Theoretical physics V
Sheet 2

SoSe 2024 **Due for the 02.05.2024**

Exercise 4 *Non-relativistic limit of the Klein-Gordon equation*

Let us consider a scalar boson described by the wave function $\phi(\boldsymbol{r}, t)$. The behaviour of the wave function is governed by the Klein-Gordon equation

$$\left(\partial_\mu \partial^\mu + \frac{m^2 c^2}{\hbar^2}\right) \phi(\boldsymbol{r}, t) = 0, \tag{1}$$

where $\partial_\mu \partial^\mu = \eta^{\mu\nu} \partial_\mu \partial_\nu$ stands for the d'Alembert operator and $\eta^{\mu\nu}$ is the flat metric tensor of special relativity, a diagonal matrix with coefficients $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

a) Assuming that the wave function has the form $\phi(\boldsymbol{r}, t) = e^{-iEt/\hbar} \phi_0(\boldsymbol{x})$, show that the behaviour of the wave function is described by the equation

$$E^2 \phi(\boldsymbol{r}, t) = \left(-\hbar^2 c^2 \nabla^2 + m^2 c^4\right) \phi(\boldsymbol{r}, t). \tag{2}$$

(1 point)

b) Show that in the non-relativistic case, namely in the limit where $E = E_{NR} + mc^2$ with $mc^2 \gg E_{NR}$, the Klein-Gordon equation is approximated by the Schrödinger equation of a free particle

$$E_{NR} \phi(\boldsymbol{r}, t) \simeq -\frac{\hbar^2}{2m} \nabla^2 \phi(\boldsymbol{r}, t). \tag{3}$$

(1 point)

Exercise 5 *Transformation between inertial reference frames*

As seen in the lectures, a change of reference frame leads to a transformation of the Dirac spinor $\psi' = S\psi$. For the transformation S to leave the Dirac equation invariant, it needs to fulfill the following conditions

$$S^\dagger \alpha^\mu S = \Lambda^\mu_\nu \alpha^\nu \quad \text{and} \quad S^\dagger \beta S = \beta, \tag{4}$$

where Λ is a transformation of the Lorentz group. The α and β matrices are defined as

$$\alpha^0 = \mathbf{1}_4 \quad , \quad \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2. \end{pmatrix} \tag{5}$$

Show that the operators S given in the following fulfill the conditions states in Eq. (4) for the corresponding reference frame transformations.

a) Consider a rotation around the z axis by an angle θ, coordinates transform as:

$$ct' = ct \; ; \; z' = z \; ;$$
$$x' = \cos\theta \, x + \sin\theta \, y \; ; \; y' = \cos\theta \, y - \sin\theta \, x \, .$$

Identify the matrix elements of Λ and show that the transformation

$$S = \cos\frac{\theta}{2}\begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix} + i\sin\frac{\theta}{2}\begin{pmatrix} \sigma^z & 0 \\ 0 & \sigma^z \end{pmatrix}, \tag{6}$$

fulfills the conditions stated in Eq. (4).

(2 points)

b) We consider now a Lorentz boost along the x axis by a rapidity χ that transforms the coordinates as:

$$ct' = \cosh\chi \, ct - \sinh\chi \, x \; ; \; x' = \cosh\chi \, x - \sinh\chi \, ct \; ;$$
$$y' = y \; ; \; z' = z \, .$$

etermine the elements of the matrix Λ and show that the transformation

$$S = \cosh\frac{\chi}{2}\begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix} - \sinh\frac{\chi}{2}\begin{pmatrix} 0 & \sigma^x \\ \sigma^x & 0 \end{pmatrix}, \tag{7}$$

fulfills the conditions stated in Eq. (4).

(2 points)

c) Let us finally consider a spatial reflection given by the transformation

$$ct' = ct \; ; \; x_i' = -x_i .$$

Identiy the form taken by the matrix Λ and show that the transformation

$$S = \beta \tag{8}$$

fulfills the conditions stated in Eq. (4).

(1 point)

Exercise 6 *Reminders on the Pauli matrices (Bonus)*

The Pauli matrices are defined as

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \; \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \; \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9}$$

a) Show that the Pauli matrices fulfill the following relations

$$\sigma_j \sigma_k = \delta_{jk} \mathbf{1}_2 + i\epsilon_{jkl}\sigma_l \quad \text{and} \quad \sigma_x\sigma_y\sigma_z = i\mathbf{1}_2, \tag{10}$$

where $j, k, l \in \{x, y, z\}$, where δ_{jk} stands for the Kronecker delta and ϵ_{jkl} as the Levi-Civita tensor, such that $\epsilon_{xyz} = \epsilon_{yzx} = \epsilon_{zxy} = 1$ and $\epsilon_{yxz} = \epsilon_{zyx} = \epsilon_{xzy} = -1$, otherwise $\epsilon_{jkl} = 0$.

(2 points)

b) Prove that the Pauli matrices follow the commutation and anti-commutation relations:

$$[\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l \quad \text{and} \quad \{\sigma_j, \sigma_k\} = 2\delta_{jk}\mathbf{1}_2, \tag{11}$$

where the commutators and anticommutators of two operators A and B are defined as $[A, B] = AB - BA$ and $\{A, B\} = AB + BA$.

(2 points)

c) Show that the vector of Pauli matrices $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ and two vectors $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$ fulfill the following relation

$$(\boldsymbol{a} \cdot \boldsymbol{\sigma})(\boldsymbol{b} \cdot \boldsymbol{\sigma}) = \boldsymbol{a} \cdot \boldsymbol{b} + i(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{\sigma}. \tag{12}$$

Use this equality on a normalized vector $\boldsymbol{n} \in \mathbb{R}^3$ to show that

$$(\boldsymbol{n} \cdot \boldsymbol{\sigma})^2 = 1. \tag{13}$$

(2 points)