Theoretical physics V
Sheet 3

SoSe 2025 **22.04.2025**

Exercise 1 *Non-relativistic limit of the Klein-Gordon equation*

Let us consider a scalar boson described by the wave function $\phi(\boldsymbol{r}, t)$. The behaviour of the wave function is governed by the Klein-Gordon equation

$$\left(\partial_\mu \partial^\mu + \frac{m^2 c^2}{\hbar^2}\right) \phi(\boldsymbol{r}, t) = 0, \tag{1}$$

where $\partial_\mu \partial^\mu = \eta^{\mu\nu} \partial_\mu \partial_\nu$ stands for the d'Alembert operator and $\eta^{\mu\nu}$ is the flat metric tensor of special relativity, a diagonal matrix with coefficients $\eta^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

a) Assuming that the wave function has the form $\phi(\boldsymbol{r}, t) = e^{-iEt/\hbar} \phi_0(\boldsymbol{r})$, show that the behaviour of the wave function is described by the equation

$$E^2 \phi(\boldsymbol{r}, t) = \left(-\hbar^2 c^2 \nabla^2 + m^2 c^4\right) \phi(\boldsymbol{r}, t)\,. \tag{2}$$

Show explicitly that Eq. (2) is fulfilled for $\phi_0(\boldsymbol{r}) = \exp(\mathrm{i}\boldsymbol{p} \cdot \boldsymbol{r}/\hbar)/N$, where $\boldsymbol{p} \in \mathbb{R}^3$ and N is a normalization constant. Subsequently, verify that $\boldsymbol{p}$ and E/c, as given by Eq. (2), are valid components of the energy-momentum four vector. *(2 points)*

b) Show that in the non-relativistic case, namely in the limit where $E = E_{NR} + mc^2$ with $mc^2 \gg E_{NR}$, the Klein-Gordon equation is approximated by the Schrödinger equation of a free particle:

$$E_{NR} \phi(\boldsymbol{r}, t) \simeq -\frac{\hbar^2}{2m} \nabla^2 \phi(\boldsymbol{r}, t). \tag{3}$$

(1 point)

Exercise 2 *Transformation between inertial reference frames*

As seen in the lectures, a change of reference frame leads to a transformation of the Dirac spinor $\psi' = S\psi$. For the transformation S to leave the Dirac equation invariant, it needs to fulfill the following conditions

$$S^\dagger \alpha^\mu S = \Lambda^\mu_\nu \alpha^\nu \quad \text{and} \quad S^\dagger \beta S = \beta, \tag{4}$$

where Λ is a transformation of the Lorentz group. The α and β matrices are defined as

$$\alpha^0 = \mathbf{1}_4 \quad , \quad \alpha^i = \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & -\mathbf{1}_2. \end{pmatrix} \tag{5}$$

Show that the operators S given in the following fulfill the conditions states in Eq. (4) for the corresponding reference frame transformations.

a) Consider a rotation around the z axis by an angle θ, coordinates transform as:

$$ct' = ct \; ; \; z' = z \, ;$$
$$x' = \cos\theta \, x + \sin\theta \, y \; ; \; y' = \cos\theta \, y - \sin\theta \, x \, .$$

Identify the matrix elements of Λ and show that the transformation

$$S = \cos\frac{\theta}{2}\begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix} + i\sin\frac{\theta}{2}\begin{pmatrix} \sigma^z & 0 \\ 0 & \sigma^z \end{pmatrix}, \tag{6}$$

fulfills the conditions stated in Eq. (4).

(2 points)

b) We consider now a Lorentz boost along the x axis by a rapidity χ that transforms the coordinates as:

$$ct' = \cosh\chi \, ct - \sinh\chi \, x \; ; \; x' = \cosh\chi \, x - \sinh\chi \, ct \, ;$$
$$y' = y \; ; \; z' = z \, .$$

etermine the elements of the matrix Λ and show that the transformation

$$S = \cosh\frac{\chi}{2}\begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix} - \sinh\frac{\chi}{2}\begin{pmatrix} 0 & \sigma^x \\ \sigma^x & 0 \end{pmatrix}, \tag{7}$$

fulfills the conditions stated in Eq. (4).

(2 points)

c) Let us finally consider a spatial reflection given by the transformation

$$ct' = ct \; ; \; x_i' = -x_i.$$

Identiy the form taken by the matrix Λ and show that the transformation

$$S = \beta \tag{8}$$

fulfills the conditions stated in Eq. (4).

(1 point)