Exercises for
Theoretical physics V

SoSe 2023 **Sheet 7** **30.05.2023**

Exercise 15 *The Yukawa potential*

Pions are scalar bosons that are involved in the binding of protons and neutrons in the nucleus. They are described by the real scalar field ϕ whose Lagrangian density reads

$$\mathcal{L} = \frac{\hbar^2}{m}\left[\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}\mu^2\phi^2 - \rho\phi\right], \tag{1}$$

where $\mu = \dfrac{mc}{\hbar}$, with m being the mass of a pion and ρ corresponding to a source term.

a) Determine the Hamiltonian density corresponding to the field ϕ.

b) Show that the equation of motion of the field ϕ takes the form

$$\partial_\mu\partial^\mu\phi + \mu^2\phi = -\rho. \tag{2}$$

(1 point)

c) In the abscence of the source term, solve the Euler-Lagrange equation of the field ϕ. We shall use the Fourier transform in a d-dimensional space defined as

$$\phi(\boldsymbol{x}) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} \mathrm{d}^d\boldsymbol{x}\, e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\hat{\phi}(\boldsymbol{k}), \tag{3}$$

where $\hat{\phi}(\boldsymbol{k})$ are the Fourier component of ϕ at vector $\boldsymbol{k}$.

Hint: We give the useful identity

$$\delta(f(x)) = \sum_i \frac{1}{|f'(x_i)|}\delta(x - x_i), \tag{4}$$

where x_i are the roots of the equation $f(x) = 0$.

(2 points)

d) In the case of a static field, determine the form of the Green function associated to the Euler-Lagrange equation Eq. (3). Then, one may see that the pion field ϕ mediates an interaction expressed in terms of the so-called Yukawa potential. Comment on its range.

Hint: You shall use that

$$\int_{-\infty}^{+\infty} \frac{k}{k^2+\mu^2}\sin(kr)\mathrm{d}k = \pi e^{-\mu r}.$$

(3 points)

Exercise 16 *Photons with circular polarization*

Let us consider a quantized electromagnetic field with a wavevector $\boldsymbol{k}$ and two possible linear polarisations $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, such that it respect the conditions $\boldsymbol{k}\cdot\boldsymbol{e}_1 = \boldsymbol{k}\cdot\boldsymbol{e}_2 = \boldsymbol{e}_1\cdot\boldsymbol{e}_2 = 0$. The vectors $\boldsymbol{e}_{1,2}$ are unit vectors.
The energy of the electromagnetic field reads

$$\hat{H} = \hbar c|\boldsymbol{k}|(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + 1), \tag{5}$$

where the operators $\hat{a}_j^\dagger$ act on the vacuum to create a photon at wavelength $\boldsymbol{k}$ and polarization $\boldsymbol{e}_j$. The annhihilation and creation operators obey bosonic commutation relations

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \tag{6}$$

$$[\hat{a}_i^{(\dagger)}, \hat{a}_j^{(\dagger)}] = 0. \tag{7}$$

a) We shall consider the circular polarization vectors

$$\boldsymbol{e}_\pm = \mp\frac{1}{\sqrt{2}}(\boldsymbol{e}_1 \pm i\boldsymbol{e}_2). \tag{8}$$

Using the definition of the quantized photon field

$$\boldsymbol{A}(\boldsymbol{x},t) = \frac{c}{\sqrt{V}}\sqrt{\frac{\hbar}{2\omega}}\sum_j\left[\hat{a}_j(t)e^{i\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{e}_j + \hat{a}_j^\dagger(t)e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\boldsymbol{e}_j^*\right], \tag{9}$$

show that it can written in terms of operators $\hat{a}_\pm$ and $\hat{a}_\pm^\dagger$ annihilating and creating photons with polarization $\boldsymbol{e}_\pm$. Give their expression and show that they commute according to bosonic commutation relations.

(2 points)

b) Let us apply to $\boldsymbol{e}_+$ and $\boldsymbol{e}_-$ a rotation around the wavevector $\boldsymbol{k}$ by an infinitesimal angle $\delta\theta$. Show that the polarization vectors are changed by a factor $\delta\boldsymbol{e}_\pm = \mp i\delta\theta\boldsymbol{e}_\pm$. What can you deduce about the spin of the photon ?

Hint: We remind that for a two-level system, the rotation operator $\hat{R}(\theta)$ reads

$$\hat{R}(\theta) = \exp(i\theta(\boldsymbol{n}\cdot\boldsymbol{\sigma})) = I_2\cos\theta + i(\boldsymbol{n}\cdot\boldsymbol{\sigma})\sin\theta,$$

where the elements of the vector $\boldsymbol{\sigma}$ are Pauli matrices.

(2 points)

Exercise 17 *Diffraction function and the Fermi golden rule*

Let us define the diffraction function as

$$\delta^{(T)}(E_f - E_i) = \frac{1}{2\pi\hbar}\int_{-T/2}^{T/2}\mathrm{d}\tau e^{i(E_f-E_i)\tau/\hbar}, \tag{10}$$

which goes toward the Dirac δ function in the limit $T \to +\infty$. In the following, we will demonstrate some properties of the diffraction function.

a) Show the following indentity

$$\int_{-\infty}^{+\infty} \mathrm{d}E \delta^{(T)}(E - E_i)\delta^{(T)}(E - E_f) = \delta^{(T)}(E_i - E_f). \tag{11}$$

(1 point)

b) Show that the diffraction function takes the form

$$\delta^{(T)}(E_f - E_i) = \frac{1}{\pi} \frac{\sin[(E_f - E_i)T/2\hbar]}{E_f - E_i}, \tag{12}$$

deduce then that

$$\int_{-\infty}^{+\infty} \mathrm{d}E_f [\delta^{(T)}(E_f - E_i)]^2 = \frac{T}{2\pi\hbar}. \tag{13}$$

Hint: We give that

$$\int_{-\infty}^{+\infty} \mathrm{d}x \frac{\sin^2 x}{x^2} = \pi$$

(2 points)

c) Let us consider a simplified model of an ionized atom consisting in a ground state $|g\rangle$ at energy $\hbar\omega_0$ and a continuum of ionized states labeled $|I\rangle$ with an energy $\hbar(\omega_1 + \omega_I)$, such that $\omega_1 > \omega_0$. The ground state and the ionized states are coupled by a time-dependent perturbative term such that the total Hamiltonian reads

$$\hat{H} = \hbar\omega_0 |g\rangle\langle g| + \sum_I \hbar(\omega_1 + \omega_I)|I\rangle\langle I| + V_0 \sum_I (|I\rangle\langle g|e^{-i\omega t} + \text{h.c.}). \tag{14}$$

Let the system be prepared in state $|g\rangle$ at time $t = 0$, use the Fermi golden rule to determine the rate at which the atom is ionized toward a state $|I\rangle$.

Hint: You shall go to the continuous limit where $\sum_I \to \int \mathrm{d}E_I \rho(E_I)$.

(2 points)