Theoretical physics V
Sheet 11

SoSe 2025 **18.06.2025**

Exercise 1 *Goeppert-Mayer transformation*

Let us consider an electron of mass m and charge $q = -e$ in a central, binding potential $V_0(\boldsymbol{r}) = V_0(|\boldsymbol{r}|)$. The electron is in a bound state and we assume that the electric dipole approximation holds. The interaction of the electron with an external electric field $\boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}$ is described by the minimal coupling Hamiltonian

$$\hat{H} = \frac{1}{2m}\left(\hat{\boldsymbol{p}} + \frac{e}{c}\hat{\boldsymbol{A}}(\boldsymbol{R})\right)^2 + V_0(\hat{\boldsymbol{r}})\,, \tag{1}$$

which is written in the electric dipole approximation and $\boldsymbol{R}$ denotes the center of the potential. In the following, we will assume that we are in the Coulomb gauge ($\boldsymbol{\nabla} \cdot \boldsymbol{A}(\boldsymbol{R}) = 0$). Because of the electric dipole approximation, moreover, $\hat{\boldsymbol{p}}$ commutes with $\hat{\boldsymbol{A}}(\boldsymbol{R})$.

a) Considering a time-dependent unitary transformation $\hat{U}(t)$, such that a state $|\tilde{\psi}\rangle$ transforms into $|\psi\rangle$ following the relation $|\psi\rangle = \hat{U}|\tilde{\psi}\rangle$. Considering that the evolution of $|\psi\rangle$ is governed by the Schrödinger equation

$$i\hbar\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = \hat{H}|\psi\rangle, \tag{2}$$

determine the effective Hamiltonian $\hat{H}'$ describing the evolution of the transformed state $|\tilde{\psi}\rangle$.

(1 point)

b) We introduce the unitary operator $\hat{U}(t) = \exp\left(\frac{i}{\hbar c}\hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{A}}(\boldsymbol{R}, t)\right)$, where $\hat{\boldsymbol{d}} = -e\hat{\boldsymbol{r}}$ is the dipolar moment of the electron. Determine the expression of the commutator $\left[\hat{\boldsymbol{p}}, \hat{U}\right]$.

Hint: You may use that $\left[\hat{p}_i, (\hat{r}_j)^k\right] = -i\hbar k(\hat{r}_j)^{k-1}\delta_{ij}$, with $k \in \mathbb{N}$.

(2 points)

c) Show then that the effective Hamiltonian associated to the Goeppert-Meyer transformation $\hat{U}(t)$ introduced in the previous question reads

$$\hat{H}' = \frac{1}{2m}\hat{\boldsymbol{p}}^2 + V_0(\hat{\boldsymbol{r}}) - \hat{\boldsymbol{d}} \cdot \hat{\boldsymbol{E}}(\boldsymbol{R}), \tag{3}$$

which is pivotal to the description of the light-matter interaction.

(2 points)

Exercise 2 *Jaynes-Cummings model*

Let us approximate an atom by a two-level system consisting of a ground state $|g\rangle$ and an excited state $|e\rangle$ with transition frequency ω_0. This atom is set in an optical cavity formed by two mirrors, such that the light field within the cavity oscillates at a single frequency ω. The Hamiltonian of this system reads

$$\hat{H}_0 = \hbar\omega_0 |e\rangle\langle e| + \hbar\omega \hat{a}^\dagger \hat{a}, \tag{4}$$

where $\hat{a}$ annihilates a quantum of excitation of the light field. The light-matter interaction in this context is modeled by an interaction Hamiltonian

$$\hat{V} = \hbar g \hat{\sigma}^x \otimes (\hat{a}^\dagger + \hat{a}), \tag{5}$$

where $\hat{\sigma}^x = |g\rangle\langle e| + |e\rangle\langle g|$. This is the celebrated Jaynes-Cummings model. In the following, we will assume that the cavity is resonant with the transition frequency of the atom $\omega = \omega_0$.

a) Determine the matrix element of the interaction term $\hat{V}$ in the basis $\{|g,n\rangle, |e,n\rangle\}$ combining the two levels $\{|g\rangle, |e\rangle\}$ of the atom and the levels $\{|n\rangle\}$ of the harmonic oscillator. Represent graphically the processes induced by the light-matter interaction and identify which one are resonant.

