Theoretical physics V
Sheet 8

SoSe 2025 **27.05.2025**

Exercise 1 *Ionization of an atom*

Let us consider a simplified model for the ionization of an atom consisting in a ground state $|g\rangle$ at energy $\hbar\omega_0$ and a continuum of ionized states labeled $|I\rangle$ at energy $\hbar(\omega_1 + \omega_I)$, such that $\omega_1 > \omega_0$ and let $\omega_I = I\delta$, with $I \in \mathbb{N}$, and $\delta \in \mathbb{R}, \delta > 0$. The ground state and the ionized states are coupled by a time-dependent perturbative term such that the total Hamiltonian reads

$$\hat{H} = \hat{H}_0 + \hat{V}(t), \tag{1}$$

where

$$\hat{H}_0 = \hbar\omega_0|g\rangle\langle g| + \sum_I \hbar(\omega_1 + \omega_I)|I\rangle\langle I|,$$
$$\hat{V}(t) = V_0 \sum_I (|I\rangle\langle g|e^{-i\omega t} + \text{h.c.}).$$

For an atom initially prepared in its ground state $|g\rangle$ at time $t = 0$, determine the rate Γ at which the atom is ionized by using the expression

$$w_{i\to f}(T) = \frac{1}{i\hbar}\int_{-T/2}^{+T/2} \mathrm{d}\tau \langle f|\hat{V}(\tau)|i\rangle e^{i(E_f - E_i)\tau/\hbar}, \tag{2}$$

where $w_{i\to f}$ is the transition amplitude over a time T from state $|i\rangle$ at energy E_i to a state $|f\rangle$ at energy E_f due to a perturbation $\hat{V}(t)$.
Hint: You shall go to the continuous limit where $\sum_I \to \int \mathrm{d}E_I\rho(E_I)$, with $\rho(E)$ being the density of state at energy E. Determine explicitly $\rho(E)$.

(2 points)

Exercise 2 *Time-dependent perturbation theory*

Let us consider a two-level system described over its energy eigenbasis by the states $\{|g\rangle, |e\rangle\}$, such that its Hamiltonian reads

$$\hat{H}_0 = E_e|e\rangle\langle e| + E_g|g\rangle\langle g|, \tag{3}$$

where $E_e > E_g$. Consider a time-dependent perturbation such that $\hat{H} = \hat{H}_0 + \lambda\hat{V}(t)$, where the perturbating term $\hat{V}(t)$ reads

$$\hat{V}(t) = \Omega_0\left(|g\rangle\langle e|e^{i\omega t} + |e\rangle\langle g|e^{-i\omega t}\right). \tag{4}$$

At initial time $t = 0$, we assume that the system is in state $|\psi_0\rangle = |g\rangle$.

a) Determine in first-order perturbation theory the probability amplitude of a transition to state $|e\rangle$ at time t. Show that, in interaction picture it takes the form

$$w_{i\to f}(t) = -2\pi i\ \lambda\Omega_0\ \delta^{(t)}(E_e - E_g - \hbar\omega) \exp\left(\frac{i(E_e - E_g - \hbar\omega)t}{2\hbar}\right) \tag{5}$$

where the function $\delta^{(t)}(E_f - E_i)$ is defined as

$$\delta^{(t)}(E_f - E_i) = \frac{1}{\pi}\frac{\sin((E_f - E_i)t/2\hbar)}{E_f - E_i}\ . \tag{6}$$

(2 points)

b) In the following, we define the transition detuning as $\Delta = E_e - E_g - \hbar\omega$. The dynamics that we have just treated perturbatively can actually be solved exactly by the means of a change of reference frame $|\psi\rangle = \hat{U}(t)|\psi'\rangle$, where

$$\hat{U}(t) = e^{-i\omega t/2}|e\rangle\langle e| + e^{+i\omega t/2}|g\rangle\langle g|. \tag{7}$$

