# Theoretical physics V
## Sheet 6

**SoSe 2025** **13.05.2025**

## Exercise 1 *Minimal coupling to the electromagnetic field*

Let us consider a particle of mass $m$ and electric charge $q$ coupled to an electromagnetic field $(\boldsymbol{E}, \boldsymbol{B})$ via the Lorentz force

$$\boldsymbol{F} = q\left(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B}\right), \tag{1}$$

with $\boldsymbol{v} = \dot{\boldsymbol{x}}$ being the velocity of the particle at position $\boldsymbol{x}$ at time $t$.

a) With the help of the relations $\boldsymbol{E} = -\boldsymbol{\nabla}\phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$ and $\boldsymbol{B} = \boldsymbol{\nabla} \times \boldsymbol{A}$, show that the Lorentz force can be written in terms of a potential $U$, such that

$$F_i = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial U}{\partial \dot{x}_i}\right) - \frac{\partial U}{\partial x_i} \tag{2}$$

and determine explicitly the form of the potential $U$.

**Hint:** Use that $\boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla}(\boldsymbol{v} \cdot \boldsymbol{A}) - (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{A}$.

*(2 points)*

b) Determine the form of the Lagrangian $L$ of the particle.

*(1 point)*

c) What is the canonically-conjugated momentum $\boldsymbol{p}$ to the position $\boldsymbol{x}$. Derive then the form of the Hamiltonian $H$ and determine the corresponding Hamilton equations. Show that they are equivalent to the Newton equation of motion.

*(3 points)*

## Exercise 2 *Fine structure of the hydrogen atom*

Let us consider an hydrogen atom consisting of a single electron of mass $m_e$ and electric charge $q = -e$ interacting with a proton. In its non-relativistic description, the electron is described by the Hamiltonian

$$\hat{H}_0 = \frac{\hat{\boldsymbol{p}}^2}{2m_e} - \frac{e^2}{4\pi\epsilon_0|\hat{\boldsymbol{r}}|}, \tag{3}$$

which has eigenvalues of the form $E_n = -\frac{1}{2}m_e c^2 \frac{\alpha^2}{n^2}$, where $\alpha$ is the fine structure constant $\alpha = \frac{e^2}{4\pi\epsilon_0}\frac{1}{\hbar c}$. These energy levels are degenerate and correspond to the eigenstates $|n, l, m_l\rangle$, where $n = 1, 2, 3, \ldots$, while $l = 0, 1, \ldots, n-1$ and $m_l = -l, \ldots, l$.

a) Show that at leading order, the relativistic correction to the kinetic energy takes the form

$$\hat{H}_r = -\frac{1}{8}\frac{\hat{\boldsymbol{p}}^4}{m_e^3 c^2}. \tag{4}$$

**Hint:** $\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \mathcal{O}(x^3)$.

*(1 point)*

b) Deduce then the expression of the first order correction of the kinetic energy $\Delta E_r$ as a perturbative expansion.

**Hint:** You shall use the following relations:

$$\left\langle \frac{1}{\hat{r}} \right\rangle_{n,l,m} = \frac{m_e c\alpha}{\hbar n^2} \quad \text{and} \quad \left\langle \frac{1}{\hat{r}^2} \right\rangle_{n,l,m} = \left(\frac{m_e c\alpha}{\hbar}\right)^2 \frac{1}{n^3(l+1/2)}$$

.

*(2 points)*

c) Another correcting term arising from relativistic effects is the spin-orbit coupling

$$\hat{H}_{SO} = \frac{1}{2m_e^2c^2r}\left(\partial_r V(r)\right) \hat{\boldsymbol{L}} \cdot \hat{\boldsymbol{S}}, \tag{5}$$

which also accounts for the spin degrees of freedom of the electron. Therefore, the state of the electron is now described by a state of the form $|n,l,m\rangle \otimes |s,m_s\rangle$, where $s = 1/2$ and $m_s \in \{-1/2, 1/2\}$.

Determine the form of the spin-orbit correction $\Delta E_{SO}$ in terms of $l$ and $j$, where $j$ are the eigenvalues of the total angular momentum operator $\hat{\boldsymbol{J}}^2 = (\hat{\boldsymbol{L}} + \hat{\boldsymbol{S}})^2$.

**Hint:** Use that $\left\langle \frac{1}{r^3} \right\rangle_{n,l,m} = \left(\frac{m_e c\alpha}{\hbar n}\right)^3 \frac{1}{l(l+1)(l+1/2)}$, for $l > 0$.

*(2 points)*

d) The final correction we account for is the so-called Darwin term

$$\hat{H}_D = \frac{\hbar^2}{8m^2c^2}\nabla^2 V(r) = \frac{\pi\hbar^3\alpha}{2m_e^2 c}\delta(r). \tag{6}$$

Determine the form of the corresponding correction $\Delta E_D$.

**Hint:** Use that $|\langle \boldsymbol{r}|n,l,m\rangle|^2_{\boldsymbol{r}=0} = \frac{1}{\pi}\left(\frac{m_e c\alpha}{\hbar n}\right)^3$, for $l = 0$, and $|\langle \boldsymbol{r}|n,l,m\rangle|^2_{\boldsymbol{r}=0} = 0$ otherwise.

*(1 point)*

e) Combining these different contributions, show that at first order in perturbation theory the energy levels $E_n$ are shifted by a factor

$$\Delta E_{n,j} = \frac{1}{2}mc^2\left(\frac{\alpha}{n}\right)^4\left(\frac{3}{4} - \frac{n}{j+1/2}\right), \tag{7}$$

for $l > 0$.

**Hint:** One shall treat the cases $j = l + 1/2$ and $j = l - 1/2$ separately.

*(2 points)*