Exercises for
Theoretical physics V

SoSe 2023 **Sheet 3** **25.04.2023**

Exercise 6 *Properties of the gamma matrices*

The Dirac-Hamiltonian operator is given by

$$\hat{H} = c\,\boldsymbol{\alpha}\cdot\boldsymbol{p} + \alpha_t mc^2, \tag{1}$$

with $\alpha_\mu = (\alpha_x, \alpha_y, \alpha_z, \alpha_t)$. It holds that $\alpha_\mu^\dagger = \alpha_\mu$, $\mathrm{Tr}[\alpha_\mu] = 0$ as well as $\{\alpha_\mu, \alpha_\nu\} = 2\delta_{\mu\nu}$, with the anticommutator defined as $\{A, B\} = AB + BA$. Beside, the Dirac Equation in the van-der-Waerden-Form is given by

$$\left(\gamma_\mu \partial^\mu + \frac{mc}{\hbar}\right)\psi(\boldsymbol{r}, t) = 0, \tag{2}$$

where the γ-matrices are defined as $\gamma_j = -i\alpha_t\alpha_j$ for $j = 1, 2, 3$ and $\gamma_4 = \alpha_t$. Show the following properties, without explicitly writing the gamma matrices:

a) the anticommutators $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$ hold,

(1 point)

b) the γ_μ are hermitian,

(1 point)

c) the trace of the γ_μ vanishes.

(1 point)

Exercise 7 *Non-relativistic limit of the Dirac equation*

Let us consider a fermion described by the spinor $\psi(\boldsymbol{r}, t)$. The behaviour of the spinor is governed by the Dirac equation

$$\left[i\hbar\gamma^\mu(\partial_\mu + i\frac{e}{\hbar}A_\mu) - mc\right]\psi(\boldsymbol{r}, t) = 0, \tag{3}$$

where the matrices γ_μ take the form

$$\gamma_0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}\ ,\ \gamma_k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}.$$

a) Assuming that A_μ is time-independent, the spinor has the form $\psi = e^{-iEt/\hbar}\psi_0(\boldsymbol{x})$. Show that the behaviour of the spinor is described by two coupled equations

$$[\boldsymbol{\sigma}\cdot(\boldsymbol{p} - e\boldsymbol{A})]\ v = \frac{1}{c}(E - e\phi - mc^2)u, \tag{4a}$$

$$-[\boldsymbol{\sigma}\cdot(\boldsymbol{p} - e\boldsymbol{A})]\ u = -\frac{1}{c}(E - e\phi + mc^2)v, \tag{4b}$$

where $\psi = \begin{pmatrix} u \\ v \end{pmatrix}$. *(2 points)*

b) Show that in the non-relativistic case, namely in the limit where $E = E_{NR} + mc^2$ with mc^2 being the largest energy scale ($mc^2 \gg E_{NR}$ and $mc^2 \gg e\phi$), the Dirac equation is approximated by the Schrödinger-Pauli equation

$$E_{NR} u \simeq \left[\frac{1}{2m} (\boldsymbol{\sigma} \cdot (\boldsymbol{p} - e\boldsymbol{A}))^2 + e\phi \right] u. \tag{5}$$

(2 points)

c) Using the relation $(\boldsymbol{\sigma} \cdot \boldsymbol{a})(\boldsymbol{\sigma} \cdot \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b} + i\boldsymbol{\sigma} \cdot (\boldsymbol{a} \times \boldsymbol{b})$, show that the Schrödinger-Pauli equation takes the form

$$E_{NR} u \simeq \left[\frac{1}{2m} (\boldsymbol{p} - e\boldsymbol{A})^2 - \frac{e\hbar}{2m} \boldsymbol{\sigma} \cdot \boldsymbol{B} + e\phi \right] u. \tag{6}$$

Hint: You may use that $\nabla \times (u\boldsymbol{A}) = u(\nabla \times \boldsymbol{A}) + (\nabla u) \times \boldsymbol{A}$. *(1 point)*