Theoretical physics V
Sheet 7

SoSe 2025 **20.05.2025**

Exercise 1 *Gauge transformation and the wave function*

We consider a single electron, $q = -e$, in the non-relativistic limit in a region of vanishing magnetic field $\boldsymbol{B} = 0$. The minimal-coupling Hamiltonian for the gauge choice $\phi = 0$ reads

$$H = \frac{1}{2m}\left(\boldsymbol{p} + \frac{e}{c}\boldsymbol{A}\right)^2 . \tag{1}$$

a) Let $\psi^{(0)}(\boldsymbol{r}, t)$ be the solution of the Schrödinger equation for the case of a vanishing vector potential ($\boldsymbol{A} = 0$):

$$\mathrm{i}\hbar\frac{\partial}{\partial t}\psi^{(0)} = -\frac{\hbar^2\nabla^2}{2m}\psi^{(0)} . \tag{2}$$

Verify that $\psi(\boldsymbol{r}, t) = \mathcal{S}(\boldsymbol{r})\psi^{(0)}(\boldsymbol{r}, t)$, with

$$\mathcal{S}(\boldsymbol{r}) = \exp\left(-\frac{\mathrm{i}e}{\hbar c}\int^{s(\boldsymbol{r})} \mathrm{d}\boldsymbol{r}' \cdot \boldsymbol{A}(\boldsymbol{r}')\right) , \tag{3}$$

satisfies the Schrödinger equation for a non-vanishing $\boldsymbol{A}$. Here, $s(\boldsymbol{r})$ denotes a generic path with endpoint $\boldsymbol{r}$. *(2 points)*

b) We now perform a gauge transformation: $\boldsymbol{A} \to \boldsymbol{A}' = \boldsymbol{A} + \nabla\Lambda$, with some scalar function $\Lambda(\boldsymbol{r})$. Show that the solution ψ' of the Schrödinger equation for $\boldsymbol{A}'$ is related to the previous solution for $\boldsymbol{A}$ via $\psi' = \mathcal{U}\psi$. Determine the explicit form of the transformation $\mathcal{U}$. *(1 point)*

Exercise 2 *About the diffraction function*

Let us define the diffraction function as

$$\delta^{(T)}(E_f - E_i) = \frac{1}{2\pi\hbar}\int_{-T/2}^{T/2} \mathrm{d}\tau e^{i(E_f - E_i)\tau/\hbar} = \frac{1}{\pi}\frac{\sin[(E_f - E_i)T/2\hbar]}{E_f - E_i} , \tag{4}$$

for some time length T and energies E_i, E_f. For $T \to \infty$, the diffraction function approaches the Dirac delta function $\delta(x)$, $\lim_{T\to+\infty} \delta^{(T)}(E_f - E_i) = \delta(E_f - E_i)$. In the following, you shall demonstrate some properties of the diffraction function.

a) Plot the diffraction function $\delta^{(T)}(x)$ as a function of x.

(1 point)

b) Show the following identity

$$\int_{-\infty}^{+\infty} \mathrm{d}E \delta^{(T)}(E - E_i)\delta^{(T)}(E - E_f) = \delta^{(T)}(E_i - E_f), \tag{5}$$

for finite T.

(1 point)

c) Given the form of $\delta^{(T)}(E)$ provided in Eq. (4), deduce that

$$\int_{-\infty}^{+\infty} \mathrm{d}E_f[\delta^{(T)}(E_f - E_i)]^2 = \frac{T}{2\pi\hbar}. \tag{6}$$

Hint: Use that

$$\int_{-\infty}^{+\infty} \mathrm{d}x \frac{\sin^2(x)}{x^2} = \pi.$$

(2 points)