(2 points)

b) Determine up to second order in perturbation theory the energy shift of the ground state $|g,0\rangle$ in the weak-coupling approximation ($g \ll \omega_0$).

Hint: The first- and second-order energy corrections for an eigenstate $|E_0\rangle$ of the unperturbed Hamiltonian $\hat{H}_0$ due to a perturbation $\hat{V}$ are given by

$$\delta E^{(1)} = \langle E_0|\hat{V}|E_0\rangle \quad \text{and} \quad \delta E^{(2)} = \lim_{\eta\to 0} \langle E_0|\hat{V}\frac{1}{E_0 - \hat{H}_0 + i\eta}\hat{V}|E_0\rangle,$$

where $\hat{H}_0|E_0\rangle = E_0|E_0\rangle$. *(1 point)*

c) We assume that at time $t = 0$, the atom and the cavity are in the state $|\psi_0\rangle = |e,0\rangle$. Determine the dynamics of the system, considering only resonant interaction processes between the atom and the cavity.

(1 point)

Exercise 3 *Parity operator*

In this exercise, we study the parity operator $\hat{\Pi}$ and its properties. Its action on the position and momentum eigenstates is defined by $\hat{\Pi}|\vec{r}\rangle = |-\vec{r}\rangle$ and $\hat{\Pi}|\vec{p}\rangle = |-\vec{p}\rangle$. As a result, for a given state $|\psi\rangle$ with wavefunction $\psi(\vec{r}) = \langle\vec{r}|\psi\rangle$, we have that $\psi(-\vec{r}) = \langle\vec{r}|\hat{\Pi}|\psi\rangle$, which illustrates that the action of $\hat{\Pi}$ corresponds to a reflection about the origin.

a) Prove the following properties of the parity operator:

1. $\hat{\Pi}^\dagger = \hat{\Pi}$
2. $\hat{\Pi}\hat{\vec{r}}\hat{\Pi} = -\hat{\vec{r}}$ and $\hat{\Pi}\hat{\vec{p}}\hat{\Pi} = -\hat{\vec{p}}$
3. $\hat{\Pi}^2 = 1$
4. Show that the eigenvalues of $\hat{\Pi}$ are ± 1.

(2 points)

Consider now a single-particle Hamiltonian of the form

$$\hat{H} = \frac{\hat{\vec{p}}^2}{2m} + V(\hat{\vec{r}})\,,$$

where the potential $V(\vec{r})$ is an analytic function of the position $\vec{r}$.

b) Determine the condition that the potential $V(\hat{\vec{r}})$ must satisfy in order for $\hat{H}$ to commute with $\hat{\Pi}$, i.e., $[\hat{H}, \hat{\Pi}] = 0$. *(1 point)*

c) Suppose that $[\hat{H}, \hat{\Pi}] = 0$, such that $\hat{H}$ and $\hat{\Pi}$ have a common eigenbasis $\{|\Psi\rangle\}$. Show that $\langle\Psi|\hat{\vec{r}}|\Psi\rangle = 0$ and $\langle\Psi|\hat{\vec{p}}|\Psi\rangle = 0$. *(1 point)*

Exercise 4 *Translation in momentum space*

Let $|\vec{p}\rangle$ be the momentum eigenstate with momentum $\vec{p}$, thus satisfying $\hat{\vec{p}}|\vec{p}\rangle = \vec{p}|\vec{p}\rangle$. Define the operator $\hat{T}(\vec{q}) = \exp(\mathrm{i}\vec{q}\cdot\hat{\vec{r}}/\hbar)$, where $\hat{\vec{r}}$ is the position operator and $\vec{q}$ is an arbitrary vector. Show that $\hat{T}(\vec{q})$ acts as a translation in momentum space, i.e., $\hat{T}(\vec{q})|\vec{p}\rangle = |\vec{p}+\vec{q}\rangle$. *(1 point)*