Theoretical physics V
Sheet 4

SoSe 2025 **29.04.2025**

Exercise 1 *Solution of the Dirac equation*

The Dirac equation is given by

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi\,, \tag{1}$$

with the Hamiltonian

$$H = c\,\boldsymbol{\alpha}\cdot\boldsymbol{p} + \beta mc^2 \tag{2}$$

and $\boldsymbol{\alpha} = (\alpha^1, \alpha^2, \alpha^3)$. The matrices α^j and β exhibit the following properties: $(\alpha^j)^\dagger = \alpha^j$, $\mathrm{Tr}[\alpha^j] = 0 = \mathrm{Tr}[\beta]$ as well as $\{\alpha^j, \alpha^k\} = 2\delta^{jk}\mathbf{1}_4$ and $\{\alpha^j, \beta\} = 0$, where the anticommutator is defined as $\{A, B\} = AB + BA$. They are explicitly given by

$$\alpha^1 \mathrel{\hat{=}} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}\,,\; \alpha^2 \mathrel{\hat{=}} \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}\,,\; \alpha^3 \mathrel{\hat{=}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}\,, \tag{3}$$

and

$$\beta \mathrel{\hat{=}} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{4}$$

In this exercise, we consider **a particle at rest**, i.e., $\boldsymbol{p} = 0$, so the Hamiltonian simplifies to $H = \beta mc^2$.

a) Determine the eigenvalues E_ν and normalized eigenstates Ψ_ν of the simplified Hamiltonian. How many distinct eigenvalues does H have and what are their degeneracies?

(1 point)

b) Using the Dirac equation, compute the time evolution of the eigenstates Ψ_ν. With this, construct a general solution $\psi(t)$ of the Dirac equation. *(1 point)*

Exercise 2 *Properties of the gamma matrices*

The Dirac Equation in the van-der-Waerden form is given by

$$(i\hbar\gamma^\mu\partial_\mu - mc)\,\psi(\boldsymbol{r}, t) = 0, \tag{5}$$

where the γ-matrices are defined by the general expression $\gamma^\mu = \beta\alpha^\mu$, namely $\gamma^0 = \beta$, while $\gamma^j = \beta\alpha^j$ for $j = 1, 2, 3$. Show the following properties, without explicitly writing the γ-matrices:

a) Anti-commutation relation $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}_4$, with the flat space-time metric

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

(1 point)

b) The matrices γ^j are anti-hermitian.

(1 point)

c) The trace of the matrices γ^μ vanishes.

(1 point)

Exercise 3 *Helicity conservation*

The helicity operator is defined as $h = \mathbf{\Sigma} \cdot \boldsymbol{p}$, with the momentum operator $\boldsymbol{p} = (p_1, p_2, p_3)$ and the spin operator $\mathbf{\Sigma} = (\Sigma^1, \Sigma^2, \Sigma^3)$, with components

$$\Sigma^j = \begin{pmatrix} \sigma_j & 0 \\ 0 & \sigma_j \end{pmatrix}.$$

Here, σ_j, with $j = 1, 2, 3$, denote the Pauli matrices. Show that the helicity is a conserved quantity of motion, namely that

$$[H, h] = 0, \tag{6}$$

where H is defined in Eq. (2).

(2 points)

Exercise 4 *Weyl fermions and helicity*

Let us consider the case of massless fermions described by the spinor $\psi(\boldsymbol{r}, t)$. Since the mass vanishes $m \to 0$, then the Dirac equation reads

$$i\hbar\gamma^\mu\partial_\mu\psi(\boldsymbol{r}, t) = 0, \tag{7}$$

where the matrices γ^μ take the explicit form

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \; \gamma^j = \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix},$$

with the Pauli matrices σ_j, $j = 1, 2, 3$.

a) Let $\psi(\boldsymbol{r}, t) = (u_1, u_2, v_1, v_2)^T$ be the 4-component spinor (column vector), with position- and time-dependent components $u_{1,2}, v_{1,2} \in \mathbb{C}$. Using Eq. (7), show that the resulting equations of motion for the components $u_{1,2}, v_{1,2}$ can be brought into the form

$$\left(\frac{i\hbar}{c}\frac{\partial}{\partial t} + i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right)\chi_1 = 0, \tag{8a}$$

$$\left(\frac{i\hbar}{c}\frac{\partial}{\partial t} - i\hbar\boldsymbol{\sigma}\cdot\boldsymbol{\nabla}\right)\chi_2 = 0, \tag{8b}$$

with two-component vectors $\chi_1, \chi_2 \in \mathbb{C}^2$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)^T$ the vector with Pauli matrices σ_j, $j = 1, 2, 3$, as entries. What are the explicit expressions for the vectors χ_1, χ_2 as a function of the components $u_{1,2}$ and $v_{1,2}$ of ψ. The fermionic fields χ describe so-called Weyl fermions.

(2 points)

b) We define the helicity operator as $(\boldsymbol{\sigma}\cdot\boldsymbol{\nabla})/|\boldsymbol{p}|$, with the momentum $\boldsymbol{p}$ and $\boldsymbol{\sigma}$ defined as in a). Assuming that the components of $\psi(\boldsymbol{r}, t)$ are of the form $u_j(\boldsymbol{r}, t) = u_{0,j}(\boldsymbol{r})e^{-iE_{\boldsymbol{p}}t/\hbar}$ and $v_j(\boldsymbol{r}, t) = v_{0,j}(\boldsymbol{r})e^{-iE_{\boldsymbol{p}}t/\hbar}$, with $j = 1, 2$, show by means of the result of a) that $\chi_{1,2}$ are eigenfunctions of the helicity operator. Determine the corresponding eigenvalues.

(2 points)