Exercises for
Theoretical physics V

SoSe 2023 **Sheet 6** **09.05.2023**

Exercise 13 *Goeppert-Mayer transformation*

Let us consider an electron of mass m and charge $q = -e$ trapped in a position-dependent potential V_0. The electron is put in interaction with an electric field $\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$, such that it is described by the Hamiltonian

$$\hat{H} = \frac{1}{2m}\left(\hat{\boldsymbol{p}} + \frac{e}{c}\boldsymbol{A}\right)^2 + V_0(\hat{\boldsymbol{r}}). \tag{1}$$

a) Considering a time-dependent unitary transformation $\hat{U}(t)$, such that a state $|\tilde{\psi}\rangle$ transforms into $|\psi\rangle = \hat{U}|\tilde{\psi}\rangle$. If the evolution of $|\psi\rangle$ is described by the Schrödinger equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = \hat{H}|\psi\rangle, \tag{2}$$

where $\hat{H}$ is the Hamiltonian operator, determine the effective Hamiltonian $\hat{H}'$ describing the evolution of $|\tilde{\psi}\rangle$.

(1 point)

b) We introduce the unitary operator $\hat{U}(t) = \exp\left(\frac{i}{\hbar c}\hat{\boldsymbol{d}}\cdot\boldsymbol{A}\right)$, where $\hat{\boldsymbol{d}} = -e\hat{\boldsymbol{r}}$ is the dipolar moment of the electron. Determine the expression of the commutator $\left[\hat{\boldsymbol{p}}, \hat{U}\right]$.

Hint: Use that $\left[\hat{p}_i, \hat{r}_j^k\right] = -i\hbar k \hat{r}_j^{k-1}\delta_{ij}$.

(1 point)

c) Show then that the effective Hamiltonian associated to the Goeppert-Meyer transformation $\hat{U}$ introduced in the previous question reads

$$\hat{H}' = \frac{1}{2m}\hat{\boldsymbol{p}}^2 + V_0(\hat{\boldsymbol{r}}) - \hat{\boldsymbol{d}}\cdot\boldsymbol{E}. \tag{3}$$

(2 points)

Exercise 14 *Fine structure of the hydrogene atom*

Let us consider an hydrogen atom consisting in a single electron of mass m_e and electric charge $q = -e$ interacting with a proton. In its non-relativistic description, electron is described by the Hamiltonian

$$\hat{H}_0 = \frac{\hat{\boldsymbol{p}}^2}{2m_e} - \frac{e^2}{4\pi\epsilon_0|\hat{r}|}, \tag{4}$$

which has the eigenvalues $E_n = -\frac{1}{2}mc^2\frac{\alpha^2}{n^2}$, where α is the fine structure constant $\alpha = \frac{e^2}{4\pi\epsilon_0}\frac{1}{\hbar c}$. These energy levels are degenerate and correspond to eigenstates $|n, l, m_l\rangle$.

a) Show that at leading order, the relativistic correction to the kinetic energy takes the form

$$\hat{H}_{1,r} = -\frac{1}{8}\frac{\hat{\boldsymbol{p}}^4}{m_e^3 c^2}. \tag{5}$$

Hint: $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \mathcal{O}(x^3)$. *(1 point)*

b) Deduce then the expression of the relativistic energy correction ΔE_r at first order in perturbation theory.

Hint: You shall use that $\left\langle \frac{1}{r} \right\rangle_{n,l,m} = \frac{m_e c\alpha}{\hbar n^2}$ and $\left\langle \frac{1}{r^2} \right\rangle_{n,l,m} = \left(\frac{m_e c\alpha}{\hbar}\right)^2 \frac{1}{n^3(l+1/2)}$.

(2 points)

c) Another correction arising from relativistic effects is the spin-orbit coupling

$$\hat{H}_{1,SO} = \frac{1}{2m_e^2 c^2 r}\left(\partial_r V(r)\right)\hat{\boldsymbol{L}}\cdot\hat{\boldsymbol{S}}, \tag{6}$$

which includes the spin degrees of freedom into the problem, therefore the state of the electron is described by $|n, l, m\rangle \otimes |s, m_s\rangle$. Determine the form of the spin-orbit correction ΔE_{SO} in terms of l and j, where j are the eigenvalues of the total angular momentum operator $\hat{\boldsymbol{J}}^2 = (\hat{\boldsymbol{L}} + \hat{\boldsymbol{S}})^2$.

Hint: Use that $\left\langle \frac{1}{r^3} \right\rangle_{n,l,m} = \left(\frac{m_e c\alpha}{\hbar n}\right)^3 \frac{1}{l(l+1)(l+1/2)}$, for $l > 0$.

(2 points)

d) the final correction is the so-called Darwin term

$$\hat{H}_D = \frac{\hbar^2}{8m^2c^2}\partial_r^2 V(r) = \frac{\pi\hbar^3\alpha}{2m_e^2 c}\delta(r). \tag{7}$$

Determine the form of the corresponding correction ΔE_D.

Hint: Use that $|\langle \boldsymbol{r}|n, l, m\rangle|^2_{\boldsymbol{r}=0} = \frac{1}{\pi}\left(\frac{m_e c\alpha}{\hbar n}\right)^3$, for $l = 0$. $|\langle \boldsymbol{r}|n, l, m\rangle|^2_{\boldsymbol{r}=0} = 0$ otherwise.

(1 point)

e) Combining these contributions, show that at first order in perturbation theory the energy levels E_n are shifted by a factor

$$\Delta E_{n,j} = \frac{1}{2}mc^2\left(\frac{\alpha}{n}\right)^4\left(\frac{3}{4} - \frac{n}{j+1/2}\right), \tag{8}$$

for $l > 0$. One shall treat the cases $j = l + 1/2$ and $j = l - 1/2$ seperately. *(2 points)*