Exercises for
Theoretical physics V

SoSe 2023 **Sheet 11** **11.07.2023**

Exercise 25 *Energy conservation and time translations*

Let us consider the Lagrangian density associated to a real-valued bosonic field

$$\mathcal{L} = \frac{1}{2}(\partial^\nu \psi)(\partial_\nu \psi) - V(\phi^2), \tag{1}$$

where V is an arbitrary continuous function. Show that for such a Lagrangian density, the conservation of the energy implies the time translation symmetry of the system.

(2 points)

Exercise 26 *First-order Lagrangian*

Let us consider a classical system a classical system with two coordinates a and b. Take the Lagrangian to be

$$L(a, b, \dot{b}) = a\dot{b} - V(a, b). \tag{2}$$

The canonical rule says that a has no canonical conjugate. It is the conjugate to b

a) Determine the Euler-Lagrange equations for variables a and b.

(1 point)

b) Find the canonical momenta, as well as the Hamilton equations of motion. Check that they are consistent with the Lagrangian ones.

(2 points)

Exercise 27 *Phase-invariance of the Dirac field*

Considering the Lagrangian density of a Dirac field ψ:

$$\mathcal{L} = \overline{\psi}(i\hbar\gamma^\mu \partial_\mu - mc)\psi, \tag{3}$$

a) Show that the system is invariant under the phase transformation $\psi \to \psi' = e^{i\alpha}\psi$.

(1 point)

b) Determine the Noether current j^μ associated to the phase-invariance of the Lagrangian density.

(1 point)

c) Show that the Noether current verify that $\partial_\mu j^\mu = 0$.

(2 points)

Exercise 28 *Quantized Dirac field*

Let us recall that the quantized Dirac field can be written in terms of fermionic creation and annihilation operators $a_{\boldsymbol{p},s}$ and $b_{\boldsymbol{p},s}$, such that for a periodic box-like space of volume Ω

$$\hat{\psi}(\boldsymbol{r},t) = \sum_{\boldsymbol{p},s} \sqrt{\frac{m}{\Omega E_{\boldsymbol{p}}}} \left[\hat{a}_{\boldsymbol{p},s} u(\boldsymbol{p},s) e^{i(\boldsymbol{p}\cdot\boldsymbol{r}-E_{\boldsymbol{p}}t)/\hbar} + \hat{b}^\dagger_{\boldsymbol{p},s} v(\boldsymbol{p},s) e^{-i(\boldsymbol{p}\cdot\boldsymbol{r}-E_{\boldsymbol{p}}t)/\hbar} \right], \tag{4}$$

with $u^\dagger(\boldsymbol{p},s)u(\boldsymbol{p},s') = v^\dagger(\boldsymbol{p},s)v(\boldsymbol{p},s') = E_{\boldsymbol{p}}\delta_{ss'}/m$. In other words, the operator $\hat{a}_{\boldsymbol{p},s}$ destroys a particle with wavefunction $u(\boldsymbol{p},s)$, while $\hat{b}_{\boldsymbol{p},s}$ annihilates an antiparticle with wavefunction $v^\dagger(\boldsymbol{p},s)$.

a) Using the following definitions,

$$\hat{H} = \int \mathrm{d}^3\boldsymbol{r}\,\hat{\psi}^\dagger(\boldsymbol{r},t)(-i\hbar\boldsymbol{\alpha}\cdot\boldsymbol{\nabla} + \beta m)\hat{\psi}(\boldsymbol{r},t), \tag{5}$$

$$\hat{\boldsymbol{P}} = -i\hbar \int \mathrm{d}^3\boldsymbol{r}\,\hat{\psi}^\dagger(\boldsymbol{r},t)\boldsymbol{\nabla}\hat{\psi}(\boldsymbol{r},t), \tag{6}$$

$$\hat{Q} = -e \int \mathrm{d}^3\boldsymbol{r}\,\hat{\bar{\psi}}(\boldsymbol{r},t)\gamma^0\hat{\psi}(\boldsymbol{r},t), \tag{7}$$

determine the form of the Hamiltonian $\hat{H}$, the momentum operator $\hat{\boldsymbol{P}}$ and the charge $\hat{Q}$ in terms of operator $\hat{a}_{\boldsymbol{p},s}$ and $\hat{b}_{\boldsymbol{p},s}$.

(3 points)

b) Considering the two-particle state $|\Psi\rangle = \hat{a}^\dagger_{\boldsymbol{p},s}\hat{a}^\dagger_{\boldsymbol{p}',s'}|0\rangle$, show that such a state is antisymmetric and determine the expectation values of $\hat{H}$, $\hat{\boldsymbol{P}}$ and $\hat{Q}$.

(1 point)

c) For a two-anti-particle state $|\Psi\rangle = \hat{b}^\dagger_{\boldsymbol{p},s}\hat{b}^\dagger_{\boldsymbol{p}',s'}|0\rangle$, perform the same calculations as in the two-particle state case.

(1 point)