Theoretical physics V e
Exam Preparation Sheet
(Not graded, points are given for reference)

SoSe 2025 **25.06.2025**

Exercise 1 *Around the Dirac equation (11 points)*

Let us consider a fermion contained in a volume Ω whose wave function at time $t = 0$ is described by the spinor field

$$\psi(\boldsymbol{r}, 0) = \frac{1}{\sqrt{\Omega}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{i\boldsymbol{p}\cdot\boldsymbol{r}/\hbar}, \tag{1}$$

with the wavevector $\boldsymbol{p} = p_3 \boldsymbol{e}_3$.

a) Show that the wave function $\psi(\boldsymbol{r}, 0)$ is normalized and that it is an eigenfunction of the momentum operator $\hat{\boldsymbol{p}} = -i\hbar\boldsymbol{\nabla}$. Show that it is not, however, an eigenstate of the Hamiltonian operator $\hat{H} = \hat{\boldsymbol{\alpha}} \cdot \hat{\boldsymbol{p}}c + \beta mc^2$.

(2 points)

b) Write the wave function $\psi(\boldsymbol{r}, 0)$ as a linear combination of the eigenfunctions of the Hamiltonian $u^{(j)}(\boldsymbol{p}) \exp(i\boldsymbol{p} \cdot \boldsymbol{r}/\hbar)$ (see Formulas):

$$\psi(\boldsymbol{r}, 0) = \frac{1}{\sqrt{\Omega}} \sum_j c_j u_j(\boldsymbol{p}) e^{i\boldsymbol{p}\cdot\boldsymbol{r}/\hbar}, \tag{2}$$

and determine the explicit form of the coefficients c_j. Determine the wavefunction at time $t > 0$ and give the explicit form of the energy $E_{\boldsymbol{p}}$ as a function of the momentum $\boldsymbol{p}$.

(3 points)

c) Determine the probability that the fermion has positive energy at time t.

(1 point)

d) Determine the probability at time t that the spin of the fermion is oriented in the up-direction along the z-axis.

(1 point)

e) Apply the operator $\hat{S} = \cosh(\chi/2)\hat{\mathbf{1}}_4 - \sinh(\chi/2)\hat{\alpha}^3$ that brings $\psi(\boldsymbol{r}, t)$ to the reference frame at rest via a Lorentz boost with rapidity χ such that $\tanh\chi = \beta$, where here $\beta = pc/E_{\boldsymbol{p}}$. Determine the form of $\psi'(\boldsymbol{r}', t') = \hat{S}\psi(\boldsymbol{r}, t)$.

Hint: Start by applying the transformation $\hat{S}$ to $\psi(\boldsymbol{r}, 0)$.

(2 points)

f) Verify that the state $\psi'(\boldsymbol{r}', t')$ is normalized. What is the volume Ω' occupied by the fermion in the new reference frame?

(2 points)

Exercise 2 *The Bremsstrahlung (10 points)*

Let us consider a single electron of mass m and charge $q = -e$ interacting with the electromagnetic field described by $A^\mu = (0, \boldsymbol{A})$ (where $\boldsymbol{A}$ is in the Coulomb gauge). The minimal-coupling Hamiltonian of the joint light-matter system reads

$$\hat{H} = \hat{\boldsymbol{\alpha}} \cdot \left(\hat{\boldsymbol{p}} + \frac{e}{c}\hat{\boldsymbol{A}}\right) c + \beta mc^2 + \sum_\lambda \hbar\omega_\lambda \hat{a}_\lambda^\dagger \hat{a}_\lambda. \tag{3}$$

The quantized vector potential takes the form

$$\hat{\boldsymbol{A}}(\boldsymbol{r}, t) = \sum_\lambda \sqrt{\frac{\hbar c^2}{2\omega_\lambda V}} \left[\hat{a}_\lambda e^{i(\boldsymbol{k}_\lambda \cdot \boldsymbol{r} - \omega_\lambda t)} \boldsymbol{e}_\lambda + \hat{a}_\lambda^\dagger e^{-i(\boldsymbol{k}_\lambda \cdot \boldsymbol{r} - \omega_\lambda t)} \boldsymbol{e}_\lambda^*\right], \tag{4}$$

where $[\hat{a}_\lambda, \hat{a}_{\lambda'}^\dagger] = \delta_{\lambda\lambda'}$.

