# Theoretical physics V
## Sheet 10

**SoSe 2025** **10.06.2025**

## Exercise 1 *Classical limit of electrodynamics*

Let us consider a polarized monochromatic electromagnetic wave of frequency $\omega$ with a wavevector $\boldsymbol{k}$. The vector potential associated to the wave is described by the operator

$$\hat{\boldsymbol{A}}(\boldsymbol{r},t) = \sqrt{\frac{2\pi c^2\hbar}{\omega V}}\left[\hat{a}e^{i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)} + \hat{a}^\dagger e^{-i(\boldsymbol{k}\cdot\boldsymbol{r}-\omega t)}\right]\boldsymbol{e}, \tag{1}$$

where $\boldsymbol{e}$ is real-valued.

a) Determine the form of the corresponding electric field operator $\hat{\boldsymbol{E}}(\boldsymbol{r},t)$ and compute the expectation value $\langle n|\hat{\boldsymbol{E}}(\boldsymbol{r},t)|n\rangle$ for any Fock state $|n\rangle$.

*(1 point)*

b) Coherent states of an harmonic oscillator are defined as eingenstates of the annihilation operator: $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, where $\alpha \in \mathbb{C}$. Defining $\alpha = \sqrt{\overline{n}}e^{i\phi}$, determine the expectation value of the electric field $\langle\alpha|\hat{\boldsymbol{E}}(\boldsymbol{r},t)|\alpha\rangle$. Give an interpretation of the number $\overline{n}$.

*(2 points)*

c) Evaluate the variance of the amplitude of the electric field $\Delta E = \sqrt{\langle\alpha|\hat{\boldsymbol{E}}^2|\alpha\rangle - \langle\alpha|\hat{\boldsymbol{E}}|\alpha\rangle^2}$, then compare it to the amplitude of the electric field. In which limit do we recover the behavior of a classical system?

*(2 points)*

## Exercise 2 *Photons with circular polarization*

Let us consider a monochromatic electromagnetic field with a wavevector $\boldsymbol{k}$ and two possible linear polarisations $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, such that it respect the conditions $\boldsymbol{k}\cdot\boldsymbol{e}_1 = \boldsymbol{k}\cdot\boldsymbol{e}_2 = \boldsymbol{e}_1\cdot\boldsymbol{e}_2 = 0$. The vectors $\boldsymbol{e}_{1,2}$ are unit vectors.
The energy of the quantized electromagnetic field reads

$$\hat{H} = \hbar c|\boldsymbol{k}|(\hat{a}_1^\dagger\hat{a}_1 + \hat{a}_2^\dagger\hat{a}_2 + 1), \tag{2}$$

where the operators $\hat{a}_j^\dagger$ act on the vacuum to create a photon at wavelength $\boldsymbol{k}$ and polarization $\boldsymbol{e}_j$. The annhihilation and creation operators obey bosonic commutation relations

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}, \tag{3}$$

$$[\hat{a}_i^{(\dagger)}, \hat{a}_j^{(\dagger)}] = 0. \tag{4}$$

a) We shall consider the circular polarization vectors

$$\boldsymbol{e}_\pm = \mp\frac{1}{\sqrt{2}}(\boldsymbol{e}_1 \pm i\boldsymbol{e}_2). \tag{5}$$

Using the definition of the quantized photon field

$$\hat{\boldsymbol{A}}(\boldsymbol{r},t) = \sqrt{\frac{2\pi c^2\hbar}{\omega V}}\sum_j \left[\hat{a}_j(t)e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{e}_j + \hat{a}_j^\dagger(t)e^{-i\boldsymbol{k}\cdot\boldsymbol{r}}\boldsymbol{e}_j^*\right], \tag{6}$$

show that it can written in terms of operators $\hat{a}_\pm$ and $\hat{a}_\pm^\dagger$ annihilating and creating photons with polarization $\boldsymbol{e}_\pm$. Determine the explicit form of $\hat{a}_\pm$ and show that they obey to bosonic commutation relations.

