Exercises for
Theoretical physics V

SoSe 2023 **Sheet 10** **04.07.2023**

Exercise 23 *The complex scalar field*

Let us consider the Lagrangian density associated to a complex-valued Klein-Gordon field ψ:

$$\mathcal{L} = (\partial^\nu \psi)(\partial_\nu \psi)^* - \mu^2 |\psi|^2, \tag{1}$$

where $\mu = mc/\hbar$.

a) Show that the Lagrangian density is invariant under the transformation $\psi \to \psi' = e^{i\lambda}\psi$.

Hint: You shall consider an infinitesimal transformation for $\lambda \ll 1$ and determine the variation of $\delta\mathcal{L}$.

(1 point)

b) Using the prescription of minimal coupling, we define the gauge covariant derivative as

$$-i\hbar\partial_\mu \to -i\hbar D_\mu = -i\hbar\partial_\mu - eA_\mu, \tag{2}$$

derive the field equation for ψ interacting with A_μ. Show that if ψ is a solution he equation of motion for $A_\mu = (A_0, 0, 0, 0)$, then ψ^* is a solution when A_0 is replaced by $-A_0$.

(2 points)

Exercise 24 *Microcausality*

In the following, we will see how the concept of causality naturally emerges from the Klein-Gordon equation and can be extracted from the commutation relation of the quantum field.

a) Show that the group velocity $v_g = \frac{\partial \omega_k}{\partial |k|}$ of a wavepacket described by a Klein-Gordon equation cannot exceed the speed of light.

(1 point)

b) Let us consider the quantum field $\phi(x) = \phi(\boldsymbol{r}, t)$, which is a solution of the Klein-Gordon equation. Show that for a fixed point of space-time y the commutator $[\phi(x), \phi(y)]$ is also a solution of the Klein-Gordon equation. Justify then that

$$[\phi(x), \phi(y)] = \langle 0|[\phi(x), \phi(y)]|0\rangle \equiv i\Delta(x-y). \tag{3}$$

Hint: You shall use that $[\phi(\boldsymbol{r}, t), \phi(\boldsymbol{r}', t)] = 0$ is a c-number, namely a multiple of the identity operator.

(2 points)

c) By the means of the Fourier decomposition of the field,

$$\phi(\boldsymbol{r}, t) = \int \frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\boldsymbol{k}}}} \left[\hat{a}_{\boldsymbol{k}} e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)} + \hat{a}^\dagger_{\boldsymbol{k}} e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega_{\boldsymbol{k}}t)}\right], \tag{4}$$

determine the form of $\Delta(x)$ and discuss the case when $x^2 = c^2t^2 - |\boldsymbol{r}|^2 < 0$.

(3 points)