# Theoretical physics V
## Sheet 9

**SoSe 2025** **4.06.2025**

## Exercise 1 *Effective cutoff to the electromagnetic field*

Let us consider an electromagnetic field contained within a quantization volume $V$ with periodic boundary conditions. The Hamiltonian of this field reads

$$\hat{H} = \frac{1}{8\pi}\int_V \mathrm{d}^3\boldsymbol{r}(\hat{\boldsymbol{E}}^2(\boldsymbol{r}) + \hat{\boldsymbol{B}}^2(\boldsymbol{r}))\,, \tag{1}$$

with $\hat{\boldsymbol{E}}(\boldsymbol{r}) = \sum_\lambda(\boldsymbol{E}_\lambda e^{i\boldsymbol{k}_\lambda\cdot\boldsymbol{r}}\hat{a}_\lambda + \text{h.c.})$ and $\boldsymbol{E}_\lambda = \sqrt{\frac{2\pi\hbar\omega_\lambda}{V}}\boldsymbol{\epsilon}_\lambda$. Each mode $\lambda$ is characterized by vectors $\boldsymbol{k}_\lambda$ and $\boldsymbol{\epsilon}_\lambda$, with $\omega_\lambda = c|\boldsymbol{k}_\lambda|$ and $\boldsymbol{k}_\lambda = \frac{2\pi}{V^{1/3}}(n_x, n_y, n_z)$, $n_j \in \mathbb{Z}$. Using the bosonic commutation relations $[\hat{a}_\lambda, \hat{a}^\dagger_{\lambda'}] = \delta_{\lambda\lambda'}$, the Hamiltonian can be brought into the form

$$\hat{H} = \sum_\lambda \hbar\omega_\lambda\left(\hat{a}^\dagger_\lambda\hat{a}_\lambda + \frac{1}{2}\right). \tag{2}$$

a) We introduce the average electric field operat $\hat{\boldsymbol{E}}_{\text{av}}(\boldsymbol{r})$ as

$$\hat{\boldsymbol{E}}_{\text{av}}(\boldsymbol{r}) = \int_V \mathrm{d}^3\boldsymbol{\rho}\, f(\boldsymbol{\rho})\hat{\boldsymbol{E}}(\boldsymbol{r}-\boldsymbol{\rho}), \quad \text{with} \quad f(\boldsymbol{\rho}) = f(\rho) = \frac{1}{(\pi\Delta^2)^{3/2}}\exp(-\rho^2/\Delta^2)\,. \tag{3}$$

Using definition (3), determine the value of the expectation value $\langle\text{vac}|\hat{\boldsymbol{E}}_{\text{av}}(\boldsymbol{r})|\text{vac}\rangle$. Show as well that

$$\langle\text{vac}|\hat{\boldsymbol{E}}_{\text{av}}(\boldsymbol{r})\cdot\hat{\boldsymbol{E}}_{\text{av}}(\boldsymbol{r})|\text{vac}\rangle = \sum_\lambda |\boldsymbol{E}_\lambda|^2\overline{f}(\boldsymbol{k}_\lambda)\,, \tag{4}$$

where $\overline{f}$ is a function of $f$ and $\boldsymbol{k}_\lambda$, and $|\text{vac}\rangle$ denotes the vacuum state of the electromagnetic field.

*(2 points)*

b) Determine the explicit form of $\overline{f}(\boldsymbol{k}_\lambda)$ introduced in Eq. (4) as a function of $\Delta$. Discuss the expectation values determined previously in the limits $\Delta \to 0$ and $\Delta \to +\infty$.

*(2 points)*

c) Using the dispersion relation $\omega_\lambda = c|\boldsymbol{k}_\lambda|$, rewrite $\overline{f}$ as a function of $\omega_\lambda$. Show that $\Delta$ defines the cutoff frequency $\omega_c$.

*(1 point)*

d) Using the cutoff function $\overline{f}(\omega_\lambda)$, demonstrate that the zero-point energy

$$E_0 = \langle\text{vac}|\sum_\lambda \overline{f}(\omega_\lambda)\hbar\omega_\lambda\left(\hat{a}^\dagger_\lambda\hat{a}_\lambda + \frac{1}{2}\right)|\text{vac}\rangle$$

takes the simple form

$$E_0 = V\frac{\hbar}{4\pi^2 c^3}\omega_c^4\,. \tag{5}$$

Note that the summation $\sum_\lambda$ is a compact notation for $\sum_{\boldsymbol{k}_\lambda}\sum_{\boldsymbol{\epsilon}\perp\boldsymbol{k}_\lambda}$. Determine the asymptotic behavior of $E_0$ as a function of $V$ and as a function of $\omega_c$. *(2 points)*

**Hints:** We provide two identities on Gaussian integrals:

$$\int_{-\infty}^{+\infty} \mathrm{d}x\, e^{-ax^2+bx} = \sqrt{\frac{\pi}{a}}e^{+\frac{b^2}{4a}}\,,\quad \mathrm{Re}[a]>0\,,$$

$$\int_{0}^{+\infty} \mathrm{d}x\, x^{2n+1}e^{-x^2/a^2} = \frac{n!}{2}a^{2n+2}\,,\quad \mathrm{Re}[a^2]>0\,,\quad n\geq 0\,.$$

