Theoretical physics V
Sheet 5

SoSe 2025 **06.05.2025**

In this exercise sheet, we consider a free fermionic particle of mass m, confined within a volume V, whose dynamics is governed by the Dirac equation:

$$\frac{\partial}{\partial t}\psi(\boldsymbol{r},t) = -\frac{i}{\hbar}\hat{H}\psi(\boldsymbol{r},t)\,, \tag{1}$$

with the Hamiltonian given by

$$\hat{H} = \hat{\boldsymbol{\alpha}}\cdot\hat{\boldsymbol{p}}c + \hat{\beta}mc^2\,, \tag{2}$$

where $\hat{\boldsymbol{p}} = (\hat{p}_1, \hat{p}_2, \hat{p}_3)$, with the momentum operators $\hat{p}_i$ corresponding to the spatial directions $i = 1, 2, 3$, and $\hat{\boldsymbol{\alpha}} = (\hat{\alpha}^1, \hat{\alpha}^2, \hat{\alpha}^3)$. The explicit form of the matrices $\hat{\alpha}^i$ and $\hat{\beta}$ are provided on the previous exercise sheet.
As discussed in the lecture, the general solution $\psi(\boldsymbol{r},t)$ to the Dirac equation for a free particle can be written as a linear combination of plane-wave solutions:

$$\psi(\boldsymbol{r},t) = \sum_{\boldsymbol{p},j} c_j(\boldsymbol{p})\psi^{(j)}_{\boldsymbol{p}}(\boldsymbol{r},t)\,, \tag{3}$$

with complex coefficients $c_j(\boldsymbol{p}) \in \mathbb{C}$ and the plane-wave solutions

$$\psi^{(j)}_{\boldsymbol{p}}(\boldsymbol{r},t) = \frac{1}{\sqrt{V}}u^{(j)}(\boldsymbol{p})e^{i(\boldsymbol{p}\cdot\boldsymbol{r}-E^{(j)}_{\boldsymbol{p}}t)/\hbar} \tag{4}$$

Here, $u^{(j)}(\boldsymbol{p})$ are the Dirac spinors for momentum $\boldsymbol{p}$ and the index $j = 1, 2$ labels the positive-energy solutions with $E^{(j)}_{\boldsymbol{p}} = E_{\boldsymbol{p}} > 0$, while $j = 3, 4$ labels the negative-energy solutions with $E^{(j)}_{\boldsymbol{p}} = -E_{\boldsymbol{p}} < 0$, where $E_{\boldsymbol{p}} = \sqrt{c^2|\boldsymbol{p}|^2 + m^2c^4}$. They are explicitly given by

$$u^{(1)}(\boldsymbol{p}) = \sqrt{\frac{E_{\boldsymbol{p}}+mc^2}{2E_{\boldsymbol{p}}}}\begin{pmatrix} 1 \\ 0 \\ \dfrac{cp_3}{E_{\boldsymbol{p}}+mc^2} \\ \dfrac{c(p_1+ip_2)}{E_{\boldsymbol{p}}+mc^2}\end{pmatrix}\,,\quad u^{(2)}(\boldsymbol{p}) = \sqrt{\frac{E_{\boldsymbol{p}}+mc^2}{2E_{\boldsymbol{p}}}}\begin{pmatrix} 0 \\ 1 \\ \dfrac{c(p_1-ip_2)}{E_{\boldsymbol{p}}+mc^2} \\ -\dfrac{cp_3}{E_{\boldsymbol{p}}+mc^2}\end{pmatrix}$$

$$u^{(3)}(\boldsymbol{p}) = \sqrt{\frac{E_{\boldsymbol{p}}+mc^2}{2E_{\boldsymbol{p}}}}\begin{pmatrix} -\dfrac{cp_3}{E_{\boldsymbol{p}}+mc^2} \\ -\dfrac{c(p_1+ip_2)}{E_{\boldsymbol{p}}+mc^2} \\ 1 \\ 0\end{pmatrix}\,,\quad u^{(4)}(\boldsymbol{p}) = \sqrt{\frac{E_{\boldsymbol{p}}+mc^2}{2E_{\boldsymbol{p}}}}\begin{pmatrix} -\dfrac{c(p_1-ip_2)}{E_{\boldsymbol{p}}+mc^2} \\ \dfrac{cp_3}{E_{\boldsymbol{p}}+mc^2} \\ 0 \\ 1\end{pmatrix}.$$

Exercise 1 *Conserved Quantities and Eigenstates*

We now explore the conserved quantities associated with the Dirac equation (1):

a) Show that the following operators correspond to conserved quantities:

   (i) energy $\hat{H}$,

   (ii) momentum $\hat{\boldsymbol{p}}$,

   (iii) helicity $\hat{h} = \hat{\boldsymbol{\Sigma}} \cdot \hat{\boldsymbol{p}}$,

   where the spin operator $\hat{\boldsymbol{\Sigma}} = (\hat{\Sigma}^1, \hat{\Sigma}^2, \hat{\Sigma}^3)$ is defined via

$$\hat{\Sigma}^i = \begin{pmatrix} \hat{\sigma}^i & 0 \\ 0 & \hat{\sigma}^i , \end{pmatrix}$$

   with $\hat{\sigma}^i$ being the Pauli matrices and the zeros denoting here 2×2 zero matrices.