In the following, we treat the light-matter interaction as a perturbation in order to describe the phenomenon dubbed as Bremsstrahlung, namely the emission of light by a decelerating charged particle. We split the Hamiltonian as:

$$\hat{H}_0 = \hat{\boldsymbol{\alpha}} \cdot \hat{\boldsymbol{p}} c + \beta mc^2 + \sum_\lambda \hbar\omega_\lambda \hat{a}_\lambda^\dagger \hat{a}_\lambda, \tag{5}$$

$$\hat{H}_{\text{int}} = e\hat{\boldsymbol{\alpha}} \cdot \hat{\boldsymbol{A}}, \tag{6}$$

where $\hat{H}_0$ is the free-particle part and $\hat{H}_{\text{int}}$ implements the light-matter interaction.

a) Determine the eigenbasis of the free Hamiltonian $\hat{H}_0$.

(1 point)

b) Assuming that the electron is initially in state $|i\rangle = |\boldsymbol{p}^{(1)}, \text{vac}\rangle$, namely in the spinor state $u^{(1)}(\boldsymbol{p})$, determine the non-vanishing matrix elements $\langle f|\hat{H}_{\text{int}}|i\rangle$.

(2 points)

c) Using the Fermi golden rule (see Formulas), determine the rate Γ out of the initial state $|i\rangle$ as a function of the density of states ρ.

(3 points)

d) Show that the total momentum $\hat{\boldsymbol{P}} = \hat{\boldsymbol{p}} + \sum_\lambda \boldsymbol{k}_\lambda \hat{a}_\lambda^\dagger \hat{a}_\lambda$ is a conserved quantity. Considering now the helicity operator $\hat{h} = \hat{\boldsymbol{\Sigma}} \cdot \hat{\boldsymbol{p}}$, show $[\hat{h}, \hat{H}] \neq 0$. Is the helicity conserved over time?

Hint: Use the following commutation relation

$$[\hat{\boldsymbol{p}}, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}] = \hbar\boldsymbol{k}\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}}.$$

(4 points)

Exercise 3 *Circular polarization of light (9 points)*

Let us consider an electromagnetic field with two possible linear polarizations defined by the unit vectors $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, such that $\boldsymbol{k} \perp \boldsymbol{e}_1,\, \boldsymbol{e}_2$. The energy of the quantized electromagnetic field reads

$$\hat{H}_{\text{em}} = \sum_{\boldsymbol{k}} \hbar c |\boldsymbol{k}| (\hat{a}^\dagger_{1,\boldsymbol{k}} \hat{a}_{1,\boldsymbol{k}} + \hat{a}^\dagger_{2,\boldsymbol{k}} \hat{a}_{2,\boldsymbol{k}} + 1), \tag{7}$$

where the operators $\hat{a}_j$ follow the bosonic commutation relations $[\hat{a}_{i,\boldsymbol{k}}, \hat{a}^\dagger_{j,\boldsymbol{k}'}] = \delta_{ij}\delta_{\boldsymbol{k}\boldsymbol{k}'}$ and $[\hat{a}^{(\dagger)}_{i,\boldsymbol{k}}, \hat{a}^{(\dagger)}_{j,\boldsymbol{k}'}] = 0$.

a) Defining the circular polarization vectors

$$\boldsymbol{e}_\pm = \mp \frac{1}{\sqrt{2}} (\boldsymbol{e}_1 \pm i \boldsymbol{e}_2). \tag{8}$$

and by the means of the definition of the quantized vector potential

$$\hat{\boldsymbol{A}}(\boldsymbol{r}, t) = \frac{c}{\sqrt{V}} \sum_{j,\boldsymbol{k}} \sqrt{\frac{\hbar}{2\omega_{\boldsymbol{k}}}} \left[\hat{a}_{j,\boldsymbol{k}}(t) e^{i\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{e}_j + \hat{a}^\dagger_{j,\boldsymbol{k}}(t) e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{e}^*_j \right], \tag{9}$$

show that one can define operators $\hat{a}_{\pm,\boldsymbol{k}}$ and $\hat{a}^\dagger_{\pm,\boldsymbol{k}}$ associated with the circular polarization $\boldsymbol{e}_\pm$ axes. Express the Hamiltonian $\hat{H}_{\text{em}}$ in terms of $\hat{a}_{\pm,\boldsymbol{k}}$.

