Exercises for
Theoretical physics V

SoSe 2023 **Sheet 5** **09.05.2023**

Exercise 10 *Interpretation of the Dirac matrices*

Let us consider $\hat{\boldsymbol{x}}$, the position operator of an electron with mass m and charge e, such that $[\hat{x}_j, \hat{p}_k] = i\hbar\delta_{jk}$ with $j, k = 1, 2, 3$.

a) Given the Hamiltonian $\hat{H} = c\hat{\boldsymbol{\alpha}} \cdot \hat{\boldsymbol{p}} + \hat{\alpha}_t mc^2$, with $\{\hat{\alpha}_\mu, \hat{\alpha}_\nu\} = 2\delta_{\mu\nu}$, with $\mu, \nu = 1, 2, 3, 4$, and $\alpha_\mu = \alpha_\mu^\dagger$. Show that in the Heisenberg picture

$$\frac{\mathrm{d}\hat{x}_k}{\mathrm{d}t} = c\hat{\alpha}_k, \tag{1}$$

therefore the Dirac matrices α_μ correspond to velocity components.

(1 point)

b) The magnetic moment of an electron is

$$\hat{\boldsymbol{\mu}} = \frac{e}{2}\hat{\boldsymbol{x}} \times \hat{\boldsymbol{\alpha}}. \tag{2}$$

Show that for an eigenstate of the energy at eigenvalue E, the expectation value of $\hat{\boldsymbol{\mu}}$ is

$$\langle\hat{\boldsymbol{\mu}}\rangle_E = \frac{ec}{2E}(\boldsymbol{L} + 2\boldsymbol{S}) \tag{3}$$

where $\boldsymbol{L} = \langle\hat{\boldsymbol{x}} \times \hat{\boldsymbol{p}}\rangle_E$ is the orbital momentum and $\boldsymbol{S} = \hbar\frac{\langle\boldsymbol{\sigma}\rangle_E}{2}$ is the spin operator.

Hint: Show that for the state $|E\rangle$ such that $\hat{H}|E\rangle = E|E\rangle$ one can write

$$\langle\hat{O}\rangle_E = \frac{1}{2E}\langle\{\hat{H}, \hat{O}\}\rangle_E, \tag{4}$$

where $\hat{O}$ is some observable.

We also remind that $\sigma_i\sigma_j = i\epsilon_{ijk}\sigma_k + \delta_{ij}I_2$, with ϵ_{ijk} being the Levi-Civita symbol such that $\epsilon_{ijk}\epsilon ijn = 2\delta_{kn}$.

(2 points)

c) Show that the Heisenberg equations of motion take the form,

$$\begin{aligned}\frac{\mathrm{d}\hat{\boldsymbol{\alpha}}}{\mathrm{d}t} &= \frac{2}{i\hbar}\left(\hat{\boldsymbol{\alpha}}\hat{H} - c\hat{\boldsymbol{p}}\right), \\ \frac{\mathrm{d}\hat{\boldsymbol{p}}}{\mathrm{d}t} &= 0, \\ \frac{\mathrm{d}\hat{H}}{\mathrm{d}t} &= 0.\end{aligned} \tag{5}$$

(1 point)

Exercise 11 *The Zitterbewegung*

We will calculate the average position $\langle \hat{\boldsymbol{x}}(t) \rangle$ of a free electron and compare this dynamics with the one of the non-relativistic motion. In order to do this we will evaluate

$$\langle \hat{\boldsymbol{x}}(t) \rangle = \int_V \boldsymbol{x}' |\psi(\boldsymbol{x}', t)|^2 \mathrm{d}\boldsymbol{x}', \tag{6}$$

with $|\psi(\boldsymbol{x}, t)|^2 = \psi^\dagger(\boldsymbol{x}, t)\psi(\boldsymbol{x}, t)$. Here

$$\psi(\boldsymbol{x}, t) = \sum_{j,\boldsymbol{p}} c_j(\boldsymbol{p}) \psi^{(j)}(\boldsymbol{x}, t), \tag{7}$$

where $\psi^{(j)}(\boldsymbol{x}, t) = \frac{1}{\sqrt{V}} u^{(j)}(\boldsymbol{p}) e^{i(\boldsymbol{p}\cdot\boldsymbol{x} - E^{(j)}t)/\hbar}$, and $j = 1, 2, 3, 4$. The $u^{(j)}$ are given by

$$u^{(1)}(\boldsymbol{p}) = \mathcal{N} \begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_z}{E + mc^2} \\ \dfrac{c(p_x + ip_y)}{E + mc^2} \end{pmatrix}, \quad u^{(2)}(\boldsymbol{p}) = \mathcal{N} \begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p_x - ip_y)}{E + mc^2} \\ -\dfrac{cp_z}{E + mc^2} \end{pmatrix}$$

