Theoretical physics V
Sheet 3

SoSe 2024 **Due for the 09.05.2024**

Exercise 7 *Properties of the gamma matrices*

The Hamiltonian corresponding to the Dirac equation is given by

$$\hat{H} = c\,\boldsymbol{\alpha} \cdot \boldsymbol{p} + \beta mc^2, \tag{1}$$

with $\boldsymbol{\alpha} = (\alpha^x, \alpha^y, \alpha^z)$. The matrices α^j and β exhibit the following properties: $(\alpha^j)^\dagger = \alpha^j$, $\mathrm{Tr}[\alpha^j] = 0 = \mathrm{Tr}[\beta]$ as well as $\{\alpha^j, \alpha^k\} = 2\delta^{jk}\mathbf{1}_4$ and $\{\alpha^j, \beta\} = 0$, where the anticommutator is defined as $\{A, B\} = AB + BA$.
Beside, the Dirac Equation in the van-der-Waerden form is given by

$$(i\hbar\gamma^\mu\partial_\mu - mc)\,\psi(\boldsymbol{r}, t) = 0, \tag{2}$$

where the γ-matrices are defined by the general expression $\gamma^\mu = \beta\alpha^\mu$, namely $\gamma^0 = \beta$, while $\gamma^j = \beta\alpha^j$ for $j = 1, 2, 3$. Show the following properties, without explicitly writing the gamma matrices:

a) The anticommutating relation $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}\mathbf{1}_4$ holds,

(1 point)

b) That the matrices γ^j are anti-hermitian,

(1 point)

c) The trace of the γ^μ vanishes.

(1 point)

Exercise 8 *Dirac spinor and Lorentz boost*

The solutions for the Dirac equation for a free particle of mass m moving with momentum $\boldsymbol{p}$ belong to a Hilbert space spanned by the basis $\psi^{(j)}$ whose elements are defined as:

$$\psi^{(j)}(\boldsymbol{x}, t) = \mathcal{N} u^{(j)}(\boldsymbol{p}) e^{i(\boldsymbol{p}\cdot\boldsymbol{x}\ -Et)/\hbar} \text{ with } \mathcal{N} = \sqrt{(|E| + mc^2)/(2|E|V)}, \tag{3}$$

where the spinors $u^{(j)}$ take the form

$$u^{(1)}(\boldsymbol{p}) = \begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_z}{E+mc^2} \\ \dfrac{c(p_x+ip_y)}{E+mc^2} \end{pmatrix}, \quad u^{(2)}(\boldsymbol{p}) = \begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p_x-ip_y)}{E+mc^2} \\ -\dfrac{cp_z}{E+mc^2} \end{pmatrix}$$

$$u^{(3)}(\boldsymbol{p}) = \begin{pmatrix} -\dfrac{cp_z}{|E|+mc^2} \\ -\dfrac{c(p_x+ip_y)}{|E|+mc^2} \\ 1 \\ 0 \end{pmatrix}, \quad u^{(4)}(\boldsymbol{p}) = \begin{pmatrix} -\dfrac{c(p_x-ip_y)}{|E|+mc^2} \\ \dfrac{cp_z}{|E|+mc^2} \\ 0 \\ 1 \end{pmatrix},$$

with p_x, p_y, p_z being the components of the momentum $\boldsymbol{p}$, while E stands for the energy of the particle and V for the volume available to the particle.

a) We have previously seen that Dirac spinors transform as $\psi' = S\psi$ for a change of reference frame. Transform the solutions for the free electron Dirac equation in the case of $\boldsymbol{p} = p\boldsymbol{e}_x$ to a moving reference frame with $v = \beta c$, where $\beta = cp/E$. The Lorentz transformation for a boost in the x-direction is given by

$$S_L = \cosh\frac{\chi}{2}\mathbf{1}_4 - \alpha_x \sinh\frac{\chi}{2}, \tag{4}$$

for $\tanh\chi = \beta$. Compare these results with the solutions of the electron at rest. Check whether the states are normalized.

(2 points)

b) Show that positive- (negative-) energy solutions of the Dirac equation for $\boldsymbol{p} = 0$ transform via a Lorentz boost into another positive- (negative-) energy solution of the Dirac equation.

Hint: Use the spinor transformation to a reference frame of rapidity χ along the direction $-\boldsymbol{n}$:

$$S_L = \cosh\frac{\chi}{2}\mathbf{1}_4 + \boldsymbol{n}\cdot\boldsymbol{\alpha}\sinh\frac{\chi}{2}.$$

(2 points)

Exercise 9 *Helicity conservation*

The helicity operator is defined as $\hat{h} = \boldsymbol{\Sigma}\cdot\hat{\boldsymbol{p}}$, where the elements of the vector $\boldsymbol{\Sigma} = (\Sigma^x, \Sigma^y, \Sigma^z)$ take the form

$$\Sigma^j = \begin{pmatrix} \sigma^j & 0 \\ 0 & \sigma^j \end{pmatrix}.$$

Show that the helicity is a conserved quantity of motion, namely that

$$[\hat{H}, \hat{h}] = 0, \tag{5}$$

where $\hat{H}$ is defined in Eq. (1).

(2 points)

Exercise 10 *Charge conjugation*

Let us introduce the charge conjugation operator $\mathcal{C} = -i\beta\alpha^2 = -i\gamma^2$, where

$$\alpha^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2 \end{pmatrix}.$$

a) Verify that, given the wave function $\psi(\boldsymbol{x}, t) = u_{\boldsymbol{p}} e^{i(\boldsymbol{p}\cdot\boldsymbol{x} - Et)/\hbar}$ of a negative-energy electron with momentum $\boldsymbol{p}$, then $v_{\boldsymbol{p}} = \mathcal{C} u^*_{\boldsymbol{p}}$ satisfy the Dirac equation with energy $E > 0$ and momentum $-\boldsymbol{p}$.

(2 points)

b) Show that $\mathcal{C} = \mathcal{C}^\dagger$ and $\mathcal{C}^2 = \mathbf{1}_4$. Write down the explicit form of $\mathcal{C}$ as a 4×4 matrix.

(1 point)

c) Verify then the validity of the relation

$$j^\mu = \phi^\dagger \alpha^\mu \phi = \psi^\dagger \alpha^\mu \psi, \qquad (6)$$

with $\phi = \mathcal{C}\psi^*$, therefore showing that the probability and flow density are identical for both the negative-energy electron and the positron.

(2 points)