(2 points)

b) Is the third component of the spin $\hat{\Sigma}^3$ also a conserved quantity? *(1 point)*

Now consider the general solution $\psi(\boldsymbol{r}, t)$ of Eq. (3).

c) Determine the condition on the coefficients $c_j(\boldsymbol{p})$ such that the state $\psi(\boldsymbol{r}, t)$ is normalized, that is, $\int_V \mathrm{d}\boldsymbol{r}\ \psi^\dagger(\boldsymbol{r}, t)\psi(\boldsymbol{r}, t) = 1$.
   Hint: Note that the spinors obey the orthonormality condition $(u^{(j)}(\boldsymbol{p}))^\dagger u^{(j')}(\boldsymbol{p}) = \delta_{jj'}$.

(1 point)

d) Determine further the momenta $\boldsymbol{p}$ and the indices j of the non-zero coefficients $c_j(\boldsymbol{p})$ such that the state $\psi(\boldsymbol{r}, t)$:

   (i) is an eigenstate of the energy $\hat{H}$,

   (ii) is an eigenstate of the momentum $\hat{\boldsymbol{p}}$,

   (iii) is an eigenstate of the spin component $\hat{\Sigma}^3$.

(2 points)

Exercise 2 *Lorentz boost*

We now explicitly examine the plane-wave solutions of the Dirac equation, denoted by $\psi^{(j)}_{\boldsymbol{p}}(\boldsymbol{r}, t)$ and defined in Eq. (4). In a previous exercise sheet, we verified that a change of reference frame via a Lorentz transformation $\hat{S}_L$ leaves the Dirac equation form-invariant. The state in the new reference frame is then given by $\psi'^{(j)}_{\boldsymbol{p}}(\boldsymbol{r}', t') = \hat{S}_L \psi^{(j)}_{\boldsymbol{p}}(\boldsymbol{r}, t)$, where the primed coordinates $(\boldsymbol{r}', t)$ are related to $(\boldsymbol{r}, t)$ by the Lorentz transformation associated with $\hat{S}_L$.

a) Consider the plane-wave solution $\psi^{(j)}_{\boldsymbol{p}}(\boldsymbol{r}, t)$ for a particle with momentum $\boldsymbol{p} = p\boldsymbol{e}_1$ (with $p > 0$) and index $j = 1$. Perform a Lorentz transformation into a reference frame moving along the x_1-direction with velocity $\boldsymbol{v} = (c^2 p / E_{\boldsymbol{p}})\boldsymbol{e}_1$, using the transformation operator

$$\hat{S}_L = \cosh\left(\frac{\chi}{2}\right) \hat{\mathbf{1}} - \hat{\alpha}^1 \sinh\left(\frac{\chi}{2}\right) , \tag{5}$$

where $\hat{\mathbf{1}}$ is the identity operator and χ denotes the rapidity, satisfying $\tanh(\chi) = pc/E_{\boldsymbol{p}}$. Compare the transformed solution with the solution for a particle at rest ($\boldsymbol{p} = 0$).

Hint: Impose $\cosh(\chi) = E_{\boldsymbol{p}}/(mc^2)$ and $\sinh(\chi) = pc/(mc^2)$. Don't forget to apply the Lorentz transformation to the spacetime coordinates $(\boldsymbol{r}, t) \to (\boldsymbol{r}', t')$ as well! *(2 points)*

b) What is the volume V' occupied by the particle in the new reference frame, expressed in terms of the original volume V? Verify that the transformed solution $\psi_{\boldsymbol{p}}^{(j)}(\boldsymbol{r}', t')$ is normalized. *(1 point)*

c) Now consider a general Lorentz transformation with spinor transformation

$$\hat{S}_L = \cosh\left(\frac{\chi}{2}\right)\hat{\mathbf{1}} + (\boldsymbol{n} \cdot \hat{\boldsymbol{\alpha}})\sinh\left(\frac{\chi}{2}\right),$$

where $\boldsymbol{n} \in \mathbb{R}^3$ is a unit vector indicating the direction of the boost, and χ is the rapidity satisfying $\tanh(\chi) = |\boldsymbol{p}|c/E_{\boldsymbol{p}}$, now for a general momentum $\boldsymbol{p}$. Show that applying this Lorentz transformation to the solutions for resting particles $\psi_{\boldsymbol{p}=0}^{(j)}(\boldsymbol{r}, t)$ maps positive-energy solutions with $j = 1, 2$ to other positive-energy solutions, and negative-energy solutions with $j = 3, 4$ to other negative-energy solutions of the Dirac equation.

(2 points)

Exercise 3 *Charge conjugation*

We now examine explicit the plane-wave solutions of the Dirac equation, denoted by $\psi_{\boldsymbol{p}}^{(j)}(\boldsymbol{r}, t)$ and defined in Eq. (4). The charge conjugate of a given solution $\psi_{\boldsymbol{p}}^{(j)}$ is defined as

$$\phi_{\boldsymbol{p}}^{(j)}(\boldsymbol{r}, t) = \hat{\mathcal{C}}(\psi_{\boldsymbol{p}}^{(j)}(\boldsymbol{r}, t))^*, \tag{6}$$

where $\hat{\mathcal{C}} = -i\hat{\gamma}^2$ is the charge-conjugation operator, and $\hat{\gamma}^2 = \hat{\beta}\hat{\alpha}^2$. Compute the explicit form of $\phi_{\boldsymbol{p}}^{(j)}(\boldsymbol{r}, t)$ for the negative-energy solutions of the Dirac equation, namely $j = 3, 4$. Then compare your result to the positive-energy solutions $\psi_{\boldsymbol{p}}^{(j=1,2)}(\boldsymbol{r}, t)$, and deduce the relation between them.

(2 points)