TPV

SoSe 2018 **Blatt 9** **21-22/06/2018**

Exercise 18 *Lorentz force*

The relativistic Hamiltonian of a charged particle with charge e and rest mass m_0 in a four-potential $A_\mu = (\mathbf{A}(\mathbf{q},t), i\phi(\mathbf{q},t))$ is given by

$$H(\mathbf{q},\mathbf{p}) = e\phi(\mathbf{q},t) + c\sqrt{m_0^2c^2 + \left(\mathbf{p} - \frac{e}{c}\mathbf{A}(\mathbf{q},t)\right)^2}\,. \tag{1}$$

Show that the Hamilton equations lead to the correct Lorentz force.
Hint: the relativistic Hamiltonian of Eq. (1) *can be rewritten as* $\hat{H} = e\phi(\mathbf{q},t) + m_0c^2\gamma$ *(3 Points)*

Exercise 19 *Particle in a central potential*

The Hamiltonian operator of one electron with mass m and charge e, in a central potential $\hat{V}(\mathbf{r}) = \hat{V}(r)$ with $r = \sqrt{x^2+y^2+z^2}$, is given by $\hat{H} = c\hat{\boldsymbol{\alpha}}\cdot\hat{\mathbf{p}} + \hat{\alpha}_t mc^2 + \hat{V}(r)$, where $\{\hat{\alpha}_\mu, \hat{\alpha}_\nu\} = 2\delta_{\mu\nu}$, $\mu,\nu = 1,2,3,4$ and $\hat{\alpha}_\mu = \hat{\alpha}_\mu^\dagger$.

a) Show that the total angular momentum $\hat{\mathbf{J}} = \hat{\mathbf{L}} + \frac{\hbar}{2}\hat{\boldsymbol{\Sigma}}$, with

$$\hat{\mathbf{L}} = \hat{\mathbf{r}} \times \hat{\mathbf{p}},$$
$$\hat{\Sigma}_n = \sum_{l,m=1}^{3} \frac{1}{2i}\epsilon_{lmn}\hat{\gamma}_l\hat{\gamma}_m, \quad \hat{\gamma}_l = -i\hat{\alpha}_t\hat{\alpha}_l \tag{2}$$

is a conserved quantity. *(1 Point)*

b) In non-relativistic quantum mechanics the eigenvalues of the operator $\hat{\boldsymbol{\sigma}}\cdot\hat{\mathbf{J}}$ determine whether the spin of the electron is aligned parallel or antiparallel to the total angular momentum $\hat{\mathbf{J}}$. In order to extend this treatment to relativistic quantum mechanics. Show that the operator $\hat{K} = \hat{\alpha}_t(\hat{\boldsymbol{\Sigma}}\cdot\hat{\mathbf{J}} - \hbar/2)$ is a conserved quantity. Show that it commutes with the total angular momentum.

Hint: Show the identity

$$\left(\hat{\boldsymbol{\alpha}}\cdot\hat{\mathbf{A}}\right)\left(\hat{\boldsymbol{\Sigma}}\cdot\hat{\mathbf{B}}\right) = -\hat{\gamma}_5\hat{\mathbf{A}}\cdot\hat{\mathbf{B}} + i\hat{\boldsymbol{\alpha}}\cdot\left(\hat{\mathbf{A}}\times\hat{\mathbf{B}}\right), \tag{3}$$

assuming $\left[\hat{A}_i, \hat{\Sigma}_j\right] = 0\ \forall i,j = 1,2,3$ *and where* $\hat{\gamma}_5$ *is defined as*

$$\hat{\gamma}_5 = \hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3\hat{\gamma}_4 = \begin{pmatrix} 0_2 & -\mathbf{1}_2 \\ -\mathbf{1}_2 & 0_2 \end{pmatrix}. \tag{4}$$

The identity $\hat{\alpha}_j = -\hat{\Sigma}_j\hat{\gamma}_5 = -\hat{\gamma}_5\hat{\Sigma}_j$ *could be useful.* *(2 Points)*

c) It follows that one can find simultaneous eigenfunctions of $\hat{H}, \hat{K}$ and $\hat{\mathbf{J}}^2$, with eigenvalues $E, -\hbar\kappa$ and $\hbar^2 j(j+1)$, respectively. Determine the relation between the eigenvalues κ and j, by first comparing the eigenvalues of $\hat{K}^2$ and $\hat{\mathbf{J}}^2$. *(1 Point)*

d) Determine the eigenvalue equations for $\hat{\psi} = \begin{pmatrix} \hat{\psi}_A \\ \hat{\psi}_B \end{pmatrix}$, where $\hat{\psi}_A, \hat{\psi}_B$ are the positive- and negative-energy components of the four-component spinor for the operators $\hat{K}$, and $\hat{\mathbf{J}}^2$. Show that the two component spinors $\hat{\psi}_A, \hat{\psi}_B$ of the eigenfunctions are also eigenfunctions to L^2 with the corresponding eigenvalues $\hbar^2 l_A(l_A+1)$ and $\hbar^2 l_B(l_B+1)$ although $\hat{\psi}$ itself is not an eigenfunction of $\hat{\mathbf{L}}^2$. How are l_A, l_B, j, κ related? *(1 Point)*