Using this change of reference frame, show that the time-evolution of the two-level system is described by a state of the form $|\psi'\rangle_t = c_e(t)|e\rangle + c_g(t)|g\rangle$, where

$$c_g(t) = \cos\left(\frac{\sqrt{\Delta^2 + 4\lambda^2\Omega_0^2}}{2\hbar}t\right) + \frac{i\Delta}{\sqrt{\Delta^2 + 4\lambda^2\Omega_0^2}}\sin\left(\frac{\sqrt{\Delta^2 + 4\lambda^2\Omega_0^2}}{2\hbar}t\right) \tag{8}$$

$$c_e(t) = -\frac{2i\lambda\Omega_0}{\sqrt{\Delta^2 + 4\lambda^2\Omega_0^2}}\sin\left(\frac{\sqrt{\Delta^2 + 4\lambda^2\Omega_0^2}}{2\hbar}t\right). \tag{9}$$

Perform a Taylor expansion in λ of the transition amplitude $c_e(t)$ and compare it to the result obtained with the perturbation theory.

(4 points)

Exercise 3 *Sine-Gordon equation*

Let us consider an infinite chain of identical pendulums of mass m and length ℓ attached to a common axis. Neighbouring pendulums are coupled via torsion springs of stiffness C. Therefore, the angles θ_n accounting for the angular deviation of the n-th pendulum from its equilibrium position is described by the Lagrangian

$$L = \sum_{n=-\infty}^{+\infty}\left[\frac{1}{2}m\ell^2\dot{\theta}_n^2 - \frac{1}{2}C(\theta_n - \theta_{n-1})^2 - mg\ell(1 - \cos\theta_n)\right] \tag{10}$$

where g is Earth's gravitational acceleration.

a) Assuming that the distance a between two successive pendulums is the smallest length scale of the system, then we can treat θ as a continuous field such that $\theta_n(t) = \theta(x = na, t)$. Show then that the Lagrangian takes the form

$$L = \int_{-\infty}^{+\infty} \mathrm{d}x \left[\frac{1}{2} J \left(\frac{\partial \theta}{\partial t} \right)^2 - \frac{1}{2} K \left(\frac{\partial \theta}{\partial x} \right)^2 - \Omega(1 - \cos\theta) \right]. \tag{11}$$

Specify the expression of the constants J, K and Ω. *(2 points)*

b) Derive the corresponding Euler-Lagrange equation and show that it takes the form of the sine-Gordon equation:

$$\frac{\partial^2 \theta}{\partial t^2} - c_0^2 \frac{\partial^2 \theta}{\partial x^2} + \omega_0^2 \sin\theta = 0, \tag{12}$$

where $c_0^2 = \frac{Ca^2}{m\ell^2}$ and $\omega_0^2 = \frac{g}{\ell}$.

(1 point)

c) Using the change of variable $z = x - vt$, show that the sine-Gordon equation boils down to

$$\frac{\mathrm{d}^2 \theta}{\mathrm{d}z^2} = \frac{\omega_0^2}{c_0^2 - v^2} \sin\theta, \tag{13}$$

where $v < c_0$ behaves as a velocity.

(1 point)

d) Solutions of the sine-Gordon equation describe a type of waves called solitons. In order to describe a single soliton, we impose boundary conditions such that $\mathrm{d}\theta/\mathrm{d}z = 0$ for $z \to \pm\infty$, as well as $\theta = 0$ for $z \to -\infty$ and $\theta = 2\pi$ for $z \to +\infty$. Show then that

$$\frac{1}{2} \left(\frac{\mathrm{d}\theta}{\mathrm{d}z} \right)^2 = \frac{\omega_0^2}{c_0^2 - v^2} (1 - \cos\theta). \tag{14}$$

(2 points)

e) Using the relation

$$\int \frac{\mathrm{d}\theta}{\sin(\frac{\theta}{2})} = 2 \ln \left(\tan \left(\frac{\theta}{4} \right) \right),$$

determine the form of the solution $\theta(x, t)$ of the sine-Gordon equation. Plot its form for a fixed time t.

(3 points)