Exercises for
Theoretical physics V

SoSe 2024 **Sheet 1** **12.04.2023**

Exercise 1 *Quantum harmonic oscillator*

Let us consider a particle of mass m in one dimension. We define the operator position $\hat{x}$ and impulsion $\hat{p}$ such that they obey the commutation relation $[\hat{x}, \hat{p}] = i\hbar\hat{\mathbf{1}}$. The particle is trapped in a harmonic potential such that it is described by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 \tag{1}$$

where $\omega \in \mathbb{R}$.

a) Let us define the annihilation and creation operators $\hat{a} = \frac{1}{\sqrt{2}}(\frac{\hat{x}}{l} + i\frac{l\hat{p}}{\hbar})$ and $\hat{a}^\dagger = \frac{1}{\sqrt{2}}(\frac{\hat{x}}{l} - i\frac{l\hat{p}}{\hbar})$, where $l = \sqrt{\frac{\hbar}{m\omega}}$. Show that $\left[\hat{a}, \hat{a}^\dagger\right] = \mathbf{1}$ and that $\hat{H} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2}\mathbf{1})$

(1 point)

b) For $\hat{N} = \hat{a}^\dagger\hat{a}$, show then that $[\hat{N}, \hat{a}] = -\hat{a}$ and $[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger$.

(1 point)

c) Let us consider the eigenvectors $|n\rangle$ of the operator $\hat{N}$ and $\hat{H}$, such that $\hat{N}|n\rangle = n|n\rangle$, where $n \in \mathbb{N}$. Show that if the ground state $|0\rangle$ is not degenerate, then so are the other states $|n\rangle$.

(1 point)

d) Let us consider the state $|\psi_0\rangle = \sum_n c_n|n\rangle$, where $c_n = e^{-\frac{|\alpha|^2}{2}}\frac{\alpha^n}{\sqrt{n!}}$ and $\alpha \in \mathbb{C}$. Show that $\langle\psi_0|\psi_0\rangle = 1$. Determine the expectation values $\langle\hat{x}\rangle$, $\langle\hat{p}\rangle$, $\langle\hat{x}^2\rangle$ and $\langle\hat{p}^2\rangle$.

(2 points)

e) Determine the expression of the evolved state $|\psi(t)\rangle = \hat{U}(t)|\psi_0\rangle$. What form takes then $\langle\hat{x}(t)\rangle$ and $\langle\hat{p}(t)\rangle$?

(2 points)

Exercise 2 *Conservation equation of quantum probability*

Let us consider a system described by the position- and time-dependent wavefunction $\psi(\boldsymbol{r}, t)$. The behaviour of the wavefunction is governed by the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}\psi(\boldsymbol{r}, t) = -\frac{\hbar^2}{2m}\nabla^2\psi(\boldsymbol{r}, t) + V(\boldsymbol{r})\psi(\boldsymbol{r}, t)\,. \tag{2}$$

a) Defining the quantum probability density $\rho(\boldsymbol{r}, t) = |\psi(\boldsymbol{r}, t)|^2$, determine the equation of motion of ρ.

(2 points)

b) Show that it can be expressed in terms of a conservation equation

$$\frac{\partial}{\partial t}\rho(\boldsymbol{r}, t) + \nabla \cdot \boldsymbol{j}(\boldsymbol{r}, t) = 0. \tag{3}$$

Give the explicit form of the current of probability $\boldsymbol{j}$.

Hint: One shall use that $\nabla \cdot (f\nabla g) = (\nabla f) \cdot (\nabla g) + f\nabla^2 g$.

(2 points)

Exercise 3 *Relativistic notations*

a) Let us define the four-momentum vector $P^\mu = mu^\mu = (E/c,\, p_x,\, p_y,\, p_z)$. Show that its pseudo-norm is constant and deduce the relation

$$E^2 = p^2c^2 + m^2c^4. \tag{4}$$

Assuming that the mass term is dominant ($m^2c^4 \gg p^2c^2$), determine the form of the energy E.

(2 points)

b) The Maxwell tensor $F_{\mu\nu}$ is defined from the four-potential $A^\mu = (\phi,\, A_x,\, A_y,\, A_z)$ (in Gauss units) as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Show that in the Lorenz gauge ($\partial_\mu A^\mu = 0$), the equation of motion for the electric potential ϕ and the vector potential $\boldsymbol{A}$ take the compact form

$$\partial_\mu \partial^\mu A^\nu = \frac{4\pi}{c} j^\nu, \tag{5}$$

where the four-current is defined as $j^\nu = (\rho c,\, j_x,\, j_y,\, j_z)$.

Hint: In Gauss units, the Maxwell equations take the form

$$\nabla \cdot \boldsymbol{E} = 4\pi\rho, \tag{6a}$$

$$\nabla \cdot \boldsymbol{B} = 0, \tag{6b}$$

$$\nabla \times \boldsymbol{E} = -\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{B}, \tag{6c}$$

$$\nabla \times \boldsymbol{B} = \frac{1}{c}\left(4\pi\boldsymbol{j} + \frac{\partial}{\partial t}\boldsymbol{E}\right). \tag{6d}$$

(2 points)