(3 points)

b) We consider a two-photon state with momenta $\boldsymbol{k}$ and $-\boldsymbol{k}$. Four states of polarization are possible:

$$|++\rangle = \hat{a}^\dagger_{+,\boldsymbol{k}} \hat{a}^\dagger_{+,-\boldsymbol{k}} |0\rangle \;\; ; \;\; |+-\rangle = \hat{a}^\dagger_{+,\boldsymbol{k}} \hat{a}^\dagger_{-,-\boldsymbol{k}} |0\rangle \; ;$$
$$|-+\rangle = \hat{a}^\dagger_{-,\boldsymbol{k}} \hat{a}^\dagger_{+,-\boldsymbol{k}} |0\rangle \;\; ; \;\; |--\rangle = \hat{a}^\dagger_{-,\boldsymbol{k}} \hat{a}^\dagger_{-,-\boldsymbol{k}} |0\rangle \, .$$

Show that these states are eigenvectors of the Hamiltonian $\hat{H}_{\text{em}}$ and determine their eigenvalues. Introducing the total momentum $\hat{\boldsymbol{P}} = \sum_{\boldsymbol{k},j} \hbar \boldsymbol{k} (\hat{a}^\dagger_{\boldsymbol{k},1} \hat{a}_{\boldsymbol{k},1} + \hat{a}^\dagger_{\boldsymbol{k},2} \hat{a}_{\boldsymbol{k},2})$, show that the states introduced above are also eigenstates of $\hat{\boldsymbol{P}}$ and determine the corresponding eigenvalues.

(3 points)

c) Express the following states in terms of the operators $\hat{a}^\dagger_{1,\pm\boldsymbol{k}}$ and $\hat{a}^\dagger_{2,\pm\boldsymbol{k}}$:

$$|++\rangle + |--\rangle \;\; ; \;\; |++\rangle - |--\rangle \;\; ; \;\; |+-\rangle \;\; ; \;\; |-+\rangle.$$

Comment on the orientation of the polarization of the photon pairs.

(3 points)

Formulas

Dirac matrices

The $\hat{\alpha}^i$ and $\hat{\beta}$ matrices are given by the following expressions

$$\hat{\alpha}^1 \triangleq \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \ , \ \hat{\alpha}^2 \triangleq \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \ , \ \hat{\alpha}^3 \triangleq \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \ ,$$

and

$$\hat{\beta} \triangleq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

These matrices obey the following anticommutation relation

$$\{\hat{\alpha}^i, \hat{\alpha}^j\} = 2\delta_{ij}\hat{\mathbf{1}}_4 \ , \ \{\hat{\alpha}^i, \hat{\beta}\} = 0 \ , \ \hat{\beta}^2 = (\hat{\alpha}^i)^2 = \hat{\mathbf{1}}_4.$$

We note that generally

$$\hat{\alpha}^i = \begin{pmatrix} 0 & \hat{\sigma}^i \\ \hat{\sigma}^i & 0 \end{pmatrix} ,$$

with the Pauli matrices $\hat{\sigma}^i$.

Spin operators

The spin operators $\hat{\Sigma}^i$, $i = 1, 2, 3$, are generally given by

$$\hat{\Sigma}^i = \begin{pmatrix} \hat{\sigma}^i & 0 \\ 0 & \hat{\sigma}^i \end{pmatrix} ,$$

with the Pauli matrices $\hat{\sigma}^i$, and explicitly

$$\hat{\Sigma}^1 \triangleq \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \ , \ \hat{\Sigma}^2 \triangleq \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \end{pmatrix} \ , \ \hat{\Sigma}^3 \triangleq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

Pauli matrices

The Pauli matrices obey the commutation relations

$$[\hat{\sigma}^i, \hat{\sigma}^j] = 2i\epsilon^{ijk}\hat{\sigma}^k,$$

where ϵ^{ijk} stands for the Levi-Civita tensor, such that

$$\epsilon^{123} = \epsilon^{231} = \epsilon^{312} = 1 \ , \ \epsilon^{132} = \epsilon^{213} = \epsilon^{321} = -1.$$

These properties lead to the relationship

$$(\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{a})(\hat{\boldsymbol{\sigma}} \cdot \boldsymbol{b}) = (\boldsymbol{a} \cdot \boldsymbol{b})\hat{\mathbf{1}}_2 + i(\boldsymbol{a} \times \boldsymbol{b}) \cdot \hat{\boldsymbol{\sigma}} \ ,$$

with $\hat{\boldsymbol{\sigma}} = (\hat{\sigma}^1, \hat{\sigma}^2, \hat{\sigma}^3)$.

