# Exercises for Theoretical physics V

**SoSe 2023** **Sheet 2** **18.04.2023**

## Exercise 4 *Minimal coupling to the electromagnetic field*

Let us consider a particle of mass $m$ and electric charge $q$ coupled to an electromagnetic field $(\boldsymbol{E}, \boldsymbol{B})$ via the Lorentz force

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right), \tag{1}$$

with $\boldsymbol{v} = \dot{\boldsymbol{x}}$ being the velocity of the particle at time $(t)$.

a) With help of the relations $\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$ and $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$, show that the Lorentz force can be written in terms of a potential $U$, such that

$$F_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{x}_i}\right) - \frac{\partial U}{\partial x_i}. \tag{2}$$

Determine $U$.

**Hint:** One shall use that $\boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{v} \times \boldsymbol{A}) - (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{A}$.

*(2 points)*

b) Determine the form of the Lagrangian $L$ of the partical.

*(1 point)*

c) Derive then the form of the Hamiltonian $H$ describing a charged particle coupled to an electromagnetic field. *(2 points)*

## Exercise 5 *Non-relativistic limit of the Klein-Gordon equation*

Let us consider a scalar boson described by the wave function $\phi(\boldsymbol{r}, t)$. The behaviour of the wave function is governed by the Klein-Gordon equation

$$\left(\partial_\mu \partial^\mu + \frac{m^2 c^2}{\hbar^2}\right)\phi(\boldsymbol{r}, t) = 0, \tag{3}$$

where $\partial_\mu \partial^\mu = \eta^{\mu\nu}\partial_\mu \partial_\nu$ stands for the d'Alembert operator.

a) Assuming that the wave function has the form $\phi(\boldsymbol{r}, t) = e^{-iEt/\hbar}\phi_0(\boldsymbol{x})$, show that the behaviour of the spinor is described by the equation

$$E^2\phi(\boldsymbol{r}, t) = \left(-\hbar^2 c^2 \nabla^2 + m^2 c^4\right)\phi(\boldsymbol{r}, t). \tag{4}$$

*(1 point)*

b) Show that in the non-relativistic case, namely in the limit where $E = E_{NR} + mc^2$ with $mc^2 \gg E_{NR}$, the Klein-Gordon equation is approximated by the Schrödinger equation of a free particle

$$E_{NR}\phi(\boldsymbol{r}, t) \simeq \frac{1}{2m}\nabla^2\phi(\boldsymbol{r}, t). \tag{5}$$

*(1 point)*