*(2 points)*

b) Let us apply to $\boldsymbol{e}_+$ and $\boldsymbol{e}_-$ a rotation around the wavevector $\boldsymbol{k}$ by an infinitesimal angle $\delta\theta$. Show that the polarization vectors are changed by a factor $\delta\boldsymbol{e}_\pm = \mp i\delta\theta\boldsymbol{e}_\pm$. What can you deduce about the spin of the photon ?

**Hint:** We remind that for a two-level system, the rotation operator $\hat{R}(\theta)$ by an angle $\theta$ around the axis $\boldsymbol{n}$ reads

$$\hat{R}(\theta) = \exp(i\theta(\boldsymbol{n}\cdot\boldsymbol{\sigma})) = I_2\cos\theta + i(\boldsymbol{n}\cdot\boldsymbol{\sigma})\sin\theta,$$

where the elements of the vector $\boldsymbol{\sigma}$ are Pauli matrices.

*(2 points)*

## Exercise 3 *The Casimir effect (12 points)*

Considering two parallel metallic plates in the vacuum, it is observed that an effective force seems to attract the plates to each other, without any explanation by classical physics. In the following, we will show that this phenomenon, coined as the Casimir effect, has a quantum origin and can be captured by quantum electrodynamics.

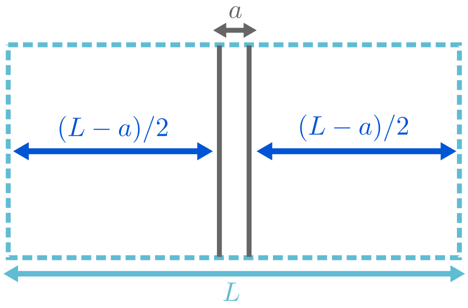


Figure 1: Schematic representation of a box of length $L$ (dashed rectangle) separated in three sections by two parallel plates (solid lines) separated by distance $a$.

For the sake of simplicity, we assume thereafter that the system is one-dimensional.

a) Let us consider a box of length $L$ containing an infinity of possible modes of the quantum electromagnetic field. Each of these modes is associated to a wave function of the form $\psi_k(x) = a\cos(kx) + b\sin(kx)$, with wave vector $k$. What are the boundary conditions imposed by the box? Show that the box imposes modes with wave vectors $k_n = \pi n/L$.

*(1 point)*

b) The modes of the electromagnetic fields are treated as a collection of independent harmonic oscillators such that the Hamiltonian reads

$$\hat{H} = \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2}\right), \tag{7}$$

where $\omega_k = ck$. To prevent the divergence of energy, we introduce a cut-off to the frequency of the electromagnetic field: $\omega \to \omega e^{-\omega/\omega_c}$. Determine the vacuum energy $E_0(L)$ and show that it can be written as

$$E_0(L) = \frac{\pi\hbar c}{8L} \frac{1}{\sinh^2\left(\dfrac{\pi c}{2L\omega_c}\right)}. \tag{8}$$

**Hint:** Use the relation

$$\sum_n n e^{-an} = -\frac{\mathrm{d}}{\mathrm{d}x}\frac{1}{1-e^{-x}}\bigg|_{x=a}.$$

*(2 points)*

c) Assuming that $\pi c/(\omega_c L) \ll 1$, perform a Taylor expansion in $1/L$ and show that

$$E_0(L) = \frac{\hbar}{4}\left[\frac{2L\omega_c^2}{\pi c} - \frac{\pi c}{6L}\right] + \mathcal{O}\left(\frac{1}{\omega_c^2}\right). \tag{9}$$

*(2 points)*

d) Assume now that, as depicted on Fig. 1 , the box is split into three sections by the insertion of two plates: two of length $(L-a)/2$ and one of length $a$, corresponding to the distance between the plates. Determine the total vacuum energy $E_{\text{tot}}$ as a function of $L$ and $a$. Show that in the limit $a \ll L$, the total vacuum energy is approximated by

$$E_{\text{tot}} \simeq \frac{\hbar\omega_c^2}{2\pi c}L - \frac{\pi\hbar c}{6L} - \frac{\pi\hbar c}{24a}. \tag{10}$$

Plot the energy as a function of $a$.

*(2 points)*

e) Show then that the energy (10) gives rise to an attractive force $F$ between the plates that is independent of the cut-off $\omega_c$.

*(1 point)*