Hyperbolic trigonometry

We give the following identities:

$$\cosh^2(x) + \sinh^2(x) = \cosh(2x)\,,$$
$$\cosh^2(x) - \sinh^2(x) = 1\,.$$

Spinor basis

Positive-energy eigenstates ($E_{\boldsymbol{p}}^{(1,2)} = E_{\boldsymbol{p}} > 0$, with $E_{\boldsymbol{p}} = \sqrt{|\boldsymbol{p}|^2c^2 + m^2c^4}$) of the Dirac equation at momentum $\boldsymbol{p}$, $u^{(1,2)}(\boldsymbol{p})e^{i(\boldsymbol{p}\cdot\boldsymbol{r}-E_{\boldsymbol{p}}t)/\hbar}$, such that

$$u^{(1)}(\boldsymbol{p}) = \sqrt{\frac{(E_{\boldsymbol{p}}+mc^2)}{2E_{\boldsymbol{p}}}}\begin{pmatrix} 1 \\ 0 \\ \dfrac{p_zc}{E_{\boldsymbol{p}}+mc^2} \\ \dfrac{(p_x+ip_y)c}{E_{\boldsymbol{p}}+mc^2} \end{pmatrix}, \; u^{(2)}(\boldsymbol{p}) = \sqrt{\frac{(E_{\boldsymbol{p}}+mc^2)}{2E_{\boldsymbol{p}}}}\begin{pmatrix} 0 \\ 1 \\ \dfrac{(p_x-ip_y)c}{E_{\boldsymbol{p}}+mc^2} \\ \dfrac{-p_zc}{E_{\boldsymbol{p}}+mc^2} \end{pmatrix}.$$

Negative-energy eigenstates ($E_{\boldsymbol{p}}^{(3,4)} = -E_{\boldsymbol{p}} < 0$) on the other hand are $u^{(3,4)}(\boldsymbol{p})e^{i(\boldsymbol{p}\cdot\boldsymbol{r}+E_{\boldsymbol{p}}t)/\hbar}$, such that

$$u^{(3)}(\boldsymbol{p}) = \sqrt{\frac{(E_{\boldsymbol{p}}+mc^2)}{2E_{\boldsymbol{p}}}}\begin{pmatrix} \dfrac{-p_zc}{E_{\boldsymbol{p}}+mc^2} \\ \dfrac{-(p_x+ip_y)c}{E_{\boldsymbol{p}}+mc^2} \\ 1 \\ 0 \end{pmatrix}, \; u^{(4)}(\boldsymbol{p}) = \sqrt{\frac{(E_{\boldsymbol{p}}+mc^2)}{2E_{\boldsymbol{p}}}}\begin{pmatrix} \dfrac{-(p_x-ip_y)c}{E_{\boldsymbol{p}}+mc^2} \\ \dfrac{p_zc}{E_{\boldsymbol{p}}+mc^2} \\ 0 \\ 1 \end{pmatrix}.$$

Spinors are an orthonormal set of vectors, such that

$$(u^{(s)}(\boldsymbol{p}))^\dagger u^{(s')}(\boldsymbol{p}) = \delta_{ss'}.$$

Fermi's golden rule

According to the Fermi golden rule, the rate of the transition between states $|i\rangle$ and $|f\rangle$ (both being eigenstates of a Hamiltonian $\hat{H}_0$ at energies E_i and E_f) due to a perturbation $\hat{V}$ reads

$$\Gamma_{i\to f}(t) = \frac{2\pi}{\hbar}\overline{|V_{fi}|}^2\rho(E_f = E_i), \tag{10}$$

where $\overline{|V_{fi}|}$ is an averaged value of $|V_{fi}| = |\langle f|\hat{V}|i\rangle|$ assuming that it is a smooth function of the final energy E_f. The density of states at energy E is represented by $\rho(E)$.