\mathbf{x}') \\
\left\langle \frac{i\hbar c}{2} \left(\hat{\alpha}_k(0) - c\hat{p}_k \hat{H}^{-1} \right) \hat{H}^{-1} \left(e^{-2i\hat{H}t/\hbar} - 1 \right) \right\rangle &= \\
\int_V \mathrm{d}\mathbf{x}'\, \psi^\dagger(\mathbf{x}') \frac{i\hbar c}{2} \left(\hat{\alpha}_k(0) - c\hat{p}_k \hat{H}^{-1} \right) \hat{H}^{-1} \left(e^{-2i\hat{H}t/\hbar} - 1 \right) \psi(\mathbf{x}'). &
\end{aligned} \tag{11}
$$

What happens if $c_{j,\mathbf{p}} = 0 \; \forall \, \mathbf{p}$ and $j = 3, 4$? *(2 Point)*

Hint: use the fact that

$$
u^{(j)}(\mathbf{p})^\dagger \hat{\alpha}_k(0) u^{(j')}(\mathbf{p}) = \begin{cases} \frac{p_k c}{E} \delta_{j\,j'}, & \textit{for } j, j' = 1, 2 \\ \frac{-p_k c}{|E|} \delta_{j\,j'}, & \textit{for } j, j' = 3, 4 \\ a^{(k)}_{j\,j'} \neq 0, & \textit{for } j = 1, 2 \textit{ and } j' = 3, 4 \textit{ or } j' = 1, 2 \textit{ and } j = 3, 4. \end{cases}
$$

Calculate $a^{(k)}_{j\,j'}$ *for* $k = 1$ *and verify it is different from* 0*.*