Exercises for
Theoretical physics V

SoSe 2023 **Sheet 4** **02.05.2023**

Exercise 8 *Transformation of the spinor*

The solutions for the Dirac equation for a free particle moving with momentum $\boldsymbol{p}$ are given by:

$$\psi(j)(\boldsymbol{x},t) = \mathcal{N} u^{(j)}(\boldsymbol{p}) e^{i(\boldsymbol{p}\cdot\boldsymbol{x}\ -Et)/\hbar} \text{ with } \mathcal{N} = \sqrt{(|E|+mc^2)/(2|E|V)}. \tag{1}$$

$$u^{(1)}(\boldsymbol{p}) = \begin{pmatrix} 1 \\ 0 \\ \frac{cp_z}{E+mc^2} \\ \frac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}, \quad u^{(2)}(\boldsymbol{p}) = \begin{pmatrix} 0 \\ 1 \\ \frac{c(p_x-ip_y)}{E+mc^2} \\ -\frac{cp_z}{E+mc^2} \end{pmatrix}$$

$$u^{(3)}(\boldsymbol{p}) = \begin{pmatrix} -\frac{cp_z}{|E|+mc^2} \\ -\frac{c(p_x+ip_y)}{|E|+mc^2} \\ 1 \\ 0 \end{pmatrix}, \quad u^{(4)}(\boldsymbol{p}) = \begin{pmatrix} -\frac{c(p_x-ip_y)}{|E|+mc^2} \\ \frac{cp_z}{|E|+mc^2} \\ 0 \\ 1 \end{pmatrix},$$

where p_x, p_y, p_z are the components of the momentum $\boldsymbol{p}$, m is the mass, E is the energy, V is the volume.
Transform the solutions for the free electron Dirac equation in the case of $\boldsymbol{p} = p\boldsymbol{e}_x$ to a moving reference frame with $v = \beta c$, where $\beta = cp/E$. The Lorentz transformation for a boost in the x-direction is given by

$$S_L = \cosh\frac{\chi}{2} - \alpha_x \sinh\frac{\chi}{2}, \tag{2}$$

for $\tanh\chi = \beta$. Compare these results with the solutions of the electron at rest. Check whether the states are normalized.

(2 points)

Exercise 9 *Weyl fermions and helicity*

Let us consider the case of massless fermions described by the spinor $\psi(\boldsymbol{r},t)$. Since the mass vanishes $m \to 0$, then the Dirac equation reads

$$i\hbar\gamma^\mu\partial_\mu\psi(\boldsymbol{r},t) = 0, \tag{3}$$

where the matrices γ^μ take the form

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}.$$

a) Show that the elements of the 4-component spinor $\psi = \begin{pmatrix} u \\ v \end{pmatrix}$ as described by the set of differential equations

$$\left(\frac{i\hbar}{c}\frac{\partial}{\partial t} + i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right)\chi_1 = 0, \tag{4a}$$

$$\left(\frac{i\hbar}{c}\frac{\partial}{\partial t} - i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right)\chi_2 = 0. \tag{4b}$$

Provide the expression of $\chi_{1,2}$ as a function of u and v. The fermionic fields χ describe so-called Weyl fermions.

(2 points)

b) We define the helicity operator as $(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})/|\boldsymbol{p}|$. Assuming that $\psi(\boldsymbol{r}, t) = \psi_0(\boldsymbol{r})e^{-iEt/\hbar}$, show that $\chi_{1,2}$ are eigenfunctions of the helicity operator. Determine the corresponding eigenvalues.

(2 points)