Exercises for
Theoretical physics V

SoSe 2023 **Sheet 9** **27.06.2023**

Exercise 21 *Bosonic and fermionic fields operators*

a) The basic commutation relation for boson annihilation and creation operator is

$$[\hat{a}, \hat{a}^\dagger] = 1 \ , \ [\hat{a}, \hat{a}] = 0, \tag{1}$$

where $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$. From this definition show that eigenstates $|n\rangle$ of $\hat{a}^\dagger\hat{a}$ have the properties

$$\begin{aligned} \hat{a}^\dagger\hat{a}|n\rangle &= n|n\rangle \ , \ (n \in \mathbb{N}) \ , \\ \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle \ , \\ \hat{a}^\dagger|n\rangle &= \sqrt{n+1}|n+1\rangle \ . \end{aligned}$$

(2 points)

b) Fermionic annihilation and creation operators are defined by an anticommutation relation

$$\{\hat{a}, \hat{a}^\dagger\} = 1 \ , \ \{\hat{a}, \hat{a}\} = 0, \tag{2}$$

where $\{\hat{A}, \hat{B}\} = \hat{A}\hat{B} + \hat{B}\hat{A}$. Show that

$$\begin{aligned} \hat{a}^\dagger\hat{a}|n\rangle &= n|n\rangle \ , \ (n = 0, 1) \ , \\ \hat{a}|n\rangle &= \sqrt{n}|n-1\rangle \ , \\ \hat{a}^\dagger|n\rangle &= \sqrt{1-n}|n+1\rangle \ . \end{aligned}$$

(2 points)

Exercise 22 *Second quantization and the Schrödinger equation*

Let us consider the N-body wave function defined as

$$\Psi_E(\boldsymbol{r}_1, \ldots, \boldsymbol{r}_N) = \frac{1}{\sqrt{N!}}\langle 0|\hat{\psi}(\boldsymbol{r}_1)\ldots\hat{\psi}(\boldsymbol{r}_N)|E, N\rangle \ , \tag{3}$$

where $|E, N\rangle$ is an N-particles energy eigenstate with eigenvalue E of the Hamiltonian

$$\hat{H} = -\frac{\hbar^2}{2m}\int \mathrm{d}^3\boldsymbol{r}\hat{\psi}^\dagger(\boldsymbol{r})\nabla^2\hat{\psi}(\boldsymbol{r}) + \frac{1}{2}\int \mathrm{d}^3\boldsymbol{r}_1\mathrm{d}^3\boldsymbol{r}_2\hat{\psi}^\dagger(\boldsymbol{r}_1)\hat{\psi}^\dagger(\boldsymbol{r}_2)v(\boldsymbol{r}_1, \boldsymbol{r}_2)\hat{\psi}(\boldsymbol{r}_2)\hat{\psi}(\boldsymbol{r}_1). \tag{4}$$

a) Show that the wave function Ψ_E is normalized to unity, namely that

$$\int \mathrm{d}^3\boldsymbol{r}_1 \dots \mathrm{d}^3\boldsymbol{r}_N |\Psi_E(\boldsymbol{r}_1, \dots, \boldsymbol{r}_N)|^2 = 1. \tag{5}$$

(1 point)

b) Show also that

$$E\Psi_E(\boldsymbol{r}_1, \dots, \boldsymbol{r}_N) = \frac{1}{\sqrt{N!}} \langle 0|\hat{\psi}(\boldsymbol{r}_1) \dots \hat{\psi}(\boldsymbol{r}_N)\hat{H}|E, N\rangle. \tag{6}$$

(1 point)

c) Show that the wave function satisfies the N-particles Schrödinger equation

$$\left[-\sum_{i=1}^{N} \frac{\hbar^2}{2m}\nabla_i^2 + \sum_{i<j} v(\boldsymbol{r}_1, \boldsymbol{r}_2)\right] \Psi_E(\boldsymbol{r}_1, \dots, \boldsymbol{r}_N) = E\Psi_E(\boldsymbol{r}_1, \dots, \boldsymbol{r}_N). \tag{7}$$

(2 points)