$$u^{(3)}(\boldsymbol{p}) = \mathcal{N} \begin{pmatrix} -\dfrac{cp_z}{|E| + mc^2} \\ -\dfrac{c(p_x + ip_y)}{|E| + mc^2} \\ 1 \\ 0 \end{pmatrix}, \quad u^{(4)}(\boldsymbol{p}) = \mathcal{N} \begin{pmatrix} -\dfrac{c(p_x - ip_y)}{|E| + mc^2} \\ \dfrac{cp_z}{|E| + mc^2} \\ 0 \\ 1 \end{pmatrix},$$

with the normalization factor $\mathcal{N} = \sqrt{(|E| + mc^2)/(2|E|)}$. The energy $E^{(j)} = \pm\sqrt{|\boldsymbol{p}|^2c^2 + m^2c^4}$ is taken such that $E^{(j)} > 0$ for $j = 1, 2$, and $E^{(j)} < 0$ for $j = 3, 4$.

a) Using the Heisenberg picture, show that

$$\ddot{\hat{\alpha}}_k(t) = \frac{2}{i\hbar} \dot{\hat{\alpha}}_k(t) \hat{H}. \tag{8}$$

Demonstrate then that the solution of this equation of motion reads

$$\hat{\alpha}_k(t) = \hat{B}_0 + \dot{\hat{\alpha}}_k(0) e^{-2i\hat{H}t/\hbar} \left(\frac{i\hbar}{2} \right) \hat{H}^{-1}, \quad \hat{B}_0 = c\hat{p}_k \hat{H}^{-1}. \tag{9}$$

(2 points)

b) Determine $\hat{x}_k(t)$ and show that

$$\hat{x}_k(t) = \hat{x}_k(0) + (c^2 \hat{p}_k \hat{H}^{-1})t + \frac{i\hbar c}{2} \left(\hat{\alpha}_k(0) - c\hat{p}_k \hat{H}^{-1} \right) \hat{H}^{-1} \left(e^{-i2\hat{H}t/\hbar} - 1 \right). \tag{10}$$

(2 points)

c) Determine explicitly $\langle c^2 \hat{p}_k \hat{H}^{-1} \rangle$ as well as $\left\langle \frac{i\hbar c}{2} \left(\hat{\alpha}_k(0) - c\hat{p}_k \hat{H}^{-1} \right) \hat{H}^{-1} \left(e^{-i2\hat{H}t/\hbar} - 1 \right) \right\rangle$, defined such that

$$\langle \hat{O} \rangle = \int_V \psi^\dagger(\boldsymbol{x}', t) \hat{O} \psi(\boldsymbol{x}', t) \mathrm{d}\boldsymbol{x}'. \tag{11}$$

What happens if $c_j(\boldsymbol{p}) = 0$ for all $\boldsymbol{p}$ and $j = 3, 4$?

Hint: One can use that

$$\mathcal{N}^2 u_j(\boldsymbol{p}) \hat{\alpha}_k(0) u_{j'}(\boldsymbol{p}) = \begin{cases} \frac{p_k c}{E} \delta_{jj'} & \text{for } j, j' = 1, 2 \\ -\frac{p_k c}{|E|} \delta_{jj'} & \text{for } j, j' = 3, 4 \\ a_{jj'}^{(k)} \neq 0 & \text{otherwise.} \end{cases} \tag{12}$$

(2 points)

Exercise 12 *Anharmonicity of the oscillations of a polar molecule*

Let us consider an harmonic oscillator with the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2, \tag{13}$$

which is perturbed by the anharmonic potential

$$V(\hat{x}) = \lambda \hbar \omega \left(\frac{m\omega}{\hbar} \right)^{3/2} \hat{x}^3, \tag{14}$$

where λ is a real dimensionless parameter such that $\lambda \ll 1$.

a) Show that

$$V(\hat{x}) = \frac{\lambda \hbar \omega}{2^{3/2}} \left[(\hat{a}^\dagger)^3 + \hat{a}^3 + 3\hat{N}\hat{a}^\dagger + 3(\hat{N}+1)\hat{a} \right], \tag{15}$$

where $\hat{a}$, $\hat{a}^\dagger$ and $\hat{N}$ are the creation, annihilation and number operators of the harmonic oscillator. *(1 point)*

b) Compute the eigenenergies and eigenstates of the perturbed harmonic oscillator up to second order in λ using perturbation theory.

(2 points)