Exercises for
Theoretical physics V

SoSe 2023 **Sheet 8** **13.06.2023**

Exercise 18 *Jaynes-Cummings model*

Let us approximate an atom by a two-level system consisting of a ground state $|g\rangle$ and an excited state $|e\rangle$. This atom is set in a cavity formed by two mirrors, such that light trapped inside the cavity can only oscillate at a single frequency ω. The Hamiltonian of this set-up reads

$$\hat{H} = \hbar\omega_0 |e\rangle\langle e| + \hbar\omega \hat{a}^\dagger \hat{a}, \tag{1}$$

where $\hat{a}$ is the annihilation operator of a quantum harmonic oscillator. The light-matter interaction in this context is modeled by an interaction Hamiltonian

$$\hat{V} = \hbar g \hat{\sigma}^x \otimes (\hat{a}^\dagger + \hat{a}). \tag{2}$$

This is the celebrated Jaynes-Cummings model. We will assume that $\omega = \omega_0$.

a) Determine the matrix element of the interaction term $\hat{V}$ in the basis $\{|g, n\rangle, |e, n\rangle\}$ combining the two levels of the atom and the ones of the harmonic oscillator. Graphically represent the processes induced by the light-matter interaction and identify which one are resonant.

(2 points)

b) Determine at second order in perturbation theory the energy shift of the ground state $|g, 0\rangle$ in the weak-coupling approximation ($g \ll \omega_0$).

(1 point)

c) Determine the dynamics of the state $|e, 0\rangle$.

Hint: State $|e, 0\rangle$ couples resonantly to $|g, 1\rangle$. Use degenerate perturbation theory and neglect the off-resonant couplings to the other states.

(1 point)

Exercise 19 *Mass renormalization*

In the lecture, you have seen that at second order in perturbation theory, the coupling of a free eectron to an electromagnetic field leads to a renormalization of its mass. In the following, we will evaluate the effect of the second-order term:

$$H^{(2)}_{\text{int}} = \frac{q^2}{2mc^2} \boldsymbol{A}(\boldsymbol{r}) \cdot \boldsymbol{A}(\boldsymbol{r}), \tag{3}$$

where the quantized electromagnetic field reads

$$\boldsymbol{A}(\boldsymbol{r}) = \sum_{\boldsymbol{k}} \sum_{\lambda} \sqrt{\frac{c^2\hbar}{2\omega_{\boldsymbol{k}} V}} \left[\hat{a}_{\boldsymbol{k},\lambda} e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{e}_\lambda + \hat{a}^\dagger_{\boldsymbol{k},\lambda} e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{e}^*_\lambda \right]. \tag{4}$$

a) Show the relation

$$e^{i\boldsymbol{k}\cdot\hat{\boldsymbol{r}}}|\boldsymbol{p}\rangle = |\boldsymbol{p}+\hbar\boldsymbol{k}\rangle, \tag{5}$$

and then determine the matrix elements of $H^{(2)}_{\text{int}}$ in the basis $\{|\boldsymbol{p}, n_{\boldsymbol{k},\lambda}\rangle\}$.

(2 points)

b) Determine the energy shift $\delta E^{(2)}$ of state $|\boldsymbol{p},0\rangle$ due to $H^{(2)}_{\text{int}}$ at first order in perturbation theory. Deduce its contribution to mass renormalization.

(2 points)

Exercise 20 *Classical limit of electrodynamics*

Let us consider a polarized monochromatic electromagnetic wave described by the operator

$$\boldsymbol{A}(\boldsymbol{r},t) = \sqrt{\frac{c^2\hbar}{2\omega V}}\left[\hat{a}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \hat{a}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right]\boldsymbol{e}. \tag{6}$$

a) Determine the form of the corresponding electric field $\boldsymbol{E}(\boldsymbol{r},t)$ and compute the expectation value $\langle n|\boldsymbol{E}(\boldsymbol{r},t)|n\rangle$ for any Fock state $|n\rangle$.

(1 point)

b) Coherent states of an harmonic oscillator are defined as eingenstates of the annihilation operator: $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, where $\alpha \in \mathbb{C}$. Defining $\alpha = \sqrt{n}e^{i\phi}$, determine the expectation value of the electric field $\langle\alpha|\boldsymbol{E}(\boldsymbol{r},t)|\alpha\rangle$. Give an interpretation of the factor n.

(2 points)

c) Evaluate the variance of the amplitude of the electric field $\Delta E = \sqrt{\langle\alpha|\boldsymbol{E}^2|\alpha\rangle - \langle\alpha|\boldsymbol{E}|\alpha\rangle^2}$, then compare it to the amplitude of the electric field. In which limit do we recover the behaviour of a classical system ?

(2 points)