## Exercise 2 *The Quantum String*

Consider the Hamiltonian

$$\hat{H} = \sum_{j=1}^{N}\left(\frac{\hat{p}_j^2}{2m} + \frac{\kappa}{2}(\hat{q}_j - \hat{q}_{j+1})^2\right)\,, \tag{6}$$

where the positions $\hat{q}_j$ and momenta $\hat{p}_j$ are canonically-conjugated variables, satisfying the commutation relations

$$[\hat{p}_j, \hat{q}_k] = -i\hbar\delta_{jk}\,. \tag{7}$$

We impose here periodic boundary conditions, such that $\hat{q}_j = \hat{q}_{j+N}$ and $\hat{p}_j = \hat{p}_{j+N}$. In this exercise, our objective is to show that $\hat{H}$ can be written as the sum of harmonic oscillators.

a) We start by expanding the operators $\hat{q}_j, \hat{p}_j$ in Fourier series:

$$\hat{q}_j(t) = \frac{1}{\sqrt{N}}\sum_{n=-N/2}^{N/2}\hat{Q}_n(t)e^{\mathrm{i}2\pi nj/N} \quad \text{and} \quad \hat{p}_j(t) = \frac{1}{\sqrt{N}}\sum_{n=-N/2}^{N/2}\hat{P}_n(t)e^{\mathrm{i}2\pi nj/N}\,. \tag{8}$$

Verify that the series in Eq. (8) obey the periodic boundary conditions $\hat{q}_j = \hat{q}_{j+N}$ and $\hat{p}_j = \hat{p}_{j+N}$. Given the hermiticity of $\hat{q}_j$ and $\hat{p}_j$ and the commutation relation (7), show that the operators $\hat{Q}_n$ and $\hat{P}_n$ shall fulfill the relations

$$[\hat{P}_n^\dagger, \hat{Q}_m] = -i\hbar\delta_{nm}\,,\quad \hat{P}_n^\dagger = \hat{P}_{-n}\,,\quad \hat{Q}_n^\dagger = \hat{Q}_{-n}\,. \tag{9}$$

*(2 points)*

b) Show that using Eqs. (8) and (9), the Hamiltonian (6) can be recast in the form

$$\hat{H} = \sum_{n=-N/2}^{N/2}\left(\frac{\hat{P}_n^\dagger\hat{P}_n}{2m} + \frac{1}{2}m\omega_n^2\hat{Q}_n^\dagger\hat{Q}_n\right)\,, \tag{10}$$

with

$$\omega_n^2 = \frac{4\kappa}{m}\sin^2\left(\frac{\pi n}{N}\right)\,. \tag{11}$$

*(2 points)*

c) We further define the operators $\hat{a}_n, \hat{a}_n^\dagger$, such that

$$\hat{Q}_n = \sqrt{\frac{\hbar}{2m\omega_n}}(\hat{a}_{-n}^\dagger + \hat{a}_n) \quad \text{and} \quad \hat{P}_n = \mathrm{i}\sqrt{\frac{\hbar m\omega_n}{2}}(\hat{a}_{-n}^\dagger - \hat{a}_n)\,. \tag{12}$$

Derive the commutation relations for $\hat{a}_n$ and $\hat{a}_n^\dagger$. Subsequently, show that the Hamiltonian (10) can be rewritten as

$$\hat{H} = \sum_{n=-N/2}^{N/2} \hbar\omega_n \left(\hat{a}_n^\dagger \hat{a}_n + \frac{1}{2}\right)\,. \tag{13}$$

What are the eigenvalues of the Hamiltonian? Discuss why $\omega_n$ in Eq. (13) is positive.

*(2 points)*

d) Consider the ground-state energy of Eq. (13) and determine its expression in the continuum limit by taking $N \to \infty$. *(1 point)*

## **Exercise 3** *Lagrange density of the electromagnetic field*

The Lagrange density of the electromagnetic field (in Gaussian units) is given by

$$\mathcal{L}_{\mathrm{EMF}} = \frac{|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2}{8\pi} - \rho\Phi + \frac{\boldsymbol{j}\cdot\boldsymbol{A}}{c}\,, \tag{14}$$

where $\boldsymbol{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$ and $\boldsymbol{B} = \nabla \times \boldsymbol{A}$. It has already been shown in the lecture that the Euler-Lagrange equation of $\mathcal{L}_{\mathrm{EMF}}$ with respect to the scalar potential $\Phi$ yields the Maxwell equation $\nabla\cdot\boldsymbol{E} = 4\pi\rho$. In this exercise, you shall explicit show that the Euler-Lagrange equation with respect to the $j$-th component of the vector potential $\boldsymbol{A}$ yields the $j$-th component of the Maxwell equation $\nabla \times \boldsymbol{B} - \frac{1}{c}\frac{\partial \boldsymbol{E}}{\partial t} = \frac{4\pi}{c}\boldsymbol{j}$.

*(3 points)*